Junior problems

J379. Prove that for any nonnegative real numbers a, b, c the following inequality holds:

$$(a-2b+4c)(-2a+4b+c)(4a+b-2c) \leq 27abc.$$

Proposed by Adrian Andreescu, Dallas, Texas

Solution by Utsab Sarkar, West Bengal, India

Set variables x, y, z such that

$$\begin{aligned} a-2b+4c &= x \\ -2a+4b+c &= y \\ 4a+b-2c &= z \end{aligned}$$

Note that $a = \dfrac{2z+x}{9}, b = \dfrac{2y+z}{9}$, and $c = \dfrac{2x+y}{9}$. Therefore, our given inequality transforms into

$$(2x+y)(2y+z)(2z+x) \geq 27xyz \tag{$\star$}$$

Now observe that the LHS of ($\star$) is always nonnegative since a, b, c are nonnegative. If $xyz \leq 0$, the inequality is trivial; so we assume $xyz > 0$. Thus either exactly two or zero of x, y, z are negative. If two of them are negative, say $x < 0, y < 0, z > 0$, then $c = \frac{2x+y}{9} < 0$, a contradiction. Hence x, y, z are positive, so by AM-GM:

$$(2x+y)(2y+z)(2z+x) \geq 3\sqrt[3]{x^2y} \cdot 3\sqrt[3]{y^2z} \cdot 3\sqrt[3]{z^2x} = 27xyz$$

Equality holds if and only if $x = y = z$, or $a = b = c$.

Also solved by Rade Krenkov, SOU "Goce Delcev", Valandovo, Macedonia; SooYoung Choi, Seoul Chung-Dam Middle School, Republic of Korea; Paul Revenant, Lycée du Parc, Lyon, France; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Daniel Lasaosa, Pamplona, Spain; Jamal Gadirov, Istanbul University; Anderson Torres, Sao Paulo, Brazil; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Polyahedra, Polk State College, USA; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Brian Zilli, Hofstra University; Arkady Alt, San Jose, CA, USA; Andrianna Boutsikou, High School of Nea Makri, Athens, Greece; Adnan Ali, A.E.C.S-4, Mumbai, India.

J380. Let $x_1, x_2, \cdots, x_n$ be nonnegative real numbers such that

$$x_1 + x_2 + \cdots + x_n = 1.$$

(a) Find the minimum value of

$$x_1\sqrt{1+x_1} + x_2\sqrt{1+x_2} + \cdots + x_n\sqrt{1+x_n}.$$

(b) Find the maximum value of

$$\frac{x_1}{1+x_2} + \frac{x_2}{1+x_3} + \cdots + \frac{x_n}{1+x_1}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Adnan Ali, A.E.C.S-4, Mumbai, India

(a) Consider the function $f(x) = x\sqrt{1+x}$, over the domain $[0,1]$. Clearly $f(x)$ is convex over the domain, as $f''(x) = (x+1)^{-1/2}(\frac{3x+4}{4x+4}) > 0$ for $x \in [0,1]$. From Jensen's inequality,

$$x_1\sqrt{1+x_1} + x_2\sqrt{1+x_2} + \cdots + x_n\sqrt{1+x_n} \geq nf(\tfrac{1}{n}) = \sqrt{1+\tfrac{1}{n}}.$$

This value is indeed achieved for $x_i = \frac{1}{n}$ for all $1 \leq i \leq n$.

(b) Note that

$$\frac{x_1}{1+x_2} + \frac{x_2}{1+x_3} + \cdots + \frac{x_n}{1+x_1} \leq x_1 + x_2 + \cdots + x_n = 1,$$

because $\frac{x_k}{1+x_{k+1}} \leq x_k$. This value is indeed achieved for $x_1 = 1$, $x_i = 0$ for $2 \leq i \leq n$.

Also solved by Wada Ali, Ben Badis College, Algeria; Daniel Lasaosa, Pamplona, Spain; Ghenghea Daniel; Arkady Alt, San Jose, CA, USA; Andrianna Boutsikou, High School of Nea Makri, Athens, Greece; Andreas Charalampopoulos, 4th Lyceum of Glyfada, Glyfada, Greece; Utsab Sarkar, West Bengal, India; Polyahedra, Polk State College, USA.

J381. Let x, y, z be positive real numbers such that $x+y+z=3$. Prove that

$$\frac{xy}{4-y}+\frac{yz}{4-z}+\frac{zx}{4-x}\leq 1.$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

Solution by Adnan Ali, A.E.C.S-4, Mumbai, India

The inequality in homogeneous form becomes

$$\frac{9xy}{4x+y+4z}+\frac{9yz}{4x+4y+z}+\frac{9zx}{x+4y+4z}\leq 3.$$

From the Cauchy-Schwarz Inequality, we know

$$\frac{9}{4x+y+4z}\leq\frac{2}{2x+z}+\frac{1}{2z+y}.$$

So,

$$\sum_{cyc}\frac{9xy}{4x+y+4z}\leq\sum_{cyc}\frac{2xy}{2x+z}+\sum_{cyc}\frac{xy}{2z+y}=\sum_{cyc}\frac{2xy}{2x+z}+\sum_{cyc}\frac{yz}{2x+z}=\sum_{cyc}\frac{2xy+yz}{2x+z}=3,$$

and the proof is complete.

Also solved by Wada Ali, Ben Badis College, Algeria; Tamoghno Kandar, AECS-4, Mumbai, India; Viet Hoang, Takapuna Grammar School, Auckland, New Zealand; Thao Le, University of Auckland, Auckland, New Zealand; Paul Revenant, Lycée du Parc, Lyon, France; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Daniel Lasaosa, Pamplona, Spain; Ghenghea Daniel; Arkady Alt, San Jose, CA, USA; Utsab Sarkar, West Bengal, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Polyahedra, Polk State College, USA.

J382. Find all triples (x, y, z) of real numbers with $x, y, z > 1$ satisfying

$$\left(\frac{x}{2}+\frac{1}{x}-1\right)\left(\frac{y}{2}+\frac{1}{y}-1\right)\left(\frac{z}{2}+\frac{1}{z}-1\right)=\left(1-\frac{x}{yz}\right)\left(1-\frac{y}{zx}\right)\left(1-\frac{z}{xy}\right).$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Polyahedra, Polk State College, FL, USA

First, if $1-\frac{x}{yz}<0$, then $x>yz$, so $1-\frac{y}{zx}>1-\frac{1}{z^2}>0$ and $1-\frac{z}{xy}>1-\frac{1}{y^2}>0$. But $\frac{x}{2}+\frac{1}{x}-1=\frac{(x-1)^2+1}{2x}>0$, so the given equation cannot be satisfied because the LHS is positive while the RHS is negative. Therefore, we must have $1-\frac{x}{yz}\geq 0$, $1-\frac{y}{zx}\geq 0$, and $1-\frac{z}{xy}\geq 0$. By the AM-GM inequality, $\frac{x}{yz}+\frac{y}{zx}\geq\frac{2}{z}$, so

$$\begin{aligned}\left(1-\frac{x}{yz}\right)\left(1-\frac{y}{zx}\right) &= 1-\left(\frac{x}{yz}+\frac{y}{zx}\right)+\frac{1}{z^2}\leq 1-\frac{2}{z}+\frac{1}{z^2}\\ &\leq 1-\frac{2}{z}+\frac{1}{z^2}+\left(\frac{z}{2}-1\right)^2=\left(\frac{z}{2}+\frac{1}{z}-1\right)^2,\end{aligned}$$

with equality if and only if $z=2$ and $x=y$. By repeating the same inequality for x and y, we see that

$$\left(\frac{x}{2}+\frac{1}{x}-1\right)\left(\frac{y}{2}+\frac{1}{y}-1\right)\left(\frac{z}{2}+\frac{1}{z}-1\right)\geq\left(1-\frac{x}{yz}\right)\left(1-\frac{y}{zx}\right)\left(1-\frac{z}{xy}\right),$$

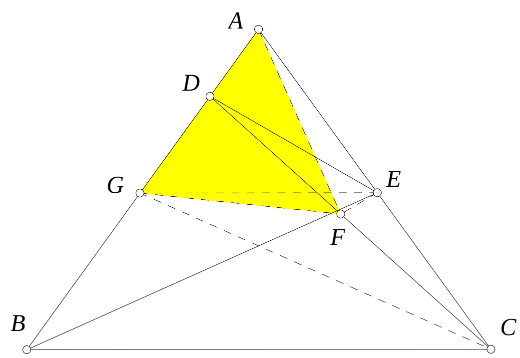
with equality if and only if $x=y=z=2$.

