Junior problems

J391. Solve the equation

$$4x^3 + \frac{127}{x} = 2016.$$

Proposed by Adrian Andreescu, Dallas, Texas

Solution by Alessandro Ventullo, Milan, Italy

The given equation is equivalent to

$$4x^4 - 2016x + 127 = 0.$$

Observe that

$$\begin{aligned} 4x^4 - 2016x + 127 &= 4x^4 - 256x^2 + (2x^2 + 16x) + 127(2x^2 - 16x) + 127 \\ &= (2x^2 - 16x + 1)(2x^2 + 16x) + 127(2x^2 - 16x + 1) \\ &= (2x^2 - 16x + 1)(2x^2 + 16x + 127) \end{aligned}$$

So, the given equation becomes

$$(2x^2 - 16x + 1)(2x^2 + 16x + 127) = 0.$$

We obtain

$$x = \frac{8 \pm \sqrt{62}}{2}, \qquad x = \frac{-8 \pm i\sqrt{190}}{2}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Joel Schlosberg, Bayside, NY, USA; Nandansai, Srigayatri College, Hyderabad, India; Albert Stadler, Herrliberg, Switzerland; Alok Kumar, New Delhi, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Andreas Charalampopoulos, 4th Lyceum of Glyfada, Greece; Haimoshri Das, South Point High School, Kolkata, India; Joseph Currier, SUNY Brockport, NY, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua; Polyahedra, Polk State College, FL, USA; Pedro Acosta, West Morris Mendham High School, Mendham, NJ, USA; Aditya Ghosh, Kolkata,West Bengal, India; Prajnanaswaroopa S, Bangalore, Karnataka, India; Dimitris Avramidis, Evaggeliki Gymnasium, Athens, Greece; Joehyun Kim, Bergen County Academies, Hackensack, NJ, USA; Vincelot Ravoson, Lycée Henri IV, Paris, France; Simon Pellicer, Paris, France; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina.

J392. Prove that in any triangle

$$\frac{(a+b+c)^3}{3abc} \le 1 + \frac{4R}{r}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

First solution by Arkady Alt, San Jose, CA, USA

Let s, R, r be semiperimeter,circumradius and inradius of the triangle. Then

$$\frac{(a+b+c)^3}{3abc} \le 1 + \frac{4R}{r} \iff \frac{8s^3}{3 \cdot 4Rrs} \le 1 + \frac{4R}{r} \iff 2s^2 \le 12R^2 + 3rR.$$

Since $s^2 \le 4R^2 + 4Rr + 3r^2$ (Gerretsen's Inequality) and $R \ge 2r$

$$12R^2 + 3rR - 2s^2 = 12R^2 + 3rR - 2\left(4R^2 + 4Rr + 3r^2\right) + 2\left(4R^2 + 4Rr + 3r^2 - s^2\right) =$$

$$(R - 2r)(4R + 3r) + 2\left(4R^2 + 4Rr + 3r^2 - s^2\right) \ge 0.$$

Second solution by Arkady Alt, San Jose, CA, USA

Let $x := s - a, y := s - b, z := s - c,\ p := xy + yz + zx, q := xyz$.Due to homogeneity of the original inequality we can assume that $s = 1$. Then $a = 1 - x, b = 1 - y, c = 1 - z$,

$$x, y, z > 0, x + y + z = 1, a + b + c = 2,\ abc = p - q, r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{q},$$

$$R = \frac{abc}{4sr} = \frac{p-q}{4\sqrt{q}}$$

and original inequality becomes

$$\frac{8}{3(p-q)} \le 1 + \frac{p-q}{q} \iff 8 \le 3\left(\frac{p^2}{q} - p\right).\mathbf{(1)}$$

Note that $p = xy + yz + zx \le \frac{(x+y+z)^2}{3} = \frac{1}{3}$ and $p^2 = (xy + yz + zx)^2 \ge 3xyz(x + y + z) = 3q$. Since $\frac{p^2}{q}$ decreasing in $q > 0$ suffices to prove inequality **(1)** for $q = \frac{p^2}{3}$ and $p \in (0, 1/3]$. We have

$$8 \le 3\left(\frac{p^2}{p^2/3} - p\right) \iff 8 \le 3\left(\frac{p^2}{p^2/3} - p\right) \iff 8 \le 3(3 - p) \iff 3p \le 1.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, FL, USA; Pedro Acosta, West Morris Mendham High School, Mendham, NJ, USA; Aditya Ghosh, Kolkata,West Bengal, India; Jamal Gadirov, Istanbul University Istanbul, Turkey; Albert Stadler, Herrliberg, Switzerland; Andreas Charalampopoulos, 4th Lyceum of Glyfada, Greece; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Nikos Kalapodis, Patras, Greece; Stanescu Florin, Serban Cioculescu School, Gaesti, Romania.

J393. Find the least multiple of 2016 whose sum of digits is 2016.

Proposed by Adrian Andreescu, Dallas, Texas

Solution by Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua.
The number is $598\underbrace{999\cdots999}_{217 \text{ times } 9}89888$.

It is clear that the number has more than 224 digits, because $2016 \div 9 = 224$. Now, to minimize the number we need to find a number with 225 digits multiple of 2016.

$$N = \overline{a_{224}a_{225}\cdots a_4a_3a_2a_1a_0}.$$

Since $2016 = 2^5 \cdot 3^2 \cdot 7$, it is sufficient and necessary that the number $k\overline{a_4a_3a_2a_1a_0}$ is multiple of 32, the number N is multiple of 7, and clearly $s(N) = \sum_{i=0}^{224} a_i = 2016$ is multiple of 9.

First, we will analyze $s(k) = a_4 + a_3 + a_2 + a_1 + a_0$. The number a_0 is even and is least equal to 8. We have several cases, but:

- If $s(k) = 44$, then k is 99998 and all its permutations.
- If $s(k) = 43$, then k is 99988 and all its permutations.
- If $s(k) = 42$, then k is $99888, 99978$, or 99996 and all its permutations.
- If $s(k) = 41$, then k is $98888, 99878$, or 99968 and all its permutations.

We discard most numbers because the numbers finished in 7, 9, 78, 86, 98, 788, 988, 996 or 8888 are not multiples of 32. The only multiples of 32 in all of these cases are 99968 and 89888 in both cases $s(k) = 41$.

If $a_{224} \leq 3$ then, $\sum_{i=0}^{224} a_i \leq 3+219\times 9+(a_4+a_3+a_2+a_1+a_0) = 1974+s(k)$, then $s(k) \geq 42$. Contradiction, because the number $s(k) \leq 41$.

