Properties of a Configuration of Repeatedly Reflected Points over Reflection-Determined Lines

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January 15, 2017

Abstract

We explore a certain configuration of reflections of points over lines determined by these points' reflections, and discover some of this configuration's properties.

Contents

1 Introduction

Reflections have simple properties in basic geometry texts, but can become quite complex after multiple iterations. This article explores a configuration of reflections with very interesting properties. The configuration is defined as follows:

> Let point P_0 lie at a distance r from point O, and let line j lie at a distance d, which is greater than r, from O. (Point A is the foot of the perpendicular from O to j.) Point P_0' is the reflection of P_0 over j, and P_1 is the reflection of P_0 over line $\overleftrightarrow{OP_0'}$. Similarly, point P_1' is the reflection of P_1 over j, and P_2 is the reflection of P_1 over line $\overleftrightarrow{OP_1'}$. Iterate this indefinitely to find P_3, P_4, $P_5, \ldots$; Figure 1 shows the first 2 iterations. Note that all points P_k (for $k \in \mathbb{Z}_{\geq 0}$) lie on circle O with radius r.
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> In this paper, angles are assumed to lie in $[0, \pi]$ and are non-directed.

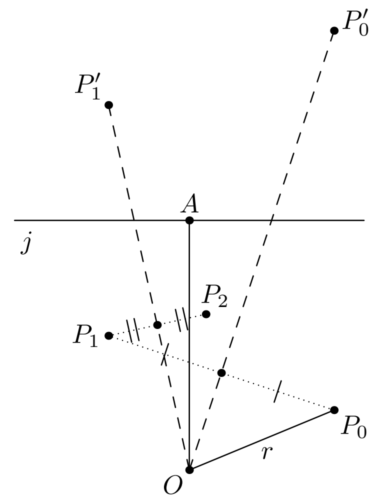


Figure 1

2 Properties

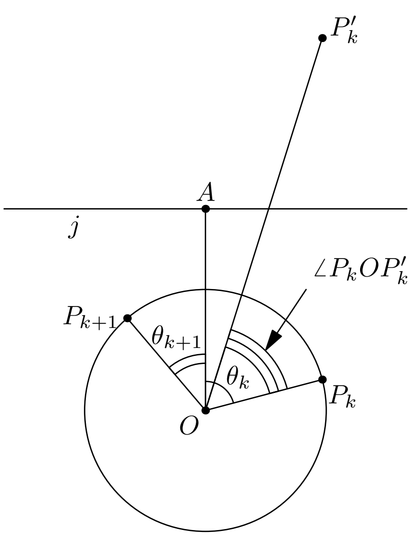


Figure 2

To explore this problem, I constructed a simulation in the dynamic geometry software Geometer's Sketchpad (GSP). I started reflecting points over lines, creating lines based on those reflections, and re-reflecting the points over those lines, when I recognized an interesting occurrence. The sequence $\{P_k\}_{k=0}^{\infty}$ appeared to converge, and it seemed that the rate of convergence depended only on the ratio of r to OA.

The first hypothesis I formed from my GSP construction was that P_k converged to the intersection point of ω and $\overline{OA}$ as $k \to \infty$. It is convenient to introduce several lemmas to prove this.

2.1 Monotonically Decreasing Angles

Lemma 1. *For all nonnegative integers k, and $m\angle AOP_k > 0$, $m\angle AOP_{k+1} < m\angle AOP_k$.*

Proof. Notice that

$$\angle P_kOP_k' \cong \angle P_{k+1}OP_k',$$

$$\theta_k = m\angle AOP_k' + m\angle P_k'OP_k,$$

and

$$\theta_{k+1} = m\angle P_{k+1}OP_k' - m\angle AOP_k' = m\angle P_k'OP_k - m\angle AOP_k'.$$

However, angles were defined earlier to lie in $[0, \pi]$, so we must take the absolute value of this so we do not receive a negative angle. Rearrange to see that

$$\theta_k - \theta_{k+1} = m\angle AOP_k' + m\angle P_k'OP_k - \left| m\angle P_k'OP_k - m\angle AOP_k' \right|.$$

Taking both cases for the absolute value sign (corresponding to whether or not P_{k+1} is on the same side of $\overleftrightarrow{AO}$ as P_k; both cases do occur), either

$$\theta_k - \theta_{k+1} = m\angle AOP_k' + m\angle P_k'OP_k - (m\angle P_k'OP_k - m\angle AOP_k') = 2m\angle AOP_k'$$

or

$$\theta_k - \theta_{k+1} = m\angle AOP_k' + m\angle P_k'OP_k + (m\angle P_k'OP_k - m\angle AOP_k') = 2m\angle P_k'OP_k.$$

Because both of these angles have positive measure, in both cases it holds that

$$\theta_k > \theta_{k+1}$$

□

We have shown $\{\theta_k\}_{k=1}^{\infty}$ to be a monotonic decreasing sequence of positive numbers. Therefore, it must converge. However, we must still show that it converges to 0.