Also solved by Daniel Lasaosa, Pamplona, Spain.

J383. Let ABC be a triangle with $AB = AC$ and $\angle BAC = 72°$. Let D and E be the points on sides AB and AC, respectively, such that $\angle ACD = 12°$ and $\angle ABE = 30°$. Prove that $DE = CE$.

Proposed by Marius Stănean, Zalău, România

Solution by Polyahedra, Polk State College, USA



As in the figure, locate point F on CD such that $FA = FC$, and locate point G on AB such that $AG = AF$. Since $\angle GAF = 60°$, $\triangle AGF$ is equilateral. Hence, $GF = FA = FC$ and $\angle DFG = 60° - \angle AFD = 36°$. So $\angle GCB = 54° - 12° - 18° = 24° = \angle CBE$. Therefore, $BG = CE$, and thus $AE = AG = AF$. Consequently, $\angle AEF = \angle EFA = 84° = \angle BDC$, which implies that A, D, F, E lie on a circle. Hence, $\angle FDE = \angle FAE = 12° = \angle ACD$, from which the claim follows.

Also solved by Daniel Lasaosa, Pamplona, Spain; Utsab Sarkar, West Bengal, India.

J384. In triangle ABC, $A < B < C$. Prove that

$$\cos\frac{A}{2}\csc\frac{B-C}{2}+\cos\frac{B}{2}\csc\frac{C-A}{2}+\cos\frac{C}{2}\csc\frac{A-B}{2}<0.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Arkady Alt, San Jose, CA, USA

Note that

$$\cos\frac{A}{2}\csc\frac{B-C}{2}=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin\frac{A}{2}\sin\frac{B-C}{2}}=\frac{\sin A}{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}=\frac{\sin A}{\sin B-\sin C}=\frac{a}{b-c}.$$

Thus, the inequality is equivalent to $\sum_{\text{cyc}}\frac{a}{b-c}<0$. Since $A<B<C$, we know $a<b<c$, and so

$$\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}=\frac{b}{c-a}-\frac{c}{b-a}-\frac{a}{c-b}=\left(\frac{b}{c-a}-\frac{a}{c-b}\right)-\frac{c}{b-a}=-\frac{(b-a)(a+b-c)}{(c-b)(c-a)}-\frac{c}{b-a}<0.$$

Also solved by Nicuşor Zlota ,"Traian Vuia" Technical College, Focşani, Romania; Daniel Lasaosa, Pamplona, Spain; Adnan Ali, A.E.C.S-4, Mumbai, India; Utsab Sarkar, West Bengal, India; Anderson Torres, Sao Paulo, Brazil; Polyahedra, Polk State College, USA.

Senior problems

S379. Prove that in any triangle ABC

$$\cos 3A + \cos 3B + \cos 3C + \cos(A-B) + \cos(B-C) + \cos(C-A) \geq 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Li Zhou, Polk State College, USA
By the sum-to-product formulas,

$$\begin{aligned} & \cos 3A + \cos 3B + \cos 3C = 2\cos\frac{3A+3B}{2}\cos\frac{3A-3B}{2} - 2\cos^2\frac{3A+3B}{2} + 1 \\ = \quad & 2\cos\frac{3A+3B}{2}\left(\cos\frac{3A-3B}{2} - \cos\frac{3A+3B}{2}\right) + 1 = 1 - 4\sin\frac{3C}{2}\sin\frac{3A}{2}\sin\frac{3B}{2}. \end{aligned}$$

Similarly,

$$\begin{aligned} & \cos(A-B) + \cos(B-C) + \cos(C-A) \\ = \quad & 2\cos\frac{A-C}{2}\cos\frac{C+A-2B}{2} + 2\cos^2\frac{C-A}{2} - 1 \\ = \quad & 2\cos\frac{C-A}{2}\left(\cos\frac{C+A-2B}{2} + \cos\frac{C-A}{2}\right) - 1 \\ = \quad & 4\cos\frac{C-A}{2}\cos\frac{A-B}{2}\cos\frac{B-C}{2} - 1 = 4\sin\frac{2A+B}{2}\sin\frac{2B+C}{2}\sin\frac{2C+A}{2} - 1. \end{aligned}$$

Adding these results, the desired inequality becomes

$$\sin\frac{2A+B}{2}\sin\frac{2B+C}{2}\sin\frac{2C+A}{2} \geq \sin\frac{3A}{2}\sin\frac{3B}{2}\sin\frac{3C}{2}.$$

If one of the angles is greater than or equal to $\frac{2\pi}{3}$, then the RHS is nonpositive while the LHS is nonnegative, and the inequality follows. From now on, assume all of the angles are less than $\frac{2\pi}{3}$.
Define $f(x) = \ln\sin x$ on $(0, \pi)$. Then $f'(x) = \cot x$ and $f''(x) = -\csc^2 x < 0$, so f is concave. Therefore,

$$2f\left(\frac{3A}{2}\right) + f\left(\frac{3B}{2}\right) \leq 3f\left(\frac{2A+B}{2}\right) \Leftrightarrow \sin^2\frac{3A}{2}\sin\frac{3B}{2} \leq \sin^3\frac{2A+B}{2}.$$

Similarly, we obtain $\sin^2\frac{3B}{2}\sin\frac{3C}{2} \leq \sin^3\frac{2B+C}{2}$ and $\sin^2\frac{3C}{2}\sin\frac{3A}{2} \leq \sin^3\frac{2C+A}{2}$, and the result follows.

Also solved by Nicuşor Zlota ,"Traian Vuia" Technical College, Focşani, Romania; Utsab Sarkar, West Bengal, India.

S380. Let a, b, c be real numbers such that $abc = 1$. Prove that

$$\frac{a+ab+1}{(a+ab+1)^2+1} + \frac{b+bc+1}{(b+bc+1)^2+1} + \frac{c+ca+1}{(c+ca+1)^2+1} \le \frac{9}{10}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Li Zhou, Polk State College, USA

Let $a = \frac{x}{y}$, $b = \frac{y}{z}$, and $c = \frac{z}{x}$. Then the given inequality is equivalent to

$$\frac{yz(xy+yz+zx)}{(xy+yz+zx)^2+(yz)^2} + \frac{zx(xy+yz+zx)}{(xy+yz+zx)^2+(zx)^2} + \frac{xy(xy+yz+zx)}{(xy+yz+zx)^2+(xy)^2} \le \frac{9}{10},$$

which is trivially true if $xy+yz+zx = 0$. So assume that $xy+yz+zx \ne 0$ and let $u = \frac{yz}{xy+yz+zx}$, $v = \frac{zx}{xy+yz+zx}$, $w = \frac{xy}{xy+yz+zx}$, and $f(t) = \frac{t}{1+t^2}$. Then $u+v+w = 1$, and it suffices to prove $f(u)+f(v)+f(w) \le \frac{9}{10}$ for the three cases below.

Case I: $u, v, w > 0$. By the concavity of $f(t)$ for $t \in (0,1)$,

$$f(u)+f(v)+f(w) \le 3f\left(\frac{u+v+w}{3}\right) = 3f\left(\frac{1}{3}\right) = \frac{9}{10}.$$

Case II: $u > 0$ and $v, w < 0$. Then $f(u)+f(v)+f(w) < f(u) \le \frac{1}{2} < \frac{9}{10}$.

Case III: $u \ge v > 0$ and $w < 0$. If $u+w \ge 0$, then

$$f(u)+f(w) = \frac{(u+w)(1+uw)}{1+u^2+w^2+u^2w^2} \le \frac{u+w}{1+u^2+w^2+2uw} = f(u+w).$$

So by the concavity of $f(t)$ for $t \in (0,1)$ again,

$$f(u)+f(v)+f(w) \le f(v)+f(u+w) \le 2f\left(\frac{u+v+w}{2}\right) = 2f\left(\frac{1}{2}\right) < \frac{9}{10}.$$

If $u+w < 0$, then $-w > u \ge v > 1$ and $2u+w \ge u+v+w = 1 > 0$. Since $f(t)$ is decreasing for $t < -1$, $f(w)+\frac{1}{2}f(u) \le f(-2u)+\frac{1}{2}f(u) = \frac{-3u}{2(1+4u^2)(1+u^2)} < 0$. Hence, $f(u)+f(v)+f(w) < f(v)+\frac{1}{2}f(u) \le \frac{1}{2}+\frac{1}{4} < \frac{9}{10}$, completing the proof.