If $a_{224} = 4$, then, $2016 = \sum_{i=0}^{224} a_i \leq 4 + 219 \times 9 + s(k) = 1975 + s(k)$, then $s(k) \geq 41$, from here that $s(k) = 41$ and the termination of N has two options, but $99968 \equiv 1 \pmod 7$ and $89888 \equiv 1 \pmod 7$, then for minimize we choose 89888

$$N = 4\underbrace{999\cdots999}_{219 \text{ times } 9}89888 = 4 \cdot 10^{224} + (10^{219} - 1) \cdot 10^5 + 89888 \equiv 4 \cdot 2 + (6-1) \cdot 5 + 1 = 6 \not\equiv 0 \pmod 7.$$

If $a_{224} = 5$, then, $2016 = \sum_{i=0}^{224} a_i \leq 5 + 219 \times 9 + s(k) = 1976 + s(k)$, then $s(k) \geq 40$. For minimize $s(k) = 41$ and in the same way as the previous case the termination is 89888

$$N = 5\underbrace{999\cdots8\cdots999}_{218 \text{ times } 9}89888 = 5 \cdot 10^{224} + (10^{219} - 1) \cdot 10^5 - 10^m + 89888 \text{ where } 5 \leq m \leq 223$$

$$N \equiv 5 \cdot 2 + (6-1) \cdot 5 - 10^m + 1 \equiv 1 - 10^m \equiv 0 \pmod 7.$$

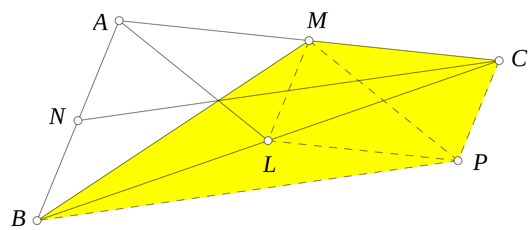
Then $10^m \equiv 1 \pmod 7 \Leftrightarrow m \equiv 0 \pmod 6$. For minimize we choose the highest value for m this is $m = 222$.

J394. Prove that in any triangle

$$am_a \leq \frac{bm_c + cm_b}{2}.$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

Solution by Polyahedra, Polk State College, FL, USA



Let L, M, N be the midpoints of BC, CA, AB, respectively. As in the figure, locate P so that $ALPM$ is a parallelogram. Then $CMLP$ is a parallelogram (because $PL\#MA\#CM$, where # denotes, obviously, $\|$ and $=$), and so is $BPCN$ (because $CP\#ML\#NB$). Applying Ptolemy's inequality in $BPCM$, we have

$$am_a = BC \cdot MP \leq CM \cdot BP + CP \cdot BM = \frac{bm_c}{2} + \frac{cm_b}{2}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Nikos Kalapodis, Patras, Greece; Stanescu Florin, Serban Cioculescu School, Gaesti, Romania; Sardor Bazarbaev, National University of Uzbekistan, Uzbekistan; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Nicusor Zlotaâš Traian Vuia Technical College, Focsani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Albert Stadler, Herrliberg, Switzerland.

J395. Let a and b be real numbers such that $4a^2 + 3ab + b^2 \leq 2016$. Find the maximum possible value of $a + b$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain
Note that

$$8 \cdot 2016 \geq 32a^2 + 24ab + 8b^2 = (5a + b)^2 + 7(a + b)^2 \geq 7(a + b)^2,$$

with equality iff $5a + b = 0$ and $4a^2 + 3ab + b^2 = 2016$. Therefore,

$$a + b \leq |a + b| = \sqrt{(a + b)^2} \leq \sqrt{\frac{8 \cdot 2016}{7}} = 48,$$

with equality iff a, b are the solutions of the system formed by $a + b = 48$ and $5a + b = 0$, ie iff $a = -12$ and $b = 60$, which results indeed in $a + b = 48$ and $4a^2 + 3ab + b^2 = 576 - 2160 + 3600 = 2016$.

Also solved by Polyahedra, Polk State College, FL, USA; Prajnanaswaroopa S, Bangalore, Karnataka, India; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Dimitris Avramidis, Evaggeliki Gymnasium, Athens, Greece; Joehyun Kim, Bergen County Academies, Hackensack, NJ, USA; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Alok Kumar, New Delhi, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Joel Schlosberg, Bayside, NY, USA; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy.

J396. Let ABC be a triangle with centroid G. The lines AG, BG, CG meet the circumcircle at A_1, B_1, C_1 respectively. Prove that

(a) $AB_1 \cdot AC_1 \cdot BC_1 \cdot BA_1 \cdot CA_1 \cdot CB_1 \leq 4R^4r^2$,

(b) $BA_1 \cdot CA_1 + CB_1 \cdot AB_1 + AC_1 \cdot BC_1 \leq 2R(2R - r)$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Let $\angle BAC = \alpha$, $\angle ABC = \beta$, $\angle BCA = \gamma$, $AB_1 = x_1$, $AC_1 = x_2$, $BA_1 = y_1$, $BC_1 = y_2$, $CA_1 = z_1$, $CB_1 = z_2$. By the law of cosine on triangle AB_1C, we have

$$\begin{aligned} b^2 &= x_1^2 + z_2^2 - 2x_1z_2\cos(\pi - \beta) \\ &= x_1^2 + z_2^2 + 2x_1z_2\cos\beta \end{aligned}$$

Using

$$x_1^2 + z_2^2 \geq 2x_1z_2$$

inequality we get,

$$2x_1z_2(1 + \cos\beta) \leq b^2 \;\Leftrightarrow\; x_1z_2 \leq \frac{b^2}{2(1 + \cos\beta)}.$$

From the other hand,

$$\cos\beta = \frac{c^2 + a^2 - b^2}{2ac}.$$

Hence we have

$$x_1z_2 \leq \frac{b^2}{2\left(1 + \frac{c^2+a^2-b^2}{2ac}\right)} = \frac{b^2ac}{(c+a)^2 - b^2} = \frac{b^2ac}{4p(p-b)}.$$

By the similarly, we get

$$x_2y_2 \leq \frac{c^2ab}{4p(p-c)}, \quad y_1z_1 \leq \frac{a^2bc}{4p(p-a)}.$$

Thus, we have

(a)

$$x_1x_2y_1y_2z_1z_2 \leq \frac{(abc)^4}{64p^3(p-a)(p-b)(p-c)} = \frac{(4Rpr)^4}{64p^2 \cdot (pr)^2} = 4R^4r^2.$$

(b)

$$\begin{aligned} y_1z_1 + z_2x_1 + x_2y_2 &\leq \frac{a^2bc}{4p(p-a)} + \frac{b^2ac}{4p(p-b)} + \frac{c^2ab}{4p(p-c)} \\ &= \frac{abc(a(p-b)(p-c) + b(p-c)(p-a) + c(p-a)(p-b))}{4p(p-a)(p-b)(p-c)} \\ &= \frac{R}{pr}(a(p-b)(p-c) + b(p-c)(p-a) + c(p-a)(p-b)). \end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, FL, USA; Nikos Kalapodis, Patras, Greece; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Albert Stadler, Herrliberg, Switzerland.

