Rational Bounds for the Logarithm Function with Applications

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Abstract

We find rational bounds for the logarithm function and we show applications to problem-solving.

1 Introduction

Let

$$a_n = \left(1 + \frac{1}{n}\right)^n,$$

solving the problem **U385** from the journal *Mathematical Reflections* I realized that I needed good lower and upper bounds for a_n depending on n. The classical $a_n < e$ was not enough. The problem **U385** was proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain:

Evaluate

$$\lim_{n\to\infty} \sqrt{n}\left(\sqrt{\frac{(n+1)^n}{n^{n-1}}} - \sqrt{\frac{n^{n-1}}{(n-1)^{n-2}}}\right).$$

More precisely, I conjectured that

$$\sqrt{\frac{a_n}{a_{n-1}}} \le 1 + \frac{1}{n^2}.$$

To prove this conjecture I used the double inequality

$$\frac{2n}{2n+1} \cdot e < a_n < \frac{2n+1}{2n+2} \cdot e, \tag{1}$$

that I found in [1]. Exactly, problem 170, page 38 with solution in page 216. The source of this problem is old, from 1872 in *Nouvelles Annales de Mathématiques*, [2], the proposer is unknown and was solved by C. Moreau in [3]. Here we present a new proof. Later we will show how this result helps to find the limit. Our proof of (1) is based on non-standard bounds for the logarithm function, these bounds are rational functions (quotient of two polynomials) and the seed inequalities (the proof is iterative) are the well-known

$$e^x \ge x + 1, \quad \text{for} \quad x \in \mathbb{R}, \tag{2}$$

$$\ln x > 1 - \frac{1}{x}, \quad \text{for} \quad x > 1. \tag{3}$$

2 Main Theorem

Let us begin with the proof of (2) and (3).

- $e^x \ge x + 1$ for all real x. Equality holds if and only if $x = 0$.

  Proof: Let $f(x) = e^x - x - 1$. Note that $f'(x) = e^x - 1 \ge 0$ for $x \ge 0$, therefore f is an increasing function for $x \geq 0$. So, $f(x) \geq f(0)$ or equivalently $e^x \geq x+1$. If $x \leq -1$ then $x+1$ is negative and e^x is positive. When $-1 < x < 0$ the function $f(x)$ is also positive because $f(-1) = e^{-1} > 0$ and $f(0) = 0$. Suppose by contradiction that exist $x_0 \in (-1, 0)$ such that $f(x_0) < 0$, then since $f(x)$ is clearly continuous by Bolzano's theorem there is a value $x_1 \in (-1, x_0)$ with $f(x_1) = 0$, contradiction since $f(x) = 0$ only for $x = 0$. To prove this last result suppose that $f(\overline{x}) = 0$ and $\overline{x} \neq 0$. By Rolle's theorem exist $\widehat{x} \in (\overline{x}, 0)$ such that $f'(\widehat{x}) = 0$, or $e^{\widehat{x}} = 1$, hence $\widehat{x} = 0$, contradiction.

- $\ln x > 1 - \frac{1}{x}$ for $x > 1$.

  Proof:

$$\int_1^x \frac{1}{t} dt > \int_1^x \frac{1}{t^2} dt.$$

Now we are ready to find the rational bounds.

Theorem:

$$\begin{aligned} I_1 : \ln(1+x) &\leq x, \quad x > -1, \\ I_2 : \ln(1+x) &\leq \frac{x(x+2)}{2(x+1)}, \quad x \geq 0, \\ I_3 : \ln(1+x) &\leq \frac{x(x+6)}{2(2x+3)}, \quad x \geq 0, \\ I_4 : \ln x &> \frac{2(x-1)}{x+1}, \quad x > 1. \end{aligned}$$

Proof:

I_1 is a direct consequence of (2). Now we integrate I_1 to obtain I_2,

$$\int_0^x \ln(1+t) dt \leq \int_0^x t dt.$$

Integrating I_2 the result is I_3,

$$\int_0^x \ln(1+t) dt \leq \int_0^x \frac{t(t+2)}{2(t+1)} dt.$$

Finally I_4 follows from integrate (3),

$$\int_1^x \ln t dt > \int_1^x \left(1 - \frac{1}{t}\right) dt.$$

Notice that in each step the new inequalities are refinements of the previous ones. Holds

$$\frac{x(x+6)}{2(2x+3)} \leq \frac{x(x+2)}{2(x+1)} \leq x,$$

for $x \geq 0$. Also

$$\frac{2(x-1)}{x+1} \leq \frac{x-1}{x},$$

for $x > 1$. The result of continuing this algorithm are not rational bounds. To see how to find new refinements that are rational bounds look at [4].

3 Applications

- We show a proof of the Arithmetic-Geometric Mean inequality using inequality (2).