Also solved by Nicuşor Zlota ,"Traian Vuia" Technical College, Focşani, Romania; Utsab Sarkar, West Bengal, India; Arkady Alt, San Jose, CA, USA; SooYoung Choi, Seoul ChungDam Middle School, Republic of Korea.

S381. Let $ABCD$ be a cyclic quadrilateral and M and N be the midpoints of the diagonals AC and BD. Prove that

$$MN \geq \frac{1}{2}|AC - BD|.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania

Denote $AB = a, BC = b, CD = c, DA = d, AC = p, BD = q, MN = v$. We have to prove the inequality

$$2v \geq |p - q|.$$

If $v = 0$, then $ABCD$ is a rectangle, so $p = q$. Now suppose that $v > 0$ and construct points G and H such that $ABCG$ and $ADCH$ are parallelograms. Then $BDGH$ is also a parallelogram, and $DG = 2MN$ since MN is a midline of triangle BDG. By the triangle inequality on triangles CDG and ADG we obtain $2v \geq |c - a|$ and $2v \geq |b - d|$, respectively. Squaring each of these inequalities and adding them together, we obtain

$$8v^2 \geq a^2 + b^2 + c^2 + d^2 - 2(ac + bd).$$

By Euler's quadrilateral theorem,

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4v^2,$$

and by Ptolemy's theorem,

$$ac + bd = pq.$$

Substituting these into the inequality above gives

$$4v^2 \geq (p - q)^2,$$

which implies the desired inequality after taking the square root of both sides. Equality holds if and only if $ABCD$ is a rectangle.

Also solved by Daniel Lasaosa, Pamplona, Spain; Andreas Charalampopoulos, 4th Lyceum of Glyfada, Glyfada, Greece; Viet Hoang, Takapuna Grammar School, Auckland, New Zealand; Thao Le, University of Auckland, Auckland, New Zealand; Utsab Sarkar, West Bengal, India; Li Zhou, Polk State College, USA; Rade Krenkov, SOU "Goce Delcev", Valandovo, Macedonia.

S382. Prove that in any triangle ABC the following inequality holds:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{r}{R}\leq 2.$$

Proposed by Florin Stănescu, Găeşti, România

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam
We will use the following well-known results:

Lemma 1. $ab+bc+ca=s^2+4Rr+r^2$.

Lemma 2. *Euler's inequality:* $R\geq 2r$.

Lemma 3. *Gerretsen's inequality:* $16Rr-5r^2\leq s^2\leq 4R^2+4Rr+3r^2$.

Coming back to the main problem, we have

$$\begin{aligned}\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}&=\frac{a(a+b)(a+c)+b(b+c)(b+a)+c(c+a)(c+b)}{(a+b)(b+c)(c+a)}\\&=\frac{(a+b+c)(a^2+b^2+c^2)+3abc}{(a+b+c)(ab+bc+ca)-abc}\\&=\frac{2(s^2-Rr-r^2)}{s^2+2Rr+r^2}.\end{aligned}$$

Hence our inequality is equivalent to

$$\frac{2(s^2-Rr-r^2)}{s^2+2Rr+r^2}+\frac{r}{R}\leq 2,$$

or

$$s^2+r^2\leq 6R^2+2Rr.$$

This is true because

$$\begin{aligned}s^2+r^2-6R^2-2Rr&=(s^2-4R^2-4Rr-3r^2)+(4r^2+2Rr-2R^2)\\&=\underbrace{(s^2-4R^2-4Rr-3r^2)}_{\leq 0}+\underbrace{(2r)^2-R^2}_{\leq 0}+\underbrace{R(2r-R)}_{\leq 0}\leq 0.\end{aligned}$$

Also solved by Nicuşor Zlota ,"Traian Vuia" Technical College, Focşani, Romania; Daniel Lasaosa, Pamplona, Spain; Andreas Charalampopoulos, 4th Lyceum of Glyfada, Glyfada, Greece; Li Zhou, Polk State College, USA; Utsab Sarkar, West Bengal, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA.

S383. Solve in positive integers the equation

$$x^6 - y^6 = 2016xy^2.$$

Proposed by Adrian Andreescu, Dallas, Texas

Solution by Utsab Sarkar, West Bengal, India

Let $\gcd(x, y) = k$. Then there exist relatively prime positive integers u, v such that $x = ku$ and $y = kv$. Now,

$$x^6 - y^6 = 2016xy^2 \implies k^3(u^6 - v^6) = 2016uv^2.$$

Taking this equation modulo u, we see that

$$-k^3v^6 \equiv 0 \pmod{u} \Rightarrow u|k^3,$$

since $(u, v) = 1$. Similarly, taking the equation modulo v^2 gives

$$-k^3u^6 \equiv 0 \pmod{v^2} \Rightarrow v^2|k^3.$$

Since $(u, v^2) = 1$, we know $uv^2|k^3$. Therefore, $u^6 - v^6 = m, k^3 = \dfrac{2016}{m}uv^2$ for some $m|2016$.

Note that $u > v$. If $u \geq 4$, see that $m = u^6 - v^6 \geq 4^6 - 3^6 = 3367 > 2016$, which contradicts $m|2016$. Therefore $1 \leq v < u \leq 3$. Thus, we need to check only the cases $(u, v) = (3, 1);\ (3, 2);\ (2, 1)$.

$(u, v) = (3, 1) \implies m = 3^6 - 1 = 728 \nmid 2016$ and $(u, v) = (3, 2) \implies m = 3^6 - 2^6 = 665 \nmid 2016$. Finally, $(u, v) = (2, 1) \implies m = 2^6 - 1 = 63|2016$ and $k^3 = \frac{2016}{63} \cdot 2 \cdot 1 = 64 \implies k = 4$. Hence the only solution is $x = 8, y = 4$.

Also solved by Rade Krenkov, SOU "Goce Delcev", Valandovo, Macedonia; Li Zhou, Polk State College, USA; Catalin Prajitura, College at Brockport, SUNY, USA; Daniel Lasaosa, Pamplona, Spain; Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania; Nicuşor Zlota ,"Traian Vuia" Technical College, Focşani, Romania; SooYoung Choi, Seoul ChungDam Middle School, Republic of Korea; Wu Qianwen, Nanjing Foreign Language School A Level Program, China.

S384. Let ABC be a triangle with circumcenter O and orthocenter H. Let D, E, F be the feet of the altitudes from A, B, C, respectively. Let K be the intersection of AO with BC and L be the intersection of AO with EF . Furthermore, let T be the intersection of AH and EF , and let S be the intersection of KT and DL. Prove that BC, EF, SH are concurrent.

Proposed by Bobojonova Latofat and Khurshid Juraev, Tashkent, Uzbekistan

Solution by Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece

We will use two well-known lemmas on harmonic divisions:

Lemma 1. Let X, A, Y, B be collinear points in that order, and let C be any point not on this line. Then any two of the following conditions implies the third condition:

i) $(A, B; X, Y) = -1$.

ii) $\angle XCY = 90°$.

iii) CY bisects $\angle ACB$.

Lemma 2. Let AB be a chord of a circle ω and select points P and Q on line AB. Then $(A, B; P, Q) = -1$ if and only if P lies on the polar of Q.

Now, let $EF \cap BC = Q$. Because $\angle ALT = 180° - (90° - \angle B) - \angle B = 90°$, we know $DLTK$ is cyclic. By Brocard's theorem for cyclic quadrilateral $DTLK$, we obtain that A is the pole of SQ with respect to the circumcircle of $DLTK$. To prove that BC, EF, SH are concurrent, it suffices to prove that $H \in SQ \Leftrightarrow H$ lies on the polar of A. By Lemma 2, it suffices to show that $(A, H; D, T) = -1$. However, $\angle AEH = 90°$, and $\angle TEH = \angle FEB = \angle FCB = \angle HCD = \angle HED$, using cyclic quadrilaterals $BCEF$ and $HDCE$. By Lemma 1, $(A, H; D, T) = -1$, and the conclusion follows.

Also solved by Daniel Lasaosa, Pamplona, Spain; Ghenghea Daniel; Andrea Fanchini, Cantù, Italy; Adnan Ali, A.E.C.S-4, Mumbai, India; Li Zhou, Polk State College, USA; Utsab Sarkar, West Bengal, India.