Senior problems

S391. Prove that in any triangle ABC

$$\min(a,b,c)+2\max(m_a,m_b,m_c)\geq \max(a,b,c)+2\min(m_a,m_b,m_c).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, Winter Haven, FL, USA
We may assume that $a \geq b \geq c$. Then

$$m_a^2=\frac{2b^2+2c^2-a^2}{4}\leq m_b^2=\frac{2c^2+2a^2-b^2}{4}\leq m_c^2=\frac{2a^2+2b^2-c^2}{4}.$$

So

$$\begin{aligned} & \min(a,b,c)+2\max(m_a,m_b,m_c)-\max(a,b,c)-2\min(m_a,m_b,m_c) \\ = \quad & c+2m_c-a-2m_a=c-a+\frac{2(m_c^2-m_a^2)}{m_c+m_a}=(a-c)\left[\frac{3(a+c)}{2(m_c+m_a)}-1\right]. \end{aligned}$$

Now by the triangle inequality, $2m_c < a+b \leq 2a$ and $2m_a < b+c \leq a+c$. Hence, $2(m_a+m_c) < 3(a+c)$, completing the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Pedro Acosta, West Morris Mendham High School, Mendham, NJ, USA; Joehyun Kim, Bergen County Academies, Hackensack, NJ, USA; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Arkady Alt, San Jose, CA, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Nicusor Zlotaâš Traian Vuia Technical College, Focsani, Romania; Nikos Kalapodis, Patras, Greece; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Albert Stadler, Herrliberg, Switzerland.

S392. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{\sqrt{(a^2+b^2)(a^2+c^2)}}+\frac{1}{\sqrt{(a^2+b^2)(a^2+c^2)}}+\frac{1}{\sqrt{(a^2+b^2)(a^2+c^2)}}\le\frac{a+b+c}{2abc}.$$

Proposed by Mircea Becheanu, University of Bucharest, România

Solution by Arkady Alt, San Jose, CA, USA

Applying Cauchy-Schwarz Inequality to pairs (a,b) and (c,a)we obtain

$$\sqrt{(a^2+b^2)(a^2+c^2)}=\sqrt{a^2+b^2}\cdot\sqrt{c^2+a^2}\ge ac+ba=a\left(b+c\right)$$

and, therefore,

$$\sum_{cyc}\frac{1}{\sqrt{(a^2+b^2)(a^2+c^2)}}\le\sum_{cyc}\frac{1}{a\left(b+c\right)}=\frac{1}{abc\left(a+b\right)\left(b+c\right)\left(c+a\right)}\sum_{cyc}bc\left(a+b\right)\left(c+a\right)=$$

$$\frac{abc\left(a+b+c\right)+\left(ab+bc+ca\right)^2}{abc\left(a+b\right)\left(b+c\right)\left(c+a\right)}.$$

Thus, suffices to prove inequality $\dfrac{abc\left(a+b+c\right)+\left(ab+bc+ca\right)^2}{abc\left(a+b\right)\left(b+c\right)\left(c+a\right)}\le\dfrac{a+b+c}{2abc}\iff$

$$2abc\left(a+b+c\right)+2\left(ab+bc+ca\right)^2\le\left(a+b+c\right)\left(a+b\right)\left(b+c\right)\left(c+a\right)\iff$$

$$2abc\left(a+b+c\right)+2\left(ab+bc+ca\right)^2\le\left(a+b+c\right)\left(\left(a+b+c\right)\left(ab+bc+ca\right)-abc\right)\iff$$

$3abc\left(a+b+c\right)+2\left(ab+bc+ca\right)^2\le\left(a+b+c\right)^2\left(ab+bc+ca\right).$

Since by AM-GM Inequality

$$\left(a+b+c\right)\left(ab+bc+ca\right)\ge3\sqrt[3]{abc}\cdot3\sqrt[3]{a^2b^2c^2}=9abc$$

and

$$\left(a+b+c\right)^2\ge3\left(ab+bc+ca\right)\iff a^2+b^2+c^2\ge ab+bc+ca$$

then

$$\left(a+b+c\right)^2\left(ab+bc+ca\right)=\frac{\left(a+b+c\right)^2\left(ab+bc+ca\right)}{3}+$$

$$\frac{2\left(a+b+c\right)^2\left(ab+bc+ca\right)}{3}\ge3abc\left(a+b+c\right)+2\left(ab+bc+ca\right)^2.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Pedro Acosta, West Morris Mendham High School, Mendham, NJ, USA; Aditya Ghosh, Kolkata,West Bengal, India; Prajnanaswaroopa S, Bangalore, Karnataka, India; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Vincelot Ravoson, Lycée Henri IV, Paris, France; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Nikos Kalapodis, Patras, Greece; Stanescu Florin, Serban Cioculescu School, Gaesti, Romania; Li Zhou, Polk State College, Winter Haven, FL, USA; Sardor Bazarbaev, National University of Uzbekistan, Uzbekistan; Albert Stadler, Herrliberg, Switzerland; Alok Kumar, New Delhi, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Andreas Charalampopoulos, 4th Lyceum of Glyfada, Greece; Jamal Gadirov, Istanbul University Istanbul, Turkey; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Devesh Rajpal, Raipur, India; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; Haimoshri Das, South Point High School, Kolkata, India; Henry Ricardo, New York Math Circle; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Vasile Giurgi, Ovidiu Pop, Colegiul National Dragos-Voda, Sighetu Marmatiei, Romania; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Pitchayut Saengrungkongka, Bangkok, Thailand; Zafar Ahmed, BARC, Mumbai, India.

S393. If n is an integer such that $n^2 + 11$ is a prime, prove that $n + 4$ is not a perfect cube.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Assume that $n + 4 = m^3$ is a perfect cube for some integer m. Then,

$$n^2 + 11 = m^6 - 8m^3 + 27 = \left(m^4 - m^3 - 2m^2 - 3m + 9\right)\left(m^2 + m + 3\right).$$

Note now that

$$m^2 + m + 3 = \frac{(2m+1)^2 + 11}{4} \geq \frac{11}{4} > 2,$$

and that

$$m^4 - m^3 - 2m^2 - 3m + 9 = \frac{\left(2m^2 - m - 3\right)^2 + 3\left(m - 3\right)^2}{4},$$

where if $m \geq 5$ or $m \leq 1$, we clearly have $m^4 - m^3 - 2m^2 - 3m + 9 > 9$, whereas for $m = 2, 3, 4$ we have $2m^3 - m - 3 = 11, 48, 121$ for $m^4 - m^3 - 2m^2 - 3m + 9 \geq \frac{11}{4} > 2$. Therefore, $n^2 + 11$ is the product of two integers, each one of them larger than 1, and it cannot be a prime. Reciprocally, it follows that if $n^2 + 11$ is prime, $n + 4$ cannot be a perfect cube.

Also solved by Pedro Acosta, West Morris Mendham High School, Mendham, NJ, USA; Aditya Ghosh, Kolkata, West Bengal, India; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Joehyun Kim, Bergen County Academies, Hackensack, NJ, USA; Vincelot Ravoson, Lycée Henri IV, Paris, France; Li Zhou, Polk State College, Winter Haven, FL, USA; Albert Stadler, Herrliberg, Switzerland; Alessandro Ventullo, Milan, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Haimoshri Das, South Point High School, Kolkata, India; Joel Schlosberg, Bayside, NY, USA; José Hernández Santiago, Morelia, Michoacán, Mexico; Kousik Sett, India, West Bengal, Hooghly; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece.