2.2 Convergence to Angle 0

Lemma 2. $\lim\limits_{k\to\infty} \theta_k = 0$

Proof. This proof will be developed through contradiction. Suppose that $\lim\limits_{k\to\infty} \theta_k = \alpha$ for some nonzero α. Then when we perform the transformation on a point at angle α, we expect the resulting point to also have angle α. However, Lemma 1 proves that $\theta_{k+1} < \theta_k$ for nonzero θ_k. Therefore, the angle of the resulting point is less than α. It is easy to see that when we perform the transformation on a point at angle 0, the result is the same point, at angle 0. It follows that

$$\lim_{k\to\infty} \theta_k = 0.$$

□

Now that it is evident that θ_k decreases for each successive k, the next step is to find the rate at which it decreases. It seems that as k increases, the ratio $Q_k = \frac{\theta_{k+1}}{\theta_k}$ converges to a value dependent on r (the radius of ω).[1] Several data points from the GSP construction were gathered to run a regression analysis, which yielded a model (Figure 3) comparing the ratio $\frac{d}{r}$ to Sketchpad's last defined Q_k value.[2] This regression suggested that the curve $Q_\infty = \left|\frac{2d-3r}{2d-r}\right|$ fit the data very well (for $r < d$). This model provides a hypothesis for the rate of convergence, so the next step is to verify this by proof.

[1]Note that GSP tends to suffer from rounding error and displays that $Q_k = \frac{0}{0}$ starting at approximately $k = 10$.

[2]The ratio generally converged quickly enough that the last few defined values differed by less than GSP's display resolution of one ten-thousandth, so in general, this last defined Q_k is indistinguishable from $\lim\limits_{k\to\infty} Q_k$ (denote this as Q_∞).