U379. Let a, b, c be nonnegative real numbers. Prove that

$$a^3 + b^3 + c^3 - 3abc \geq k\,|(a-b)(b-c)(c-a)|\,,$$

where $k = \left(\frac{27}{4}\right)^{\frac{1}{4}}\left(1+\sqrt{3}\right)$ and that k is the best possible constant.

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Li Zhou, Polk State College, USA

We may assume that $a > b > c \geq 0$ and let $f(a,b,c) = \frac{a^3+b^3+c^3-3abc}{(a-b)(b-c)(a-c)}$. Note that

$$a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (a-c)^2\right].$$

So

$$\frac{2f(a,b,c)}{a+b+c} = \frac{a-b}{(b-c)(a-c)} + \frac{1}{a-b}\left(\frac{b-c}{a-c} + \frac{a-c}{b-c}\right).$$

Since $g(t) = t + \frac{1}{t}$ is increasing for $0 < t < 1$ and $0 < \frac{b-c}{a-c} \leq \frac{2b+c}{2a+c} < 1$, $g\left(\frac{b-c}{a-c}\right) \geq g\left(\frac{2b+c}{2a+c}\right)$. Also, $\frac{1}{(b-c)(a-c)} \geq \frac{4}{(2b+c)(2a+c)}$. Therefore, $f(a,b,c) \geq f\left(a+\frac{c}{2}, b+\frac{c}{2}, 0\right)$. Hence, it suffices to show that $h(x) = f(x,1,0) = \frac{x^3+1}{(x-1)x} \geq k$ for $x > 1$. Now $h'(x) = \frac{x^4-2x^3-2x+1}{(x^2-x)^2}$. Solving the symmetric equation $x^4 - 2x^3 - 2x + 1 = 0$, we get $x + \frac{1}{x} = 1 + \sqrt{3}$ and then $x = r = \frac{1+\sqrt{3}+\sqrt{2\sqrt{3}}}{2}$. So $h(x)$ attains its minimum at $x = r$. Finally, since $r + \frac{1}{r} = 1 + \sqrt{3}$ and $r - \frac{1}{r} = \sqrt{\left(r+\frac{1}{r}\right)^2 - 4} = \sqrt{2\sqrt{3}}$,

$$\begin{aligned} h(r) &= \frac{r^3+1}{r(r-1)} = \frac{(r^2-1)(r^3+1)}{\sqrt{2\sqrt{3}}r^2(r-1)} = \frac{(r+1)(r^3+1)}{\sqrt{2\sqrt{3}}r^2} \\ &= \frac{1}{\sqrt{2\sqrt{3}}}\left[\left(r-\frac{1}{r}\right)^2 + r + \frac{1}{r} + 2\right] = \frac{1}{\sqrt{2\sqrt{3}}}\left(3\sqrt{3}+3\right) = k, \end{aligned}$$

completing the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Paul Revenant, Lycée du Parc, Lyon, France; Utsab Sarkar, West Bengal, India.

U380. Prove that for all positive real numbers a, b, c the following inequality holds:

$$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{2a+b+c}+\frac{1}{2b+c+a}+\frac{1}{2c+a+b}\geq\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

By the Schur's inequalit we have,

$$x^4+y^4+z^4+xyz(x+y+z)\geq x^3(y+z)+y^3(z+x)+z^3(x+y) \tag{1}$$

Using AM-GM inequality we have,

$$\begin{aligned} x^3(y+z)+y^3(z+x)+z^3(x+y) &= (x^3y+y^3x)+(x^3z+z^3x)+(y^3z+z^3y) \\ &\geq 2x^2y^2+2y^2z^2+2z^2x^2 \qquad (2) \end{aligned}$$

From the (1) and (2) we get

$$x^4+y^4+z^4+x^2yz+xy^2z+xyz^2\geq 2(x^2y^2+y^2z^2+z^2x^2).$$

If we choosing $x=t^{a-\frac{1}{4}},\ y=t^{b-\frac{1}{4}},\ z=t^{c-\frac{1}{4}}$:

$$\begin{aligned} &t^{4a-1}+t^{4b-1}+t^{4c-1}+t^{2a+b+c-1}+t^{a+2b+c-1}+t^{a+b+2c-1} \\ &\geq 2t^{2(a+b)-1}+2t^{2(b+c)-1}+2t^{2(c+a)-1} \end{aligned}$$

Hence we have

$$\begin{aligned} &\int_0^1 t^{4a-1}dt+\int_0^1 t^{4b-1}dt+\int_0^1 t^{4c-1}dt \\ &+\int_0^1 t^{2a+b+c-1}dt+\int_0^1 t^{a+2b+c-1}dt+\int_0^1 t^{a+b+2c-1}dt \\ &\geq 2\int_0^1 t^{2(a+b)-1}dt+2\int_0^1 t^{2(b+c)-1}dt+2\int_0^1 t^{2(c+a)-1}dt \end{aligned}$$

Thus we get

$$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{2a+b+c}+\frac{1}{2b+c+a}+\frac{1}{2c+a+b}\geq\frac{2}{2(a+b)}+\frac{2}{2(b+c)}+\frac{2}{2(c+a)}.$$

Equality holds only when $a=b=c$.

Also solved by Anderson Torres, Sao Paulo, Brazil; Li Zhou, Polk State College, USA; Utsab Sarkar, West Bengal, India; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Chinthalagiri Venkata Sriram, Chennai Mathematical Institute, India; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania.

U381. Find all positive integers n such that

$$\sigma(n) + d(n) = n + 100.$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

Let $n = p_1^{e_1} \cdots p_k^{e_k}$ where $p_1 < \cdots < p_k$ are k distinct prime factors of n, and e_i $(1 \le i \le k)$ are positive integers. Suppose that n is odd. Then $\sigma(n) + d(n)$ is even while $n + 100$ is odd. Thus $2|n$.

If $k = 1$, then $p_1 = 2$ and so $\sigma(n) + d(n) = 2^{e_1+1} + e_1 = n + 100 = 2^{e_1} + 100 \Rightarrow 2^{e_1} + e_1 = 100$, which yields no solution.

If $k \ge 4$, then $n \ge 2{\cdot}3{\cdot}5{\cdot}7$. So we must have $\sigma(n)+d(n)-n \ge \sigma(2{\cdot}3{\cdot}5{\cdot}7)+d(2{\cdot}3{\cdot}5{\cdot}7)-2{\cdot}3{\cdot}5{\cdot}7 > 3{\cdot}5{\cdot}7 > 100$. So $k \le 3$.

Now, if $k = 3$, then $n = 2^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3}$. If $e_1 \ge 2$, then $\sigma(n)+d(n)-n \ge \sigma(2^2 \cdot p_2p_3)+d(2^2 \cdot p_2p_3)-2^2 \cdot p_2p_3 = 3p_2p_3+7(p_2+p_3)+19 \ge 120$, for $p_2 = 3, p_3 = 5$. So, $e_1 = 1$. Similarly $e_2, e_3 < 2$. So, $e_1 = e_2 = e_3 = 1$. Then $\sigma(n) + d(n) - n = 3(p_2 + 1)(p_3 + 1) + 8 - 2p_2p_3 = p_2p_3 + 3(p_2 + p_3) + 9 + 2 = 100 \Rightarrow (p_2 + 3)(p_3 + 3) = 98$, which clearly has no solutions because the left hand side is divisible by at least 4 while the right hand side is not divisible by 4.

Next, if $k = 2$, then $n = 2^{e_1} \cdot p_2^{e_2}$. If $e_1 \ge 5$, then $\sigma(n)+d(n)-n \ge \sigma(2^5 \cdot 3)+d(2^5 \cdot 3)-2^5 \cdot 3 = 168 > 100$. Thus $e_1 \le 4$ and we consider the cases for e_1:

(1) $e_1 = 1$

If $e_2 \ge 4$, then $\sigma(n) + d(n) - n \ge \sigma(2 \cdot 3^4) + d(2 \cdot 3^4) - 2 \cdot 3^4 = 211 > 100$. So, $e_2 \in \{1, 2, 3\}$. It is easily verified that none of the pairs $(e_1, e_2) = (1, 1), (1, 2)$ and $(1, 3)$ yield a solution for n.