S394. Prove that in any triangle, inscribed in a circle of radius R, the following inequality holds:

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\leq\left(\frac{R}{a}+\frac{R}{b}+\frac{R}{c}\right)^2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain

Multiplying both sides by $2a^2b^2c^2$ and remembering that the area of ABC is $S=\frac{abc}{4S}$, the inequality is equivalent to

$$2R^2(ab+bc+ca)^2\geq 2abc\left(a^3+b^3+c^3\right)=6a^2b^2c^2+$$
$$+abc(a+b+c)\left((a-b)^2+(b-c)^2+(c-a)^2\right),$$

or using also that $S=\frac{r(a+b+c)}{2}$, to

$$(a-b)^2+(b-c)^2+(c-a)^2\leq\frac{R\left(a^2b^2+b^2c^2+c^2a^2\right)}{r(a+b+c)^2}+4R^2-12Rr.$$

Now, using the Sine Law and after some algebra, we find that

$$(a-b)^2+(b-c)^2+(c-a)^2=2R^2-6Rr-2r^2+$$
$$+2R^2\left(\cos A\cos B+\cos B\cos C+\cos C\cos A\right),$$

and using Heron's formula, that

$$a^2b^2+b^2c^2+c^2a^2\geq 2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4=16S^2,$$

or it suffices to show that

$$R^2\left(\cos A\cos B+\cos B\cos C+\cos C\cos A\right)\leq R^2-Rr+r^2.$$

Now,

$$\cos A\cos B+\cos B\cos C+\cos C\cos A\leq\frac{(\cos A+\cos B+\cos C)^2}{3}=\frac{(R+r)^2}{3R^2},$$

or again it suffices to show that

$$(R+r)^2\leq 3R^2-3Rr+3r^2,\qquad 0\leq 2R^2-5Rr+2r^2=(R-2r)(2R-r),$$

clearly true because $R\geq 2r$. The conclusion follows, equality holds iff ABC is equilateral.

Also solved by Nikos Kalapodis, Patras, Greece; Arkady Alt, San Jose, CA, USA; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Albert Stadler, Herrliberg, Switzerland.

S395. Let a, b, c be positive integers such that

$$a^2b^2 + b^2c^2 + c^2a^2 - 69abc = 2016.$$

Find the least possible value of $\min(a, b, c)$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, USA

We show that the answer is 2. First, it is easy to check that $(a, b, c) = (12, 12, 2)$ is a solution. Next, suppose that $a \geq b \geq c = 1$. Then the equation can be written as $(ab - 90)(ab + 23) + (a - b)^2 + 54 = 0$. Hence $b \leq 9$. Viewing the equation as quadratic in a: $(b^2 + 1)a^2 - 69ba + b^2 - 2016 = 0$, we then check that the discriminant $(69b)^2 - 4(b^2 + 1)(b^2 - 2016)$ is not a perfect square for $b = 1, 2, \ldots, 9$. Hence, there is no solution if $a \geq b \geq c = 1$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland; Alessandro Ventullo, Milan, Italy; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina.

S396. Let $P(X)=a_nX^n+\cdots+a_1X+a_0$ be a polynomial with complex coefficients. Prove that if all roots of $P(X)$ have modulus 1, then

$$|a_0+a_1+\cdots+a_n|\le\frac{2}{n}\,|a_1+2a_2+\cdots+na_n|\,.$$

When does the equality hold?

Proposed by Florin Stănescu, Găeşti, România

Solution by Li Zhou, Polk State College, USA

Note that $a_0+a_1+\cdots+a_n=P(1)$ and $a_1+2a_2+\cdots+na_n=P'(1)$. If $P(1)=0$ then the inequality is trivially true, with equality if and only if $P'(1)=0$, that is, if and only if 1 is a zero of $P(X)$ with multiplicity ≥ 2.

Now consider that $P(1)\neq 0$. Let $z_k=e^{it_k}$, with $0<t_k<2\pi$ for $1\le k\le n$, be the zeros of $P(X)$. Then $P'(X)/P(X)=1/(X-z_1)+\cdots+1/(X-z_n)$. Since

$$\frac{1}{1-z_k}=\frac{1}{(1-\cos t_k)-i\sin t_k}=\frac{(1-\cos t_k)+i\sin t_k}{2(1-\cos t_k)}=\frac{1}{2}+\frac{i}{2}\cot\frac{t_k}{2},$$

$$\left|\frac{P'(1)}{P(1)}\right|=\left|\frac{1}{1-z_1}+\cdots+\frac{1}{1-z_n}\right|=\left|\frac{n}{2}+\frac{i}{2}\left(\cot\frac{t_1}{2}+\cdots+\frac{t_n}{2}\right)\right|\ge\frac{n}{2},$$

with equality if and only if $\cot(t_1/2)+\cdots+\cot(t_n/2)=0$.

Alternative solution by Joel Schlosberg, Bayside, NY, USA

By the fundamental theorem of algebra,

$$P(X)=a_n\prod_{k=1}^{n}(X-r_k)$$

for roots $r_1,\ldots,r_n\in\mathbb{C}$. For any $X\in\mathbb{C}$ that is not a root of $P(X)$,

$$\ln(P(X))=\ln a_n+\sum_{k=1}^{n}\ln(X-r_k)$$

$$\frac{P'(X)}{P(X)}=\sum_{k=1}^{n}\frac{1}{X-r_k}.$$

If $P(1)=a_0+a_1+\cdots+a_n=0$, the inequality is trivially true. If $P(1)=a_0+a_1+\cdots+a_n\neq 0$, 1 is not a root of $P(X)$, so

$$\frac{a_1+2a_2+\cdots+na_n}{a_0+a_1+\cdots+a_n}=\frac{P'(1)}{P(1)}=\sum_{k=1}^{n}\frac{1}{1-r_k}.$$

If $|z|=|x+yi|=1$ for $z\neq 1\in\mathbb{C}$, $x,y\in\mathbb{R}$, then

$$\frac{1}{1-z}=\frac{1}{1-x-yi}=\frac{1-x+yi}{(1-x)^2+y^2}=\frac{1}{2}+\frac{y}{2-2x}i,$$

($(1-x)^2+y^2=2-2x\neq 0$ since otherwise $z=1$). Thus

$$\Re\left(\tfrac{a_1+2a_2+\cdots+na_n}{a_0+a_1+\cdots+a_n}\right)=\Re\left(\sum_{k=1}^{n}\frac{1}{1-r_k}\right)=\sum_{k=1}^{n}\Re\left(\frac{1}{1-r_k}\right)=\frac{n}{2}$$

$$\left|\frac{a_1+2a_2+\cdots+na_n}{a_0+a_1+\cdots+a_n}\right|=\sqrt{\left(\frac{n}{2}\right)^2+\left[\Im\left(\tfrac{a_1+2a_2+\cdots+na_n}{a_0+a_1+\cdots+a_n}\right)\right]^2}\ge\frac{n}{2},$$

which is equivalent to the inequality.