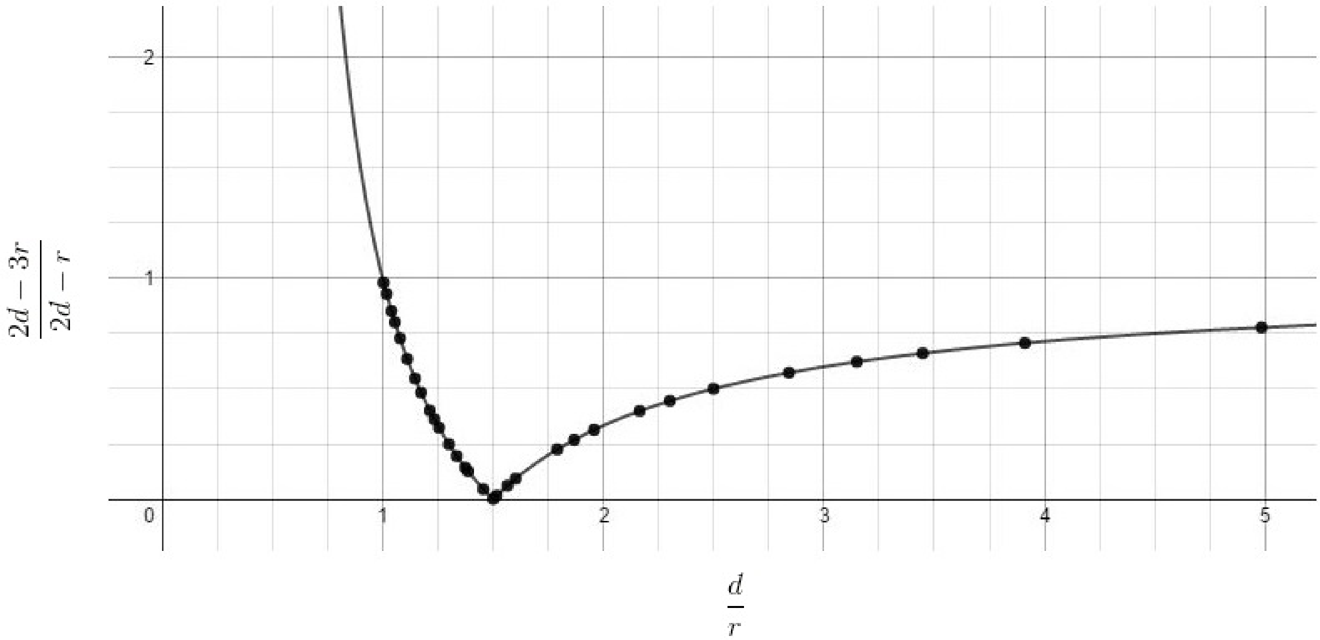


Figure 3

2.3 Rate of Convergence

Theorem 1. *As k tends towards infinity, Q_k approaches $\left|\frac{2d-3r}{2d-r}\right|$; that is, $Q_\infty = \left|\frac{2d-3r}{2d-r}\right|$*

Proof. As in the proof of Lemma 1, place the figure in the $(x,\ y)$ plane with O as the origin, and set j as the line $y = d$ (Figure 4). This gives P_k the coordinates

$$(r\,sin(\theta_k),\ r\,cos(\theta_k))$$

Then, we find that P'_k, as the reflection of P_k over j, has coordinates

$$(r\,sin(\theta_k),\ 2d - r\,cos(\theta_k)).$$

As shown above,

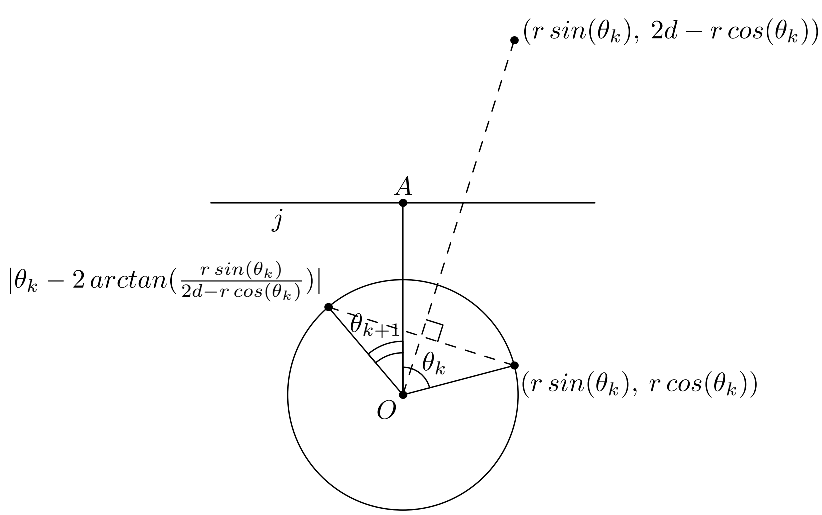
$$\theta_{k+1} = \left|m\angle P'_kOP_k - m\angle AOP'_k\right|$$

and

$$m\angle P'_kOP_k = \theta_k - m\angle AOP'_k.$$

Therefore,

$$\theta_{k+1} = \left|\theta_k - 2m\angle AOP'_k\right|.$$



Since the coordinates of P'_k are $(r\,sin(\theta_k),\ 2d - r\,cos(\theta_k))$, we see that

$$tan(m\angle AOP'_k) = \frac{r\,sin(\theta_k)}{2d - r\,cos(\theta_k)},$$

so

$$m\angle AOP'_k = arctan(\frac{r\,sin(\theta_k)}{2d - r\,cos(\theta_k)})$$
$$\theta_{k+1} = \left|\theta_k - 2\,arctan(\frac{r\,sin(\theta_k)}{2d - r\,cos(\theta_k)})\right|$$
$$Q_k = \frac{\left|\theta_k - 2\,arctan(\frac{r\,sin(\theta_k)}{2d - r\,cos(\theta_k)})\right|}{\theta_k}.$$

Since $Q_\infty = \lim_{k\to\infty} Q_k$, we know that

$$Q_\infty = \lim_{k\to\infty} \frac{\left|\theta_k - 2\,arctan(\frac{r\,sin(\theta_k)}{2d - r\,cos(\theta_k)})\right|}{\theta_k}.$$

However, since $\lim_{k\to\infty} \theta_k = 0$ (Lemma 2),

$$Q_\infty = \lim_{\theta_k\to 0} \frac{\left|\theta_k - 2\,arctan(\frac{r\,sin(\theta_k)}{2d - r\,cos(\theta_k)})\right|}{\theta_k}.$$

A symbolic processor in Mathematica was used to simplify that limit to $Q_\infty = \left|\frac{2d-3r}{r-2d}\right|$, equivalent to the result of the regression analysis[3].