(2) $e_1 = 2$

If $e_2 \ge 3$, then $\sigma(n) + d(n) - n \ge \sigma(2^2 \cdot 3^3) + d(2^2 \cdot 3^3) - 2^2 \cdot 3^3 = 184 > 100$. So, $e_2 \in \{1, 2\}$. If $e_2 = 1$, then $\sigma(n) + d(n) - n = 7(p_2 + 1) + 6 - 4p_2 = 100 \Rightarrow 3p_2 = 87$. So, $p_2 = 29$ and so $n = 2^2 \cdot 29 = 116$ is a solution. If $e_2 = 2$, then $\sigma(n) + d(n) - n = 7(p_2^2 + p_2 + 1) + 9 - 4p_2^2 = 100 \Rightarrow 3p_2^2 + 7p_2 = 84$. So, p_2 can be either 3 or 7, neither of which gives a solution.

(3) $e_1 = 3$

If $e_2 \ge 2$, then $\sigma(n) + d(n) - n \ge \sigma(2^3 \cdot 3^2) + d(2^3 \cdot 3^2) - 2^3 \cdot 3^2 = 135 > 100$. So $e_2 = 1$. This means that $n = 2^3 \cdot p_2$ and so $\sigma(n) + d(n) - n = 100 \Rightarrow 15(p_2 + 1) + 8 - 8p_2 = 100 \Rightarrow p_2 = 11$. So, $n = 2^3 \cdot 11 = 88$ is a solution.

(4) $e_1 = 4$

If $e_2 \ge 2$, then $\sigma(n) + d(n) - n \ge \sigma(2^4 \cdot 3^2) + d(2^4 \cdot 3^2) - 2^4 \cdot 3^2 = 274 > 100$. So here also $e_2 = 1$. Then $n = 2^4 \cdot p_2$ and $\sigma(n) + d(n) - n = 100 \Rightarrow 31(p_2 + 1) + 10 - 16p_2 = 100 \Rightarrow 15p_2 = 59$, which clearly gives no solution for p_2 and hence this case as well gives no solution for n.

So, in summary we have only two solutions namely $n = 2^3 \cdot 11 = 88$ and $n = 2^2 \cdot 29 = 116$.

Also solved by Li Zhou, Polk State College, USA; Wu Qianwen, Nanjing Foreign Language School A Level Program, China; Utsab Sarkar, West Bengal, India; Chinthalagiri Venkata Sriram, Chennai Mathematical Institute, India.

U382. Prove that

$$\int_0^1 \prod_{k=1}^{\infty}\left(1-x^k\right)dx = \frac{4\pi\sqrt{3}\sinh\frac{\pi\sqrt{23}}{3}}{\sqrt{23}\cosh\frac{\pi\sqrt{23}}{2}}.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Li Zhou, Polk State College, USA

By Euler's pentagonal number theorem, $\prod_{k=1}^{\infty}\left(1-x^k\right) = \sum_{n=-\infty}^{\infty}(-1)^n x^{n(3n-1)/2}$. Hence,

$$I = \int_0^1 \prod_{k=1}^{\infty}\left(1-x^k\right)dx = 2\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n(3n-1)+2} = \frac{2}{3}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(n+c)(n+\overline{c})},$$

where $c = \frac{-1+i\sqrt{23}}{6}$. By partial fraction decomposition and Eisenstein's series of $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n+z} = \frac{\pi}{\sin\pi z}$, we get

$$\begin{aligned} I &= \frac{2}{3(c-\overline{c})}\sum_{n=-\infty}^{\infty}\left(\frac{(-1)^n}{n+\overline{c}} - \frac{(-1)^n}{n+c}\right) = \frac{2\pi}{i\sqrt{23}}\left(\frac{1}{\sin\pi\overline{c}} - \frac{1}{\sin\pi c}\right) \\ &= \frac{4\pi}{i\sqrt{23}}\cdot\frac{\sin\pi c - \sin\pi\overline{c}}{\cos\pi(\overline{c}-c) - \cos\pi(\overline{c}+c)} = \frac{4\pi\sqrt{3}\sin\frac{\pi i\sqrt{23}}{6}}{i\sqrt{23}\left(\cos\frac{\pi i\sqrt{23}}{3} - \frac{1}{2}\right)} \\ &= \frac{4\pi\sqrt{3}\sinh\frac{\pi\sqrt{23}}{6}}{\sqrt{23}\left(\cosh\frac{\pi\sqrt{23}}{3} - \frac{1}{2}\right)}. \end{aligned}$$

Finally,

$$\sinh\frac{\pi\sqrt{23}}{6}\cosh\frac{\pi\sqrt{23}}{2} = \frac{1}{2}\left(\sinh\frac{2\pi\sqrt{23}}{3} - \sinh\frac{\pi\sqrt{23}}{3}\right) = \sinh\frac{\pi\sqrt{23}}{3}\left(\cosh\frac{\pi\sqrt{23}}{3} - \frac{1}{2}\right),$$

completing the proof.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Utsab Sarkar, West Bengal, India.

U383. Let $n \geq 2$ be an integer and A and B be two $n \times n$ matrices with complex entries such that $A^2 = B^2 = O$ with $A + B$ being invertible. Prove that n is even and $\operatorname{rank}(AB)^k = n/2$ for all $k \geq 1$.

Proposed by Florin Stanescu, Gaesti, România

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

From the given condition, $\operatorname{rank}(A+B) = n$. Choosing $A = B$, for the Sylvester inequality we have:

$n = 0 + n = \operatorname{rank}(A^2) + n \geq 2\operatorname{rank}(A)$. From here we have $\operatorname{rank}(A) \leq \frac{n}{2}$.

$n = 0 + n = \operatorname{rank}(B^2) + n \geq 2\operatorname{rank}(B)$. From here have $\operatorname{rank}(B) \leq \frac{n}{2}$.

Thus,

$$n = \operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B) \leq \frac{n}{2} + \frac{n}{2} = n.$$

Hence we get

$$\operatorname{rank}(A) = \operatorname{rank}(B) = \frac{n}{2}.$$

From the given condition we get,

$$B^2(AB)^k = 0, \; A(AB)^k = 0.$$

Hence we have,

$$\begin{aligned}
\operatorname{rank}(AB)^{k+1} &= \operatorname{rank}\Big((AB)(AB)^k + B^2(AB)^k\Big) \\
&= \operatorname{rank}\Big((A+B)B(AB)^k\Big) = \operatorname{rank}(B(AB)^k) \\
&= \operatorname{rank}\Big(A(AB)^k + B(AB)^k\Big) = \operatorname{rank}\Big((A+B)(AB)^k\Big) \\
&= \operatorname{rank}(AB)^k.
\end{aligned}$$

Thus we get,

$$\operatorname{rank}(AB)^k = \operatorname{rank}(AB)^{k-1} = \ldots = \operatorname{rank}(AB).$$

Now, we prove that

$$\operatorname{rank}(AB) = \frac{n}{2}.$$

$$\begin{aligned}
\operatorname{rank}(AB) &= \operatorname{rank}(A^2) + \operatorname{rank}(AB) \geq \operatorname{rank}(A^2 + AB) \\
&= \operatorname{rank} A(A+B) = \operatorname{rank}(A) = \frac{n}{2}.
\end{aligned}$$

and

$$\operatorname{rank}(AB) \leq \operatorname{rank}(A) = \frac{n}{2}.$$

Hence we have $\operatorname{rank}(AB) = \frac{n}{2}$.

Also solved by Li Zhou, Polk State College, USA; Kosmidis Dimitrios, National and Kapodistrian University of Athens, Greece.

U384. Let m and n be positive integers. Evaluate

$$\lim_{x\to 0}\frac{(1+x)\left(1+\frac{x}{2}\right)^2\cdots\left(1+\frac{x}{m}\right)^m-1}{(1+x)\sqrt{1+2x}\cdots\sqrt[n]{1+nx}-1}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Henry Ricardo, New York Math Circle

We note that for any positive integer k

$$\left(1+\frac{x}{k}\right)^k \;=\; 1 \;+\; k\cdot\frac{x}{k} \;+\; O(x^2) \;=\; 1+x+O(x^2).$$

Thus

$$\begin{aligned}(1+x)\left(1+\frac{x}{2}\right)^2\cdots\left(1+\frac{x}{m}\right)^m-1 &= \left(1+x+O(x^2)\right)^m-1\\ &= \left(1+mx+O(x^2)\right)-1\\ &= mx+O(x^2).\end{aligned}$$

Also, for any x such that $|x|<1$ and for any $k=2,3,\ldots,n$, we have

$$\sqrt[k]{1+kx} \;=\; 1+\frac{1}{k}\cdot kx+O(x^2) \;=\; 1+x+O(x^2),$$

so that

$$\begin{aligned}(1+x)\sqrt{1+2x}\cdots\sqrt[n]{1+nx}-1 &= \left(1+x+O(x^2)\right)^n-1\\ &= \left(1+nx+O(x^2)\right)-1\\ &= nx+O(x^2).\end{aligned}$$

Thus

$$\frac{\prod_{k=1}^m\left(1+\frac{x}{k}\right)^k}{\prod_{k=1}^n\sqrt[k]{1+kx}} \;=\; \frac{mx+O(x^2)}{nx+O(x^2)} \;=\; \frac{m+O(x)}{n+O(x)} \;\to\; \frac{m}{n} \quad \text{as } x\to 0.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Anderson Torres, Sao Paulo, Brazil; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Li Zhou, Polk State College, USA; Utsab Sarkar, West Bengal, India; Adnan Ali, A.E.C.S-4, Mumbai, India; Leeza Kerr, College at Brockport, SUNY, USA; Arkady Alt, San Jose, CA, USA; Joel Schlosberg, Bayside, NY, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania.