Equality holds iff $\frac{a_1+2a_2+\cdots+na_n}{a_0+a_1+\cdots+a_n} \in \mathbb{R}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Sardor Bazarbaev, National University of Uzbekistan, Uzbekistan; Albert Stadler, Herrliberg, Switzerland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy.

U391. Find all positive integers n such that

$$\varphi(n)^3 \leq n^2.$$

Proposed by Alessandro Ventulo, Milan, Italy

Solution by Li Zhou, Polk State College, Winter Haven, FL, USA

It is easy to check that $n = 1, 2, 3, 4, 6, 8, 10, 12, 18, 24, 30, 42$ are solutions. We show that they are the only solutions.

Suppose that $n \geq 5$. Let $f(n) = \varphi(n)^3/n^2$. Note that f is multiplicative and $f(p^e) = p^{e-3}(p-1)^3$ for all prime p and integer $e \geq 1$. In particular, $f(2^e) = 2^{e-3}$, $f(3^e) = 8(3^{e-3})$, $f(5^e) = 64(5^{e-3})$, and for $p \geq 7$ and $e \geq 1$, $f(p^e) \geq 6^3(7^{-2}) = \frac{216}{49}$. Now let $p_1^{e_1} \cdots p_k^{e_k}$ be the prime factorization of n, with $p_1 < \ldots < p_k$.

If $k = 1$, then $(p_1, e_1) = (2, 3)$, so $n = 8$.

Now consider $k = 2$. If $p_1 \geq 3$, then $f(n) \geq \frac{8}{9} \cdot \frac{64}{25} > 1$. So $p_1 = 2$. If $e_1 \geq 4$, then $f(n) \geq 2 \cdot \frac{8}{9} > 1$. If $e_1 \in \{2, 3\}$, then (p_2, e_2) can only be $(3, 1)$, that is $n \in \{12, 24\}$. If $e_1 = 1$, then $(p_2, e_2) \in \{(3, 1), (3, 2), (5, 1)\}$, that is, $n \in \{6, 18, 10\}$.

Next consider $k = 3$. Again we must have $p_1 = 2$. If $e_1 \geq 2$, then $f(n) \geq \frac{1}{2} \cdot \frac{8}{9} \cdot \frac{64}{25} > 1$. So $e_1 = 1$. If $p_2 \geq 5$, then $f(n) \geq \frac{1}{4} \cdot \frac{64}{25} \cdot \frac{216}{49} > 1$. So $p_2 = 3$. If $e_2 \geq 2$, then $f(n) \geq \frac{1}{4} \cdot \frac{8}{3} \cdot \frac{64}{25} > 1$. So $e_2 = 1$. Then we must have $(p_3, e_3) = (5, 1)$ or $(p_3, e_3) = (7, 1)$, that is, $n = 30, 42$. The reason is because if $p_3 \geq 11$ then $f(n) \geq \frac{2000}{9} \cdot 11^{e_3 - 1} > 1$.

Finally, if $k \geq 4$, then

$$f(n) \geq \frac{1}{4} \cdot \frac{8}{9} \cdot \frac{64}{25} \cdot \frac{216}{49} > 1.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Simon Pellicer, Paris, France; Sardor Bazarbaev, National University of Uzbekistan, Uzbekistan; Albert Stadler, Herrliberg, Switzerland; Joel Schlosberg, Bayside, NY, USA; José Hernández Santiago, Morelia, Michoacán, Mexico.

U392. Let $f(x) = x^4 + 3x^3 + ax^2 + bx + c$ be a polynomial with real coefficients which has four real roots in the interval $(-1, 1)$. Prove that

$$(1 - a + c)^2 + (3 - b)^2 \geq \left(\frac{5}{4}\right)^8.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Mircea Becheanu, University of Bucharest, Romania

Let x_1, x_2, x_3, x_4 be the roots of $f(x)$. Then we have

$$f(i) = 1 - 3i - a + bi + c = (i - x_1)(i - x_2)(i - x_3)(i - x_4)$$

and

$$|f(i)|^2 = (1 - a + c)^2 + (3 - b)^2 = (1 + x_1^2)(1 + x_2^2)(1 + x_3^2)(1 + x_4^2).$$

Now, we have to show that

$$(1 + x_1^2)(1 + x_2^2)(1 + x_3^2)(1 + x_4^2) \geq \left(\frac{5}{4}\right)^8.$$

This is equivalent to

$$\ln(1 + x_1^2) + \ln(1 + x_2^2) + \ln(1 + x_3^2) + \ln(1 + x_4^2) \geq 8 \ln \frac{5}{4}.$$

Now, using the fact that the function $g(x) = \ln(1 + x^2)$ is convex on the interval $(-1, 1)$ and the sum $x_1 + x_2 + x_3 + x_4 = -3$, the result comes from Jensen inequality.

Also solved by Daniel Lasaosa, Pamplona, Spain, Li Zhou, Polk State College, Winter Haven, FL, USA; Sardor Bazarbaev, National University of Uzbekistan, Uzbekistan; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland.

U393. Evaluate

$$\lim_{x\to 0}\frac{\cos x\sqrt{\cos 2x}\cdots\sqrt[n]{\cos nx}-1}{\cos x\cos 2x\cdots\cos nx-1}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

First of all,

$1-\cos x\cos 2x\cdot\ldots\cdot\cos nx=\sum_{m=0}^{n-1}\left(\prod_{k=0}^{m}\cos kx\right)\cdot\left(1-\cos(m+1)x\right).$

Then,

$$\begin{aligned}&1-\cos x\cdot\sqrt{\cos 2x}\cdot\ldots\cdot\sqrt[n]{\cos nx}\\&=\sum_{m=0}^{n-1}\left(\prod_{k=0}^{m}\sqrt[k]{\cos kx}\right)\left(1-\sqrt[m+1]{\cos(m+1)x}\right)\\&=\sum_{m=0}^{n-1}\left(\prod_{k=0}^{m}\sqrt[k]{\cos kx}\right)\cdot\frac{\left(1-\cos(m+1)x\right)}{\sum_{j=0}^{m}\sqrt[m+1]{(\cos(m+1)x)^j}}.\end{aligned}$$

$$\lim_{x\to 0}\frac{1-\cos(m+1)x}{x^2}=\lim_{x\to 0}\left(\frac{\sin\frac{m+1}{2}x}{\frac{m+1}{2}x}\right)^2\cdot\frac{(m+1)^2}{2}=\frac{(m+1)^2}{2}.$$

and

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x\cdot\sqrt{\cos 2x}\cdot\ldots\cdot\sqrt[n]{\cos nx}-1}{\cos x\cdot\cos 2x\cdot\ldots\cdot\cos nx-1}&=\frac{\sum_{m=0}^{n-1}\frac{(m+1)^2}{2}\cdot\frac{1}{m+1}}{\sum_{m=0}^{n-1}\frac{(m+1)^2}{2}}\\&=\frac{\frac{n(n+1)}{2}}{\frac{n(n+1)(2n+1)}{6}}=\frac{3}{2n+1}.\end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Pedro Acosta, West Morris Mendham High School, Mendham, NJ, USA; Li Zhou, Polk State College, Winter Haven, FL, USA; Sardor Bazarbaev, National University of Uzbekistan, Uzbekistan; Adam Krause, student, SUNY Brockport, NY, USA; Albert Stadler, Herrliberg, Switzerland; Arkady Alt, San Jose, CA, USA; Joel Schlosberg, Bayside, NY, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Vasile Giurgi, Ovidiu Pop, Colegiul National Dragos-Voda, Sighetu Marmatiei, Romania; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Zafar Ahmed, BARC, Mumbai, India.

