Junior Problems

J391. Solve the equation

$$4x^3 + \frac{127}{x} = 2016.$$

Proposed by Adrian Andreescu, Dallas, Texas

J392. Prove that in any triangle

$$\frac{(a+b+c)^3}{3abc} \le 1 + \frac{4R}{r}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J393. Find the least multiple of 2016 whose sum of digits is 2016.

Proposed by Adrian Andreescu, Dallas, Texas

J394. Prove that in any triangle

$$am_a \le \frac{bm_c + cm_b}{2}.$$

Proposed by Dorin Andrica, Babeș-Bolyai University, Cluj-Napoca, România

J395. Let a and b be real numbers such that $4a^2 + 3ab + b^2 \le 2016$. Find the maximum possible value of $a + b$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J396. Let ABC be a triangle with centroid G. The lines AG, BG, CG meet the circumcircle at $A_1, B_!, C_1$, respectively. Prove that
(a) $AB_1.AC_1.BC_1.BA_1.CA_1.CB_1 \le 4R^4r^2$,
(b) $BA_1.CA_1 + CB_1.AB_1 + AC_1.BC_1 \le 2R(2R - r)$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Senior Problems

S391. Prove that in any triangle ABC

$$\min(a,b,c) + 2\max(m_a, m_b, m_c) \geq \max(a,b,c) + 2\min(m_a, m_b, m_c).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S392. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{\sqrt{(a^2+b^2)(a^2+c^2)}} + \frac{1}{\sqrt{(b^2+c^2)(b^2+a^2)}} + \frac{1}{\sqrt{(c^2+a^2)(c^2+b^2)}} \leq \frac{a+b+c}{2abc}.$$

Proposed by Mircea Becheanu, University of Bucharest, România

S393. If n is an integer such that $n^2 + 11$ is a prime, prove that $n + 4$ is not a perfect cube.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S394. Prove that in any triangle, inscribed in a circle of radius R, the following inequality holds:

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \leq \left(\frac{R}{a} + \frac{R}{b} + \frac{R}{c}\right)^2$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S395. Let a, b, c be positive integers such that

$$a^2b^2 + b^2c^2 + c^2a^2 - 69abc = 2016.$$

Find the least possible value of $\min(a, b, c)$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S396. Let $P(X) = a_nX^n + \cdots + a_1X + a_0$ be a polynomial with complex coefficients. Prove that if all roots of $P(X)$ have modulus 1, then

$$|a_0 + a_1 + \cdots + a_n| \leq \frac{2}{n}|a_1 + 2a_2 + \cdots + na_n|.$$

When does the equality hold?

Proposed by Florin Stănescu, Găeşti, România

Undergraduate Problems

U391. Find all positive integers n such that

$$\varphi(n)^3 \leq n^2.$$

Proposed by Alessandro Ventulo, Milan, Italy

U392. Let $f(x) = x^4 + 3x^3 + ax^2 + bx + c$ be a polynomial with real coefficients which has four real roots in the interval $(-1, 1)$. Prove that

$$(1 - a + c)^2 + (3 - b)^2 \geq \left(\frac{5}{4}\right)^8.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U393. Evaluate

$$\lim_{x \to 0} \frac{\cos x \sqrt{\cos 2x} \dots \sqrt[n]{\cos nx} - 1}{\cos x \cos 2x \dots \cos nx - 1}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U394. Let $x_0 > 1$ be an integer and define $x_{n+1} = d^2(x_n)$, where $d(k)$ denotes the number of positive divisors of k. Prove that

$$\lim_{n \to \infty} x_n = 9.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

U395. Evaluate $\displaystyle\int \frac{x^2 + 6}{\left(x \cos x - 3 \sin x\right)^2}\, dx.$

Proposed by Abdelouahed Hamdi, Doha, Qatar

U396. Let S_8 be the symmetric group of permutations of an 8-element set and let k be the number of its abelian subgroups of order 16. Prove that $k \geq 1050$.

Proposed by Mircea Becheanu, University of Bucharest, România

Olympiad Problems

O391. Find all 4-tuples (x, y, z, w) of positive integers such that

$$(xy)^3 + (yz)^3 + (zw)^3 - 252yz = 2016.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O392. Let ABC be a triangle with area Δ. Prove that

$$\frac{1}{3r^2} \geq \frac{1}{r_a h_a} + \frac{1}{r_b h_b} + \frac{1}{r_c h_c} \geq \frac{\sqrt{3}}{\Delta}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O393. Let a, b, c, d be nonnegative real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$\frac{1}{5 - \sqrt{ab}} + \frac{1}{5 - \sqrt{bc}} + \frac{1}{5 - \sqrt{cd}} + \frac{1}{5 - \sqrt{da}} \leq 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O394. Let a, b, c be positive real numbers with $a + b + c = 3$. Prove that

$$\frac{1}{(b+2c)^a} + \frac{1}{(c+2a)^b} + \frac{1}{(a+2b)^c} \geq 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O395. Let a, b, c, d be nonnegative real numbers such that $ab + bc + cd + da + ac + bd = 6$. Prove that

$$a^4 + b^4 + c^4 + d^4 + 8abcd \geq 12.$$

Proposed by Marius Stănean, Zalău, România

O396. Find all polynomials $P(X)$ with positive integer coefficients having the following property: for any positive integer n and every prime p such that n is a quadratic residue modulo$p$, $P(n)$ is also a quadratic residue modulo p.

Proposed by Vlad Matei, University of Wisconsin, Madison, USA