O379. Let a, b, c, d are real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$\frac{2}{3}(ab + bc + cd + da + ac + bd) \leq (3 - \sqrt{3})abcd + 1 + \sqrt{3}$$

Proposed by Marius Stanean, Zalau, România

Solution by Utsab Sarkar, West Bengal, India

Let $f(a,b,c,d) = 1 + \sqrt{3} + (3 - \sqrt{3})abcd - \frac{2(ab+bc+cd+da+ac+bd)}{3} = \frac{7}{3} + \sqrt{3} + (3 - \sqrt{3})abcd - \frac{(a+b+c+d)^2}{3}$. Therefore our given inequality becomes $f(a,b,c,d) \geq 0$.

In other words we need to minimize $f(a,b,c,d)$ wrt $a^2 + b^2 + c^2 + d^2 = 4$. Lets set the lagrangian,

$$\mathcal{L} = f(a,b,c,d) + \lambda(a^2 + b^2 + c^2 + d^2 - 4)$$

We denote $\frac{\delta \mathcal{L}}{\delta a} := \mathcal{L}_a$ and similarly for other variables.

Now note that $a^2 + b^2 + c^2 + d^2 = 4$ corresponds to the surface of the 4−sphere centered at origin with radius 2 and its a compact(closed and bounded) subset of $\mathbb{R}^4$ therefore by the Extreme Value Theorem f being continuous attains its maximum and minimum on this compact set and the extreme points of f are the critical points of $\mathcal{L}$.

To see the set $S = \{(a,b,c,d) \in \mathbb{R}^4 \big| a^2 + b^2 + c^2 + d^2 = 4\}$ is closed note that the function $g(a,b,c,d) = a^2 + b^2 + c^2 + d^2$ is a continuous(as polynomial!) and $g^{-1}(\{4\}) = S$. And boundedness of S is trivial. Therefore S is compact.

Thus we see,

$$0 = \mathcal{L}_a = f_a + 2a\lambda \implies \lambda = \frac{-f_a}{2a}$$
$$f_a = (3 - \sqrt{3})\frac{abcd}{a} - \frac{2(a+b+c+d)}{3}$$
$$\implies \lambda = \frac{(a+b+c+d)}{3a} - (3 - \sqrt{3})\frac{abcd}{2a^2} \tag{1}$$

Similarly

$$\mathcal{L}_b = 0 \implies \lambda = \frac{(a+b+c+d)}{3b} - (3 - \sqrt{3})\frac{abcd}{2b^2} \tag{2}$$
$$\mathcal{L}_c = 0 \implies \lambda = \frac{(a+b+c+d)}{3c} - (3 - \sqrt{3})\frac{abcd}{2c^2} \tag{3}$$
$$\mathcal{L}_d = 0 \implies \lambda = \frac{(a+b+c+d)}{3d} - (3 - \sqrt{3})\frac{abcd}{2d^2} \tag{4}$$
$$\mathcal{L}_\lambda = 0 \implies a^2 + b^2 + c^2 + d^2 = 4 \tag{5}$$

After analysing the system of equations in (1),(2),(3),(4),(5) we see $f_{\max} = \frac{16}{3}$ occurs at $(a,b,c,d) = (1,1,-1,-1)$ with permutations and $f_{min} = 0$ which occurs at $(a,b,c,d) = (\pm 1, \pm 1, \pm 1, \pm 1), \Big(\pm\frac{\sqrt{3}-1}{\sqrt{2}}, \pm\frac{\sqrt{3}+1}{\sqrt{6}}, \pm\frac{\sqrt{3}+1}{\sqrt{6}}, \pm\sqrt{}$ and permutations. Therefore $f(a,b,c,d) \geq f_{\min} = 0$.

Analysis of Critical Points

First **WLOG** lets assume that $(a,b,c,0)$ be a point satisfying the system above. Then note that,

$$\lambda = \frac{a+b+c}{3a} = \frac{a+b+c}{3b} = \frac{a+b+c}{3c}$$
$$f_d = (3 - \sqrt{3})abc - \frac{2(a+b+c)}{3} = 0$$
$$a^2 + b^2 + c^2 = 4$$

Suppose now $a+b+c \neq 0$ that would imply,

$$a=b=c;\ \lambda=\frac{1}{3}$$

Therefore,

$$\begin{aligned} f_d=0 &\implies a^2=\frac{2}{3-\sqrt{3}} \\ g(a,a,a,0)=4 &\implies a^2=\frac{4}{3} \end{aligned}$$

A contradiction. Thus $a+b+c=0$, then,

$$\lambda=0;\ f_d=0 \implies abc=0$$

Again contradiction as we assumed a,b,c are non-zero. Now lets assume $(a,b,0,0)$ be a feasible point. Then note,

$$\begin{aligned} &f_c=f_d=0 \implies a+b=0 \\ &\lambda=\frac{a+b}{3a}=\frac{a+b}{3b}=0 \\ &4=g(a,b,0,0)=g(a,-a,0,0) \implies a=\pm\sqrt{2} \end{aligned}$$

Hence we get $(\pm\sqrt{2},\mp\sqrt{2},0,0)$ with permutations are one set of critical points of $\mathcal{L}$. Also note $f(\pm\sqrt{2},\mp\sqrt{2},0,0)=\frac{7}{3}+\sqrt{3}$.
Now lets take $(a,0,0,0)$ be a solution point of the above system but this is clearly false as then $f_b=f_c=f_d=0 \implies a=0$. Therefore we now assume none of a,b,c,d are zero for any solution point. Lets set $3u=a+b+c+d;\ w=\delta abcd\neq 0;\ \delta=\frac{3-\sqrt{3}}{2}$, then observe,

$$\begin{aligned} &\lambda=\frac{u}{a}-\frac{w}{a^2}=\frac{u}{b}-\frac{w}{b^2}=\frac{u}{c}-\frac{w}{c^2}=\frac{u}{d}-\frac{w}{d^2} \qquad (\star) \\ &g(a,b,c,d)=4 \end{aligned}$$

Now note for any $x,y\in\{a,b,c,d\}$ from $(\star)$ we see

$$\begin{aligned} &\frac{u}{x}-\frac{w}{x^2}=\frac{u}{y}-\frac{w}{y^2} \\ \implies &(x-y)\left(w\left(\frac{x+y}{xy}\right)-u\right)=0 \end{aligned}$$

Hence this suggests the following cases with permutations,

$$\begin{aligned} (I)\ & a=b=c=d \\ (II)\ & a=b\neq c=d \\ (III)\ & a\neq b=c=d \\ (IV)\ & a=b\neq c\neq d \\ (V)\ & a\neq b\neq c\neq d \end{aligned}$$

Case (*I*)

$$a=b=c=d \implies g(a,a,a,a)=4 \implies a=\pm 1$$

Thus we get a solution set as $(\pm1,\pm1,\pm1,\pm1)$. Also note $f(\pm1,\pm1,\pm1,\pm1)=0$.
Case (*II*)

$$\begin{aligned} &x=a=b\neq c=d=y \implies u=\frac{2(x+y)}{3};\ w=\delta x^2y^2 \\ &\left(w\left(\frac{x+y}{xy}\right)-u\right)=0 \implies \left(\delta xy-\frac{2}{3}\right)(x+y)=0 \\ &g(a,b,c,d)=4 \implies x^2+y^2=2 \end{aligned}$$

Note $2 = x^2 + y^2 = (x-y)^2 + 2xy \geq 2xy \implies \delta xy \leq \delta$ and also note $\delta = \frac{3-\sqrt{3}}{2} < \frac{2}{3}$ thus we get $x + y = 0 \implies x = -y = \pm 1$ therefore another set of solutions is $(1, 1, -1, -1)$ with permutations. Also see $f(1, 1, -1, -1) = \frac{16}{3}$.