U394. Let $x_0 > 1$ be an integer and define $x_{n+1} = d^2\left(x_n\right)$, where $d(k)$ denotes the number of positive divisors of k. Prove that

$$\lim_{n\to\infty} x_n = 9.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Li Zhou, Polk State College, Winter Haven, FL, USA

Since $x_0 > 1$, $x_1 = y^2$ for some integer $y \geq 2$. Let $p_1^{e_1}\cdots p_m^{e_m}$ be the prime factorization of y. Then $x_2 = d^2(y^2) = (2e_1+1)^2\cdots(2e_m+1)^2$. Now we claim that for any odd prime p and $e \geq 1$, $p^{2e} \geq (2e+1)^2$, with equality if and only if $p = 3$ and $e = 1$. Clearly, $p^2 \geq 3^2$, with equality if only if $p = 3$. As an induction hypothesis, assume that the claim is true for some $e \geq 1$. Then

$$p^{2(e+1)} = p^2(p^{2e}) \geq p^2(2e+1)^2 = (2pe+p)^2 > (2e+3)^2,$$

completing the induction and establishing the claim. Therefore, $x_n \geq x_{n+1}$ for all $n \geq 2$, with equality if and only if $x_n = 3^2$. Since the sequence $\langle x_n\rangle_{n\geq 2}$ cannot strictly decreases forever, there exists $N \geq 2$ such that $x_2 > x_3 > \cdots > x_N$ and $x_N = 3^2$. Finally, $d^2(3^2) = 9$, completing the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Joel Schlosberg, Bayside, NY, USA.

U395. Evaluate

$$\int \frac{x^2+6}{(x\cos x - 3\sin x)^2}dx.$$

Proposed by Abdelouahed Hamdi, Doha, Qatar

Solution by Li Zhou, Polk State College, USA

Noticing

$$(x\cos x - 3\sin x)^2 = \left(x^2+9\right)\cos^2\left(x + \arctan\frac{3}{x}\right),$$

we let $u = x + \arctan\frac{3}{x}$. Then

$$du = \left(1 + \frac{-3}{x^2+9}\right)dx = \frac{x^2+6}{x^2+9}dx,$$

and

$$\begin{aligned}\int \frac{x^2+6}{(x\cos x - 3\sin x)^2}dx &= \int \frac{1}{\cos^2 u}du = \tan u + C = \frac{\tan x + \frac{3}{x}}{1 - \frac{3}{x}\tan x} + C \\ &= \frac{x\sin x + 3\cos x}{x\cos x - 3\sin x} + C.\end{aligned}$$

Alternative solution by Daniel Lasaosa, Pamplona, Spain;

Denote $f(x) = 3\cos x + x\sin x$ and $g(x) = x\cos x - 3\sin x$. Note that $f'(x) = -2\sin x + x\cos x$ and $g'(x) = -x\sin x - 2\cos x$, or

$$f'(x)g(x) - f(x)g'(x) =$$

$$= \left(x^2\cos^2 x + 6\sin^2 x - 5x\sin x\cos x\right) + \left(6\cos^2 x + x^2\sin^2 x + 5x\sin x\cos x\right) =$$

$$= \left(x^2+6\right)\left(\cos^2 x + \sin^2 x\right) = x^2+6,$$

or

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} = \frac{x^2+6}{(x\cos x - 3\sin x)^2},$$

for

$$\int \frac{x^2+6}{(x\cos x - 3\sin x)^2}dx = \frac{f(x)}{g(x)} + C = \frac{3\cos x + x\sin x}{x\cos x - 3\sin x} + C,$$

where C is an integration constant, and we are done.

Also solved by Albert Stadler, Herrliberg, Switzerland; Pedro Acosta, West Morris Mendham High School, Mendham, NJ, USA; Aditya Ghosh, Kolkata,West Bengal, India; Alessandro Ventullo, Milan, Italy; Alok Kumar, New Delhi, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Joel Schlosberg, Bayside, NY, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Ovidiu Pop, Colegiul National Dragos-Voda, Sighetu Marmatiei, Romania; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Zafar Ahmed, BARC, Mumbai, India.

U396. Let S_8 be the symmetric group of permutations of an 8-element set and let k be the number of its abelian subgroups of order 16. Prove that $k \geq 1050$.

Proposed by Mircea Becheanu, University of Bucharest, România

Solution by Li Zhou, Polk State College, Winter Haven, FL, USA

Let $a_1, a_2, \ldots, a_8$ be a permutation of $1, 2, \ldots, 8$. We count three types of abelian subgroups of S_8.

Type I: Let $a = (a_1 a_2)$, $b = (a_3 a_4)$, $c = (a_5 a_6)$, and $d = (a_7 a_8)$. Then $a^2 = b^2 = c^2 = d^2 = e$ and a, b, c, d pairwise commute. Let $G = \{a^i b^j c^m d^n : 0 \leq i, j, m, n \leq 1\}$. Then G is an abelian subgroup of order 16. The number of such subgroups (all isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$) is the number of ways to partition $1, 2, \ldots, 8$ into four 2-cycles: $\frac{8!}{2^4 4!} = 105$.

Type II: Let $a = (a_1 a_2 a_3 a_4)$, $b = (a_5 a_6)$, and $c = (a_7 a_8)$. Then $a^4 = b^2 = c^2 = e$ and a, b, c pairwise commute. Let $G = \{a^i b^m c^n : 0 \leq i \leq 3, 0 \leq m, n \leq 1\}$. Then G is an abelian subgroup of order 16. Note that $a^{-1} = (a_1 a_4 a_3 a_2) \neq a$ and $\{a^{-1}, b, c\}$ generates the same group G. So the number of such subgroups (all isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$) is $\frac{1}{2}$ of the number of ways to partition $1, 2, \ldots, 8$ into one 4-cycle and two 2-cycles: $\frac{1}{2}\left(\frac{8!}{4 \cdot 2^2 2!}\right) = 630$.

Type III: Let $a = (a_1 a_2 a_3 a_4)$ and $b = (a_5 a_6 a_7 a_8)$. Then $a^4 = b^4 = e$ and $ab = ba$. Let $G = \{a^i b^j : 0 \leq i, j \leq 3\}$. Then G is an abelian subgroup of order 16. Note again that G can also be generated by $\{a^{-1}, b\}$, $\{a, b^{-1}\}$, or $\{a^{-1}, b^{-1}\}$. So the number of such subgroups (all isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$) is $\frac{1}{4}$ of the number of ways to partition $1, 2, \ldots, 8$ into two 4-cycles: $\frac{1}{4}\left(\frac{8!}{4^2 2!}\right) = 315$.

