Junior Problems

J397. Find all positive integers n for which $3^4 + 3^5 + 3^6 + 3^7 + 3^n$ is a perfect square.

Proposed by Adrian Andreescu, Dallas, Texas

J398. Let a, b, c be real numbers. Prove that

$$(a^2 + b^2 + c^2 - 2)(a + b + c)^2 + (1 + ab + bc + ca)^2 \geq 0.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J399. Two nine-digit numbers m and n are called *cool* if

(a) they have the same digits but in different order,

(b) no digit appears more than once,

(c) m divides n or n divides m.

Prove that if m and n are cool, then they contain digit 8.

Proposed by Titu Andreescu, Dallas, Texas

J400. Prove that for all real numbers a, b, c the following inequality holds:

$$\frac{|a|}{1 + |b| + |c|} + \frac{|b|}{1 + |c| + |a|} + \frac{|c|}{1 + |a| + |b|} \geq \frac{|a + b + c|}{1 + |a + b + c|}.$$

When does the equality occur?

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J401. Find all integers n for which $n^2 + 2^n$ is a perfect square.

Proposed by Adrian Andreescu, Dallas, Texas

J402. Consider a nonisosceles triangle ABC. Let I be its incenter and G its centroid. Prove that GI is perpendicular to BC if and only if $AB + AC = 3BC$.

Proposed by Proposed by Bazarbaev Sardar, National University of Uzbekistan

Senior Problems

S397. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}+\frac{3(ab+bc+ca)}{2(a+b+c)} \geq a+b+c.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S398. The tetrahedron $ABCD$ lies inside a unit cube. Let M and N be the midpoint of the side AB and CD, respectively. Prove that $AB \cdot CD \cdot MN \leq 2$.

Proposed by Nairi Sedrakian,Yerevan, Armenia

S399. Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2=1$. Prove that

$$\sqrt{2} \leq \sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}} \leq \sqrt[4]{27}.$$

When do equalities occur?

Proposed by Marcel Chiriță †, Bucharest, România

S400. Find all n for which $(n-4)!+\frac{1}{36n}(n+3)!$ is a perfect square.

Proposed by Proposed by Titu Andreescu, University of Texas at Dallas, USA

S401. Let a, b, c, d be nonnegative real numbers such that $ab+ac+ad+bc+bd+cd=6$. Prove that

$$a+b+c+d+(3\sqrt{2}-4)abcd \geq 3\sqrt{2}.$$

Proposed by Marius Stănean, Zalău, România

S402. Prove that

$$\sum_{k=1}^{31} \frac{k}{(k-1)^{4/5}+k^{4/5}+(k+1)^{4/5}} < \frac{3}{2}+\sum_{k=1}^{31}(k-1)^{1/5}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Undergraduate Problems

U397. Let T_n be the $n-$th triangular number. Evaluate

$$\sum_{n\geq 1} \frac{1}{(8T_n - 3)(8T_{n+1} - 3)}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U398. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{4a} + \frac{1}{4b} + \frac{1}{4c} + \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq 3\left(\frac{1}{3a+b} + \frac{1}{3b+c} + \frac{1}{3c+a}\right)$$

Proposed by Sardor Bazarbaev, National University of Uzbekistan

U399. Consider the functional equation $f(f(x)) = f(x)^2$, where $f : \mathbb{R} \longrightarrow \mathbb{R}$.

(a) Find all real analytic solutions of the equation.

(b) Prove that there exist infinitely many differentiable solutions of the equation.

(c) Do only finitely many infinitely differentiable solutions exist?

Proposed by David Rose and Li Zhou, Polk State College, Florida, USA

U400. Let A and B be 3×3 matrices with integer entries such that $AB = BA$, $\det(B) = 0$, and $\det(A^3 + B^3) = 1$. Find all possible polynomials $f(x) = \det(A + xB)$.

Proposed by Florin Stănescu, Găeşti, România

U401. Let P be a polynomial of degree n such that $P(k) = \frac{1}{k^2}$, for all $k = 1, 2, \ldots, n+1$. Determine $P(n+2)$.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

U402. Let n be a positive integer and let $P(x)$ be a polynomial of degree at most n such that $|P(x)| \leq x + 1$ for all $x \in [0, n]$. Prove that

$$|P(n+1)| + |P(-1)| \leq (n+2)\left(2^{n+1} - 1\right).$$

Proposed by Alessandro Ventulo, Milan, Italy

Olympiad Problems

O397. Solve in integers the equation:

$$(x^3-1)(y^3-1)=3(x^2y^2+2).$$

Proposed by Proposed by Titu Andreescu, University of Texas at Dallas, USA

O398. Let a, b, c, d be positive real numbers such that $abcd \geq 1$. Prove that

$$\frac{1}{a+b^5+c^5+d^5}+\frac{1}{b+c^5+d^5+a^5}+\frac{1}{c+d^5+a^5+b^5}+\frac{1}{d+a^5+b^5+c^5} \leq 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O399. Let a, b, c be positive real numbers. Prove that

$$\frac{a^5+b^5+c^5}{a^2+b^2+c^2} \geq \frac{1}{2}(a^3+b^3+c^3-abc).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O400. Let ABC be a triangle and let BD be the angle bisector of $\angle ABC$. The circumcircle of triangle BCD intersects the side AB at E such that E lies between A and B. The circumcircle of triangle ABC intersects he line CE at F. Prove that

$$\frac{BC}{BD}+\frac{BF}{BA}=\frac{CE}{CD}$$

Proposed by Florin Stănescu, Găeşti, România

O401. Let a, b, c be positive real numbers. Prove that

$$\sqrt{\frac{9a+b}{9b+a}}+\sqrt{\frac{9b+c}{9c+b}}+\sqrt{\frac{9c+a}{9a+c}} \geq 3.$$

Proposed by An Zhenping, Xianyang Normal University, China

O402. Prove that in any triangle ABC the following inequality holds:

$$\sin^2 2A+\sin^2 2B+\sin^2 2C \geq 2\sqrt{3}\sin 2A \sin 2B \sin 2C.$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România