Case (*IV*)

$$a = b \neq c \neq d \implies \frac{u}{w} = \frac{1}{a} + \frac{1}{c} = \frac{1}{c} + \frac{1}{d} = \frac{1}{d} + \frac{1}{a} \implies a = c = d$$

Contradiction follows. Similarly,

Case (*V*)

$$\begin{aligned} & a \neq b \neq c \neq d \\ \implies & \frac{u}{w} = \frac{1}{a} + \frac{1}{b} = \frac{1}{b} + \frac{1}{c} = \frac{1}{c} + \frac{1}{d} = \frac{1}{a} + \frac{1}{d} = \frac{1}{a} + \frac{1}{c} = \frac{1}{b} + \frac{1}{d} \\ \implies & a = b = c = d \end{aligned}$$

Contradiction follows.

Case (*III*)

$$\begin{aligned} & a \neq b = c = d \implies u = \frac{a+3b}{3};\ w = \delta ab^3; \\ & g(a,b,b,b) = 4 \implies a^2 + 3b^2 = 4 \\ & \left(w \left(\frac{a+b}{ab} \right) - u \right) = 0 \implies \left(\delta b^2 (a+b) - \frac{a+3b}{3} \right) = 0 \\ & \quad \implies a = \frac{3b(1-\delta b^2)}{3\delta b^2 - 1} \end{aligned}$$

Now note $b^2 \neq \frac{1}{\delta}, \frac{1}{3\delta}$ as $a, b \neq 0$; Set $\delta b^2 = t > 0$ then notice,

$$\begin{aligned} & a^2 + 3b^2 = 4 \implies \frac{9b^2(1-\delta b^2)^2}{(3\delta b^2 - 1)^2} + 3b^2 = 4 \\ & \implies \frac{3t(1-t)^2}{(3t-1)^2} + t = \frac{4\delta}{3} \\ & \implies 18t^3 + 9\left(\sqrt{3} - 5\right) t^2 - 6\left(\sqrt{3} - 4\right) t + \sqrt{3} - 3 = 0 \\ & \implies t = \frac{\sqrt{3}+3}{6},\ \frac{3 - \sqrt{3} \pm \sqrt{6 - 3\sqrt{3}}}{3} \end{aligned}$$

Or,

$$\begin{aligned} & \delta b^2 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right),\ \left(1 - \frac{1}{\sqrt{3}}\right)\left(1 \pm \frac{1}{\sqrt{2}}\right) \\ & \implies b^2 = \frac{2+\sqrt{3}}{3},\ \frac{2}{3}\left(1 \pm \frac{1}{\sqrt{2}}\right) \end{aligned}$$

Here we get a complicated solution set. Now note,

$$\begin{aligned} & w = \delta ab^3 = \frac{3\delta b^4(1-\delta b^2)}{3\delta b^2 - 1} \\ & u^2 = \left(\delta b^2 (a+b)\right)^2 = \left(\delta b^2 \left(\frac{3b(1-\delta b^2)}{3\delta b^2 - 1} + b \right) \right)^2 = \frac{4\delta^2 b^6}{(3\delta b^2 - 1)^2} \end{aligned}$$

Now note $f(a,b,c,d) = \frac{7}{3} + \sqrt{3} + 2w - 3u^2$, thus at this solution point we see,

$$\begin{aligned} f &= \frac{7}{3} + \sqrt{3} + \frac{6\delta b^4(1-\delta b^2)}{3\delta b^2 - 1} - \frac{12\delta^2 b^6}{(3\delta b^2 - 1)^2} \\ &= \frac{7}{3} + \sqrt{3} - \frac{6b^4\delta\left(3b^4\delta^2 - 2b^2\delta + 1\right)}{\left(3b^2\delta - 1\right)^2};\ \delta = \frac{3-\sqrt{3}}{2} \end{aligned}$$

Now see,

$$
\begin{aligned}
b^2 &= \frac{2+\sqrt{3}}{3} \longrightarrow f = 0\\
b^2 &= \frac{2+\sqrt{2}}{3} \longrightarrow f = \frac{-3+5\sqrt{3}-4\sqrt{2}}{3} \approx 0.0011\\
b^2 &= \frac{2-\sqrt{3}}{3} \longrightarrow f = \frac{-3+5\sqrt{3}+4\sqrt{2}}{3} \approx 3.7723
\end{aligned}
$$

Thus we see $f_{\min} = 0$ which in this case occurs at $c = d = b = \pm\sqrt{\frac{2+\sqrt{3}}{3}} = \pm\frac{\sqrt{3}+1}{\sqrt{6}}$; $a = \pm\frac{\sqrt{3}-1}{\sqrt{2}}$.

O380. Let ABC be a triangle with orthocenter H. Let X and Y be points on side BC such that $\angle BAX = \angle CAY$. Let E and F be the feet of the altitudes from B and C, respectively. Let T and S be the intersections of EF with AX and AY, respectively. Prove that X, Y, S, T are concyclic. Furthermore, prove that H lies on the polar of A with respect to this circle.

Proposed by Bobojonova Latofat and Khurshid Juraev, Tashkent, Uzbekistan

Solution by Adnan Ali, Student in A.E.C.S-4, Mumbai, India

For the first part, we use the fact that $\triangle AEF \sim \triangle ABC$. This similarity implies that $\triangle AFT \sim \triangle ACY$ (we assume WLOG that X is closer to B than C). So, $\angle ATF = \angle AYC$ which implies that $\angle ATS = \angle AYX$. The equality obtained clearly implies that X, Y, T, S are concyclic.

For the second part, we have a useful lemma from which the problem follows directly.

Lemma: Let XYZ be a triangle and P, Q, R be points on YZ, ZX, XY respectively such that XP, YQ, ZR are concurrent. Let QR meet YZ at U. Let ℓ_1 and ℓ_2 be any two arbitrary lines from X. Let ℓ_1 and ℓ_2 meet $\overline{QRU}$ at K, L and $\overline{YZ}$ at M, N, respectively. Then the points $U, KN \cap LM$ and $YQ \cap ZR$ are collinear.

Proof: Let $YQ \cap ZR = V$ and let $UV \cap XY = U_0, UV \cap \ell_1 = U_1, UV \cap \ell_2 = U_2$ (WLOG we assume that U is closer to Z than Y). Clearly the pencil $U(Y, R, U_0, X)$ is harmonic. Thus, the pencils $U(M, K, U_1, X)$ and $U(N, L, U_2, X)$ must also be harmonic. This implies that in the triangle UMK, the cevians UU_1, ML and KN are concurrent. Thus, the points $U, KN \cap LM$ and $YQ \cap ZR$ are collinear and the proof is complete.

□

Back to the problem, we note that from our lemma, the points $TY \cap SX$, $BE \cap CF$ and $EF \cap BC$ are collinear. But from Brocard's Theorem, we know that the line joining the points $TY \cap SX$ and $ST \cap XY$ is the polar of $XT \cap SY = A$ w.r.t the circle $\odot(XYST)$. Thus H lies on the polar of A w.r.t the circle $\odot(XYST)$.

Also solved by Utsab Sarkar, West Bengal, India; Minh Pham Hoang, High School for the Gifted, Vietnam National University, Ho Chi Minh City, Vietnam; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Li Zhou, Polk State College, USA.

O381. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3+b^3+c^3}{3} \geq \frac{a^2+bc}{b+c} \cdot \frac{b^2+ca}{c+a} \cdot \frac{c^2+ab}{a+b} \geq abc.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

Solution by Daniel Lasaosa, Pamplona, Spain

Multiplying both sides by $3(a+b)(b+c)(c+a)$ and after some algebra, the left inequality is found to be equivalent to

$$a^2b^2(a-b)^2 + b^2c^2(b-c)^2 + c^2a^2(c-a)^2 + \frac{ab\left(a^2-b^2\right)^2}{2} + \frac{bc\left(b^2-c^2\right)^2}{2} + \frac{ca\left(c^2-a^2\right)^2}{2} +$$

$$+\left(\frac{4a^5b+4a^5c+b^5c+c^5b}{10} - a^4bc\right) + \left(\frac{4b^5c+4b^5a+c^5a+a^5c}{10} - b^4ca\right) +$$

$$+\left(\frac{4c^5a+4c^5b+a^5b+b^5a}{10} - c^4ab\right) +$$

$$+abc\left(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b-6abc\right) \geq 0.$$

Now, the first six terms are clearly non-negative, being simultaneously zero iff $a=b=c$. The next three terms are also non-negative because of the weighted AM-GM inequality, being each one of them zero iff $a=b=c$. The last term is also non-negative because of the AM-GM inequality, with equality again iff $a=b=c$.