Therefore, $k \geq 105 + 630 + 315 = 1050$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland.

O391. Find all 4-tuples (x, y, z, w) of positive integers such that

$$(xy)^3 + (yz)^3 + (zw)^3 - 252yz = 2016.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

The given equation can be written as

$$(xy)^3 + (zw)^3 + yz(y^2z^2 - 252) = 2016.$$

Observe that it must be $yz(y^2z^2 - 252) \leq 2016$, so $yz \leq 18$. It follows that

$$\begin{aligned}(xy)^3 + (zw)^3 \quad \in \quad & \{2267, 2512, 2745, 2960, 3151, 3312, 3437, 3520, 3555, \\ & 3536, 3457, 3312, 3095, 2800, 2421, 1952, 1387, 720\}.\end{aligned}$$

A case by case analysis gives $(xy)^3 + (zw)^3 \in \{2745, 2960\}$. If $(xy)^3 + (zw)^3 = 2745$, we get $xy = 1, zw = 14, yz = 3$ or $xy = 14, zw = 1, yz = 3$, which gives no solutions. If $(xy)^3 + (zw)^3 = 2960$, then $xy = 6, zw = 14, yz = 4$ or $xy = 14, zw = 6, yz = 4$. We get $(x, y, z, w) \in \{(3, 2, 2, 7), (7, 2, 2, 3)\}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland; Li Zhou, Polk State College, Winter Haven, FL, USA.

O392. Let ABC be a triangle with area Δ. Prove that

$$\frac{1}{3r^2} \geq \frac{1}{r_a h_a} + \frac{1}{r_b h_b} + \frac{1}{r_c h_c} \geq \frac{\sqrt{3}}{\Delta}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy

First of all,

$$r_a h_a = \frac{2s(s-b)(s-c)}{a}, \quad \Delta^2 = s(s-a)(s-b)(s-c)$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \qquad s = (a+b+c)/2$$

Let's change variables $a = y+z$, $b = x+z$, $c = x+y$, $x, y, z \geq 0$. The inequality becomes

$$\frac{x+y+z}{3xyz} \geq \frac{xy+yz+zx}{(x+y+z)xyz} \geq \frac{\sqrt{3}}{\sqrt{(x+y+z)xyz}}$$

At left we have $(x+y+z)^2 \geq 3(xy+yz+zx)$ which is a direct consequence of $(x-y)^2+(y-z)^2+(z-x)^2 \geq 0$.

At right we get squaring

$$(xy+yz+zx)^2 \geq 3(x+y+z)xyz \iff (xy)^2 + (yz)^2 + (zx)^2 \geq xyz(x+y+z)$$

and this follows by

$$((xy)^2 + (yz)^2)/2 \geq x^2yz$$

and cyclic. This concludes the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Pedro Acosta, West Morris Mendham High School, Mendham, NJ, USA; Aditya Ghosh, Kolkata,West Bengal, India; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Sardor Bazarbaev, National University of Uzbekistan, Uzbekistan; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Arpon Basu, AECS-4, Mumbai, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Nicusor Zlotaâš Traian Vuia Technical College, Focsani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Albert Stadler, Herrliberg, Switzerland; Telemachus Baltsavias, Keramies Junior High School, Kefalonia, Greece; Li Zhou, Polk State College, Winter Haven, FL, USA.

O393. Let a, b, c, d be nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$. Prove that

$$\frac{1}{5-\sqrt{ab}}+\frac{1}{5-\sqrt{bc}}+\frac{1}{5-\sqrt{cd}}+\frac{1}{5-\sqrt{da}}\le 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Using AM-GM inequality we get,

$$\begin{aligned}4=a^2+b^2+c^2+d^2 \ \Rightarrow\ 6&=(1+1+a^2+b^2)+(c^2+d^2)\\ &\ge 4\cdot\sqrt[4]{1\cdot 1\cdot a^2\cdot b^2}+(c^2+d^2)\\ &=4\sqrt{ab}+(c^2+d^2)\ge 4\sqrt{ab}.\end{aligned}$$

Hence we have,

$$\sqrt{ab}\le\frac{3}{2} \qquad (1)$$

Using (1) we get

$$\begin{aligned}\frac{1}{5-\sqrt{ab}}&=\frac{1}{4}+\frac{1}{16}(\sqrt{ab}-1)+\frac{(\sqrt{ab}-1)^2}{16(5-\sqrt{ab})}\\ &\le\frac{1}{4}+\frac{1}{16}(\sqrt{ab}-1)+\frac{(\sqrt{ab}-1)^2}{16\left(5-\frac{3}{2}\right)}\\ &=\frac{1}{112}(2\cdot ab+3\sqrt{ab}+23).\end{aligned}$$

Similarly,

$$\frac{1}{5-\sqrt{bc}}\le\frac{1}{112}(2\cdot bc+3\sqrt{bc}+23) \qquad (2)$$

$$\frac{1}{5-\sqrt{cd}}\le\frac{1}{112}(2\cdot cd+3\sqrt{cd}+23) \qquad (3)$$

$$\frac{1}{5-\sqrt{da}}\le\frac{1}{112}(2\cdot da+3\sqrt{da}+23) \qquad (4)$$

Adding $(1),(2),(3),(4)$ and using AM-GM inequality we get,

$$\begin{aligned}&\frac{1}{5-\sqrt{ab}}+\frac{1}{5-\sqrt{bc}}+\frac{1}{5-\sqrt{cd}}+\frac{1}{5-\sqrt{da}}\\ &\le\frac{1}{112}\left(2(ab+bc+cd+da)+3(\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da})+92\right)\\ &\le\frac{1}{112}\left((a^2+b^2)+(b^2+c^2)+(c^2+d^2)+(d^2+a^2)+3\left(\sum\sqrt[4]{a^2b^2\cdot 1\cdot 1}\right)+92\right)\\ &\le\frac{1}{112}\left(8+\frac{3}{4}\left(2a^2+2b^2+2c^2+2d^2+8\right)+92\right)=1.\end{aligned}$$

Equality holds only if $a=b=c=d=1$.