  Let $\alpha = \frac{x_1+x_2+\cdots+x_n}{n}$. By (2),

  $$e^{\left(\frac{x_1}{\alpha}-1\right)} \cdot e^{\left(\frac{x_2}{\alpha}-1\right)} \cdots e^{\left(\frac{x_n}{\alpha}-1\right)} \geq \frac{x_1}{\alpha} \cdot \frac{x_2}{\alpha} \cdots \frac{x_n}{\alpha}.$$

  After simple transformations this inequality is equivalent to AM-GM:

  $$\alpha^n \geq x_1 x_2 \cdots x_n.$$

- Let us see the proof of the double inequality (1). Taking logarithms on both sides we need to prove

  $$\ln\left(\frac{2n}{2n+1}\right) + 1 < n \ln\left(1+\frac{1}{n}\right) < \ln\left(\frac{2n+1}{2n+2}\right) + 1.$$

  The left hand inequality is

  $$n \ln\left(1+\frac{1}{n}\right) + \ln\left(\frac{2n+1}{2n}\right) > 1.$$

  The lower bound provided by (3) is not effective here, but I_4 it is,

  $$\frac{2n}{2n+1} + \frac{2}{4n+1} > 1 \Leftrightarrow 2 > 1.$$

  The right hand inequality becomes

  $$n \ln\left(1+\frac{1}{n}\right) + \ln\left(1+\frac{1}{2n+1}\right) < 1.$$

  I_1 and I_2 are not good enough, but I_3 it is,

  $$\frac{6n+1}{2(3n+2)} + \frac{12n+7}{2(6n+5)(2n+1)} < 1 \Leftrightarrow 3n+1 > 0.$$

  A nice consequence of the double inequality (1) is

  $$\lim_{n\to\infty} a_n = e.$$

- The problem **U373** from *Mathematical Reflections* is

  Prove the following inequality holds for all positive integers $n \geq 2$,

  $$\left(1+\frac{1}{1+2}\right)\left(1+\frac{1}{1+2+3}\right)\cdots\left(1+\frac{1}{1+2+3+\cdots+n}\right) < 3.$$

  Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam.

The published solution is due to *Albert Stadler, Herrliberg, Switzerland* using the inequality I_1.

$$\begin{aligned}\prod_{k=2}^{n}\left(1+\frac{1}{1+2+\cdots+k}\right) &= \prod_{k=2}^{n}\left(1+\frac{2}{k(k+1)}\right), \\ &= \exp\left(\sum_{k=2}^{n}\ln\left(1+\frac{2}{k(k+1)}\right)\right), \\ &\leq \exp\left(2\sum_{k=2}^{n}\frac{1}{k(k+1)}\right), \\ &= \exp\left(2\sum_{k=2}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)\right), \\ &= \exp\left(1-\frac{2}{n+1}\right) \leq e < 3.\end{aligned}$$

Now, we want to show how applying I_2 we obtain a refinement.

We need to show that

$$\prod_{k=1}^{n}\left(1+\frac{2}{k(k+1)}\right) < 6.$$

Or equivalently

$$\sum_{k=1}^{n}\ln\left(1+\frac{2}{k(k+1)}\right) < \ln 6.$$

By I_2 it is enough to prove that

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)}+\sum_{k=1}^{\infty}\frac{1}{k^2+k+2} \leq \ln 6.$$

The first series telescopes to 1 and the second one can be found from the series for $z\cot(\pi z)$ for an appropriate z, the value using `Wolfram Alpha` is

$$\sum_{k=1}^{\infty}\frac{1}{k^2+k+2} = \frac{\pi\tanh\left(\frac{\sqrt{7}\pi}{2}\right)}{\sqrt{7}} - \frac{1}{2} \approx 0.686827.$$

Finally $1.686827 < 1.791759 = \ln 6$.

Notice that $e^{1.686827} \approx 5.4023 < 5.4365 \approx 2e$.

- To finish let us see our solution to problem **U385**, the before mentioned limit.

  The value of the limit is $\frac{\sqrt{e}}{2}$. Denoting

$$a_n = \left(1+\frac{1}{n}\right)^n \to e,$$

the expression to find the limit is

$$\sqrt{a_{n-1}}\left(\sqrt{\frac{a_n}{a_{n-1}}}n-\sqrt{n(n-1)}\right).$$

It only remains to show that the expression inside parenthesis tend to $\frac{1}{2}$. We shall use the double inequality

$$1 \leq \sqrt{\frac{a_n}{a_{n-1}}} \leq 1+\frac{1}{n^2}.$$

The lower bound is because the sequence a_n is increasing, the upper bound is hard and is true because the double inequality

$$\frac{2n}{2n+1}\cdot e < a_n < \frac{2n+1}{2n+2}\cdot e.$$

We obtain

$$\sqrt{\frac{a_n}{a_{n-1}}} < \sqrt{\frac{4n^2-1}{4n^2-4}} < 1+\frac{1}{n^2}.$$

To finish, note that

$$\lim_{n\to\infty} n-\sqrt{n(n-1)} = \lim_{n\to\infty}\frac{1-\sqrt{1-1/n}}{1/n},$$

and doing $x=1/n$, we have

$$\lim_{x\to 0}\frac{1-\sqrt{1-x}}{x} = \lim_{x\to 0}\frac{1}{2\sqrt{1-x}} = \frac{1}{2}.$$

By L'Hopital's rule.

References

[1] G. Pólya - G. Szegő. *Problems and Theorems in Analysis. Vol. I.* Springer, (1998).

[2] Problem 1098. *Nouv. Annls. Math. Ser. 2, Vol. 11,* p. 480. (1872)

[3] C. Moreau. *Nouv. Annls. Math. Ser. 2, Vol. 13,* p. 61. (1874)

[4] Flemming Topsoe. *Some bounds for the logarithmic function.* University of Copenhagen. (available on the Internet.)

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