Multiplying by $(a+b)(b+c)(c+a)$ and also after some algebra, the right inequality is found to be equivalent to

$$\frac{a^3(b+c)(b-c)^2 + b^3(c+a)(c-a)^2 + c^3(a+b)(a-b)^2}{2} +$$

$$+abc\frac{(a+b)(a-b)^2 + (b+c)(b-c)^2 + (c+a)(c-a)^2}{2} \geq 0.$$

where both terms in the LHS are clearly non-negative, being zero iff $a=b=c$.

The conclusion follows, equality holds in each one of the inequalities iff $a=b=c$.

Also solved by Rade Krenkov, SOU "Goce Delcev", Valandovo, Macedonia; Utsab Sarkar, West Bengal, India; Li Zhou, Polk State College, USA; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Catalin Prajitura, College at Brockport, SUNY, USA; Ghenghea Daniel; Paul Revenant, Lycée du Parc, Lyon, France.

O382. Prove that in any triangle ABC

$$\left(\frac{m_a+m_b+m_c}{3}\right)^2-\frac{m_am_bm_c}{m_a+m_b+m_c}\leq\frac{a^2+b^2+c^2}{6}.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Li Zhou, Polk State College, USA

For $x,y,z>0$, Schur's inequality yields

$$\begin{aligned}&2(x+y+z)\left(x^2+y^2+z^2\right)+9xyz-(x+y+z)^3\\=\;&x(x-y)(x-z)+y(y-z)(y-x)+z(z-x)(z-y)\geq 0.\end{aligned}$$

Hence,

$$\left(\frac{x+y+z}{3}\right)^2-\frac{xyz}{x+y+z}\leq\frac{2\left(x^2+y^2+z^2\right)}{9}.$$

Letting $x=m_a$, $y=m_b$, $z=m_c$, and recalling $m_a^2=\frac{2b^2+2c^2-a^2}{4}$, we see that $\frac{2}{9}\left(x^2+y^2+z^2\right)=\frac{1}{6}\left(a^2+b^2+c^2\right)$.

Also solved by Daniel Lasaosa, Pamplona, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Utsab Sarkar, West Bengal, India; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece.

O383. Let a, b, c be positive real numbers. Prove that

$$\frac{a+b}{6c}+\frac{b+c}{6a}+\frac{c+a}{6b}+2 \geq \sqrt{\frac{a+b}{2c}}+\sqrt{\frac{b+c}{2a}}+\sqrt{\frac{c+a}{2b}}.$$

Proposed by Marius Stănean, Zalău, România

Solution by Arkady Alt, San Jose, CA, USA

Since $2+\sum_{cyc}\frac{a+b}{6c} \geq \sum_{cyc}\sqrt{\frac{a+b}{2c}} \iff \left(2+\sum_{cyc}\frac{a+b}{6c}\right)^2 \geq \sum_{cyc}\frac{a+b}{2c}+\sum_{cyc}\sqrt{\frac{(a+b)(b+c)}{ca}}$

and

$$\sqrt{\frac{(a+b)(b+c)}{ca}} \leq \frac{1}{2}\left(\frac{a+b}{a}+\frac{b+c}{c}\right)=1+\frac{1}{2}\left(\frac{b}{a}+\frac{b}{c}\right)$$

then

$$\sum_{cyc}\sqrt{\frac{(a+b)(b+c)}{ca}} \leq 3+\frac{1}{2}\sum_{cyc}\left(\frac{b}{a}+\frac{b}{c}\right)=3+\sum_{cyc}\frac{a+b}{2c}$$

and

$$\sum_{cyc}\frac{a+b}{2c}+\sum_{cyc}\sqrt{\frac{(a+b)(b+c)}{ca}} \leq 3+\sum_{cyc}\frac{a+b}{c}.$$

Thus, suffice to prove inequality

(1)

$$\left(2+\sum_{cyc}\frac{a+b}{6c}\right)^2 \geq 3+\sum_{cyc}\frac{a+b}{c}.$$

Let $t:=\sum_{cyc}\frac{a+b}{6c}$.Since

$$t+\frac{1}{2}=\frac{1}{6}\sum_{cyc}\left(\frac{a+b}{c}+1\right)=\frac{1}{6}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

and $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$ then $t+\frac{1}{2} \geq \frac{9}{6} \iff t \geq 1$ and **(1)** becomes

$(2+t)^2 \geq 3+6t \iff (t-1)^2 \geq 0.$

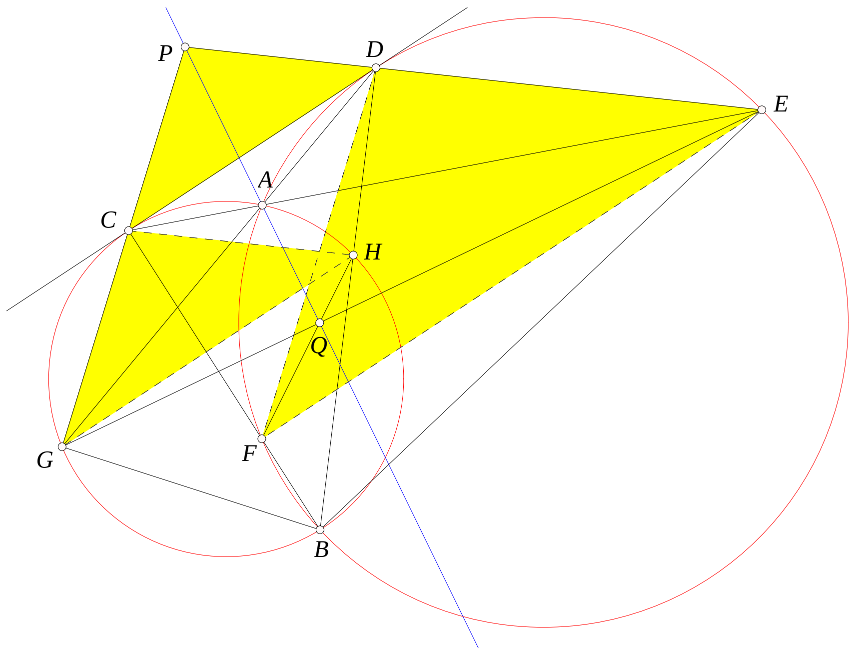
Since $t=1 \iff a=b=c$ then equality in original inequality occurs iff $a=b=c$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Utsab Sarkar, West Bengal, India; Ghenghea Daniel; Nicuşor Zlota,"Traian Vuia" Technical College, Focşani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Sushanth Sathish Kumar, Jeffery Trail Middle School, Irvine, CA, USA; Li Zhou, Polk State College, USA.

O384. Let ω_1 and ω_2 be circles intersecting at points A and B. Let CD be their common tangent such that C, D lie on ω_1, ω_2, respectively; and A is closer to CD than B. Let CA and CB intersect ω_2 at A, E and B, F, respectively. Lines DA and DB intersect ω_1 at A, G and B, H, respectively. Let P be the intersection of CG and DE and Q be the intersection of EG and FH. Prove that A, P, Q lie on the same line.

Proposed by Anton Vassilyev, Astana, Kazakhstan

Solution by Li Zhou, Polk State College, USA



By construction,

$$\angle CHG = \angle DCP = \angle DGC + \angle CDG = \angle ABC + \angle DBA = \angle DBC = \angle HGC.$$

Similarly, $\angle EFD = \angle CDP = \angle DBC = \angle DEF$. Hence, $\triangle PCD$, $\triangle CGH$, and $\triangle DFE$ are three similar isosceles triangles and $CD \parallel GH \parallel FE$. Thus, $\frac{GQ}{EQ} = \frac{GH}{EF} = \frac{CG}{DE}$, which implies that $\frac{PC}{CG} \cdot \frac{GQ}{QE} \cdot \frac{ED}{DP} = 1$. Therefore, A, P, Q are collinear by Ceva's theorem.

Also solved by Soo Young Choi, Seoul ChungDam Middle School, Republic of Korea; Utsab Sarkar, West Bengal, India; Andrianna Boutsikou, High School of Nea Makri, Athens, Greece; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Utsab Sarkar, West Bengal, India.