Alternatively, near $x = 0$, $tan(x) \approx x$, so we may discard the arctangent to obtain that

$$Q_\infty = \lim_{\theta_k\to 0} \frac{\left|\theta_k - \frac{2r\,sin(\theta_k)}{2d - r\,cos(\theta_k)}\right|}{\theta_k}.$$

Since angles are nonnegative (and thus cannot approach 0 from below), we may also write that

$$\begin{aligned}
Q_\infty &= \lim_{\theta_k\to 0^+} \frac{\left|\theta_k - \frac{2r\,sin(\theta_k)}{2d - r\,cos(\theta_k)}\right|}{\theta_k} \\
&= \lim_{\theta_k\to 0^+} \left|\frac{\theta_k - \frac{2r\,sin(\theta_k)}{2d - r\,cos(\theta_k)}}{\theta_k}\right| \\
&= \left|\lim_{\theta_k\to 0^+} \left[1 - \frac{2r\,sin(\theta_k)}{\theta_k(2d - r\,cos(\theta_k))}\right]\right| \\
&= \left|1 - \lim_{\theta_k\to 0^+} \left[\frac{2r\,sin(\theta_k)}{\theta_k} \frac{1}{2d - r\,cos(\theta_k)}\right]\right| \\
&= \left|1 - 2r \lim_{\theta_k\to 0^+} \frac{sin(\theta_k)}{\theta_k} \cdot \lim_{\theta_k\to 0^+} \frac{1}{2d - r\,cos(\theta_k)}\right| \\
&= \left|1 - 2r \cdot 1 \cdot \frac{1}{2d - r}\right| \\
&= \left|\frac{2d - 3r}{2d - r}\right|
\end{aligned}$$

[3]This may be verified by differentiating the numerator and denominator to use L'Hôpital's Rule, then substituting in 1 for $cos(\theta_k)$ and 0 for $sin(\theta_k)$ (since θ_k is tending towards 0).

□

Notice that the model (see Figure 3) predicts that $Q_\infty = 0$ if $\frac{d}{r} = \frac{3}{2}$. This does indeed appear to be the case: when viewing the geometric model with $\frac{d}{r} = \frac{3}{2}$, small angles θ_k result in much smaller angles θ_{k+1} after the iteration. This very fast convergence may be related to the fact that $\frac{d}{r} = \frac{3}{2}$ is the point where the absolute value changes signs; that is, when $\frac{d}{r} < \frac{3}{2}$, an infinitesimally small θ_k lies on the same side of $\overleftrightarrow{AO}$ as the θ_{k+1} that it determines, and $\frac{d}{r} > \frac{3}{2}$, an infinitesimally small θ_k lies on the opposite side of $\overleftrightarrow{AO}$ as the θ_{k+1}that it determines, so when $\frac{d}{r} = \frac{3}{2}$, an infinitesimally small θ_k would result in a θ_{k+1} very close to $\overleftrightarrow{AO}$.

3 Conclusion

We have proved that θ_k monotonically decreases as k increases, and $\lim\limits_{k\to\infty} \theta_k = 0$. We have also proved that $\lim\limits_{k\to\infty} \dfrac{\theta_{k+1}}{\theta_k} = \left|\dfrac{2d-3r}{2d-r}\right|$. In order to make these conjectures, we used dynamic geometry software for visualization, and we used a symbolic processor for algebraic computations.

4 Acknowledgements

I would like to thank Dr. Evan Glazer, principal of Thomas Jefferson High School for Science and Technology, for advising me as I wrote this. I would also like to thank Dr. Jonathan Osborne for reviewing various drafts and providing feedback.

5 About the Author

Matthew J. Cox is a freshman at the Thomas Jefferson High School for Science and Technology. He enjoys doing contest and olympiad math and is a member of the school's math and physics teams. He also plays percussion in the school's band and marching band.