Junior Problems

J385. If the equalities

$$2(a+b)-6c-3(d+e)=6$$
$$3(a+b)-2c+6(d+e)=2$$
$$6(a+b)+3c-2(d+e)=-3$$

hold simultaneously, evaluate $a^2-b^2+c^2-d^2+e^2$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J386. Find all real solutions to the system of equations

$$x+yzt=y+ztx=z+txy=t+xyz=2.$$

Proposed by Mohamad Kouroshi, Tehran, Iran

J387. Find all digits a,b,c,x,y,z for which $\overline{abc}$, $\overline{xyz}$, and $\overline{abcxyz}$ are all perfect squares (no leading zeros allowed).

Proposed by Adrian Andreescu, Dallas, Texas

J388. Let $ABCD$ be a cyclic quadrilateral with $AB=AD$. Points M and N are taken on sides CD and BC, respectively, such that $DM+BN=MN$. Prove that the circumcenter of triangle AMN lies on segment AC.

Proposed by Hayk Sedrakyan, Paris, France

J389. Solve in real numbers the system of equations

$$(x^2-y+1)(y^2-x+1)=2[(x^2-y)^2+(y^2-x)^2]=4.$$

Proposed by Alessandro Ventullo, Milan, Italy

J390. Let ABC be a triangle. Points D,D' lie on side BC, points E,E' lie on side AC and points F,F' lie on side AB such that $AD=AD'=BE=BE'=CF=CF'$. Prove that if AD,BE,CF are concurrent, then so are AD',BE',CF'.

Proposed by Josef Tkadlec, Vienna, Austria

Senior Problems

S385. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a^3+8abc}+\frac{1}{b^3+8abc}+\frac{1}{c^3+8abc}\le\frac{1}{3abc}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S386. Evaluate

$$\cos\frac{\frac{\pi}{4}}{2}+\cos\frac{\frac{2\pi}{4}}{2^2}+\cdots+\cos\frac{\frac{n\pi}{4}}{2^n}$$

Proposed by Mohamad Kouroshi, Tehran, Iran

S387. Find all nonnegative real numbers k such that

$$\sum a(a-b)(a-kb)\ge 0,$$

for all nonnegative numbers a, b, c.

Proposed by Mehtaab Sawhney, Commack High School, New York, USA

S388. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{11a-6}{c}+\frac{11b-6}{a}+\frac{11c-6}{b}\le\frac{15}{abc}$$

Proposed by Marius Stănean, Zalău, România

S389. Let n be a positive integer. Prove that for any integers $a_1, a_2, \ldots, a_{2n+1}$ there is a rearrangement $b_1, b_2, \ldots, b_{2n+1}$ such that $2^n n!$ divides

$$(b_1-b_2)(b_3-b_4)\ldots(b_{2n-1}-b_{2n}).$$

Proposed by Cristinel Mortici, Valahia University, Târgovişte, România

S390. Let ABC be a triangle and G be its centroid. Lines AG, BG, CG meet the circumcircle of triangle ABC at A_1, B_1, C_1, respectively. Prove that

$$\sqrt{a^2+b^2+c^2}\le GA_1+GB_1+GC_1\le 2R+\frac{1}{6}\left(\frac{a^2}{m_a}+\frac{b^2}{m_c}+\frac{c^2}{m_c}\right)$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Undergraduate Problems

U385. Evaluate

$$\lim_{n\to\infty} \sqrt{n}\left(\sqrt{\frac{(n+1)^n}{n^{n-1}}} - \sqrt{\frac{n^{n-1}}{(n-1)^{n-2}}}\right)$$

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

U386. Given a convex quadrilateral $ABCD$, denote by S_A, S_B, S_C, S_D the area of triangles BCD, CDA, DAB, ABC, respectively. Determine the point P in the plane of the quadrilateral such that

$$S_A.\overrightarrow{PA} + S_B.\overrightarrow{PB} + S_C.\overrightarrow{PC} + S_D.\overrightarrow{PD} = 0.$$

Proposed by Dorin Andrica, Babeș-Bolyai University, Cluj-Napoca, România

U387. A polynomial with complex coefficients is called special if all its roots lie on the unit circle. Is any complex polynomial the sum of two special polynomials?

Proposed by Gabriel Dospinescu, Ecole Normale Supérieure, Lyon, France

U388. Evaluate

$$\sum_{n=1}^{\infty} \frac{\sin^{2n}\theta + \cos^{2n}\theta}{n^2}$$

Proposed by Li Zhou, Polk State College, USA

U389. Let P be a nonconstant polynomial whose zeros $x_1, x_2, \ldots, x_n$ are all real. Prove that

$$\exp\left(\int_a^b \frac{P'''(x)P(x)}{P'(x)}\,dx\right) < \left|\frac{P(a)^2P'(b)^3}{P'(a)^3P(b)^2}\right|,$$

whenever $a < b < \min(x_1, x_2, \ldots, x_n)$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U390. Prove that there is a unique representation of $\sin(\pi z)$ as a series of the form

$$\sin(\pi z) = \sum_{k=1}^{\infty} a_k z^k (1-z)^k$$

that converges for all complex numbers z, wherein the coefficients a_k are real number satisfying $|a_k| \le c \cdot \dfrac{\pi^{2k}}{(2k)!}$ for some absolute constant c.

Olympiad Problems

O385. Let $f(x,y) = \frac{x^3-y^3}{6} + 3xy + 48$. Let m and n be odd integers such that

$$|f(m,n)| \leq mn + 37.$$

Evaluate $f(m,n)$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O386. Find all pairs (m,n) of positive integers such that $3^m - 2^n$ is a perfect square.

Proposed by Alessandro Ventullo, Milan, Italy

O387. Are there integers n for which $3^{6n-3} + 3^{3n-1} + 1$ is a perfect cube?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O388. Prove that in any triangle ABC with area S,

$$\frac{m_a m_b m_c (m_a + m_b + m_c)}{\sqrt{m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2}} \geq 2S.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O389. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a^2(b+c)}{b^2+c^2} + \frac{b^2(c+a)}{c^2+a^2} + \frac{c^2(a+b)}{a^2+b^2} \geq \sqrt{3(a+b+c)}.$$

Proposed by Bazarbaev Sardar, National University of Uzbekistan

O390. Let $p > 2$ be a prime. Find the number of $4p$ element subsets of the set $\{1, 2, \ldots, 6p\}$ for which the sum of elements is divisible by $2p$.

Proposed by Vlad Matei, University of Wisconsin, Madison, USA