Alternative solution by Daniel Lasaosa, Pamplona, Spain

By symmetry in the problem, it suffices to show that

$$\frac{1}{5-\sqrt{ab}}+\frac{1}{5-\sqrt{cd}}\le\frac{1}{2}.$$

After multiplying by the product of denominators (clearly positive since $ab, cd \leq \frac{a^2+b^2+c^2+d^2}{2} = 2$) and rearranging terms, this inequality is equivalent to

$$5 - 3\sqrt{ab} - 3\sqrt{cd} + \sqrt{abcd} \geq 0.$$

Now, using the AM-GM and the AM-QM inequalities, we have

$$\sqrt{ab} + \sqrt{cd} \leq \frac{a+b+c+d}{2} \leq 2\sqrt{\frac{a^2+b^2+c^2+d^2}{4}} = 2,$$

or we can define $\delta = 2 - \sqrt{ab} - \sqrt{cd}$, and

$$\sqrt{abcd} = 2 - 2\delta + \frac{\delta^2 - ab - cd}{2},$$

or it suffices to show that

$$2 + 2\delta + \delta^2 - ab - cd \geq 0,$$

clearly true since, by the AM-GM inequality, we have

$$ab + cd \leq \frac{a^2+b^2+c^2+d^2}{2} = 2,$$

and $2\delta + \delta^2 \geq 0$. Note that equality requires $\delta = 0$, and simultaneously $a = b$ and $c = d$, which results in $a = b = c = d$.

The conclusion follows, equality holds iff $a = b = c = d = 1$, in which case all terms in the LHS are $\frac{1}{4}$.

Also solved by Joehyun Kim, Bergen County Academies, Hackensack, NJ, USA; Sardor Bazarbaev, National University of Uzbekistan, Uzbekistan; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Albert Stadler, Herrliberg, Switzerland.

O394. Let a, b, c be positive real numbers with $a+b+c=3$. Prove that

$$\frac{1}{(b+2c)^a}+\frac{1}{(c+2a)^b}+\frac{1}{(a+2b)^c}\geq 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that we may write

$$\frac{1}{(b+2c)^a}=\exp\left(-a\ln(b+2c)\right),$$

or since $\exp(x)$ is a strictly convex function, we have

$$\frac{1}{(b+2c)^a}+\frac{1}{(c+2a)^b}+\frac{1}{(a+2b)^c}\geq 3\exp\left(-\frac{a\ln(b+2c)+b\ln(c+2a)+c\ln(a+2b)}{3}\right),$$

with equality iff $a\ln(b+2c)=b\ln(c+2a)=c\ln(a+2b)$, or since $\exp(x)$ is a strictly increasing function, it suffices to show that

$$a\ln(b+2c)+b\ln(c+2a)+c\ln(a+2b)\leq 3\ln(3).$$

Now, since $\ln(x)$ is a concave function, we have

$$a\ln(b+2c)+b\ln(c+2a)+c\ln(a+2b)\leq (a+b+c)\ln\left(\frac{a(b+2c)+b(c+2a)+c(a+2b)}{a+b+c}\right)=$$

$$=3\ln(ab+bc+ca),$$

and since $\ln(x)$ is a strictly increasing function, it finally suffices to show that $ab+bc+ca\leq 3$, or equivalently, that $3(ab+bc+ca)\leq 9=a^2+b^2+c^2+2(ab+bc+ca)$, ie that $a^2+b^2+c^2\geq ab+bc+ca$, clearly true by the scalar product inequality, and where equality holds iff $a=b=c=1$. The conclusion follows, substitution shows that the necessary condition $a=b=c=1$ is also sufficient.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nermin Hodžić, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Albert Stadler, Herrliberg, Switzerland; Li Zhou, Polk State College, Winter Haven, FL, USA; Zafar Ahmed, BARC, Mumbai, India.

O395. Let a, b, c, d be nonnegative real numbers such that $ab + bc + cd + da + ac + bd = 6$. Prove that

$$a^4 + b^4 + c^4 + d^4 + 8abcd \geq 12.$$

Proposed by Marius Stănean, Zalău, România

Solution by Daniel Lasaosa, Pamplona, Spain

Note that it suffices to show that

$$(ab + bc + cd + da + ac + bd)^2 = 36 \leq 3\left(a^4 + b^4 + c^4 + d^4\right) + 24abcd.$$

Define

$$f(a, b, c, d) = 3\left(a^4 + b^4 + c^4 + d^4\right) + 24abcd - (ab + bc + cd + da + ac + bd)^2,$$

and since we may assume wlog by symmetry in the problem that $a \geq b \geq c \geq d$, it suffices to show that $f(a, b, c, d) \geq 0$ when $a \geq b \geq c \geq d \geq 0$. Note that we may define $x = a - c$ and $y = b - c$, such that $x \geq y \geq 0$, and with this notation,

$$f(a, b, c, d) - f(c, c, c, d) = 6cd(c - d)(x + y) + 5c(c - d)\left(x^2 + y^2\right) + 9c^2(x - y)^2 +$$

$$+8cdxy + d(c - d)(x + y)^2 + 6c\left(x^3 + y^3\right) + 6c(x + y)(x - y)^2 + 2(c - d)xy(x + y) +$$

$$+3\left(x^2 - y^2\right)^2 + 5x^2y^2.$$

Since $c - d \geq 0$, all terms in the RHS are non-negative. The last term being zero requires $x^2y^2 = 0$, whereas the previous one requires $x^2 = y^2$. Since setting $x = y = 0$ makes the RHS zero regardless of the values of c and d, we conclude that $f(a, b, c, d) \geq f(c, c, c, d)$, with equality iff $a = b = c$. Note finally that

$$f(c, c, c, d) = 6c^3d - 9c^2d^2 + 3d^4 = 3d(2c + d)(c - d)^2,$$

clearly non-negative, being zero iff either $c = d$, or $d = 0$, or $c = d = 0$. This last option results in $ab+bc+cd+da+ac+bd = 0$, absurd. On the other hand, $d = 0$ results in $ab+bc+cd+da+ac+bd = 3c^2 = 6$ for $a = b = c = \sqrt{2}$ and $d = 0$, whereas $c = d$ results in $ab+bc+cd+da+ac+bd = 6c^2 = 6$ for $a = b = c = d = 1$. Substitution of these two sets of values results indeed in equality in the proposed inequality. The conclusion follows, equality holds iff either $a = b = c = d = 1$ or (a, b, c, d) is a permutation of $\left(\sqrt{2}, \sqrt{2}, \sqrt{2}, 0\right)$.

Also solved by Arkady Alt, San Jose, CA, USA; Sardor Bazarbaev, National University of Uzbekistan, Uzbekistan; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Roma, Italy; Albert Stadler, Herrliberg, Switzerland.

O396. Find all polynomials $P(X)$ with positive integer coefficients having the following property: for any positive integer n and every prime p such that n is a quadratic residue modulo p, $P(n)$ is also a quadratic residue modulo p.

Proposed by Vlad Matei, University of Wisconsin, Madison, USA

Solution by Li Zhou, Polk State College, Winter Haven, FL, USA

We show that the solutions are $P(X) = Q(X)^2$ or $XQ(X)^2$, where $Q(X) \in \mathbb{Z}[X]$ and $Q(X)^2$ has positive integer coefficients.

First, if n is a quadratic residue modulo p, then $n \equiv m^2 \pmod{p}$, so both $Q(n)^2$ and $nQ(n)^2 \equiv m^2Q(n)^2$ are quadratic residues modulo p.

To show that these polynomials are the only solutions, we use two known theorems.

Theorem 1. If a is a nonsquare integer, then there are infinitely many primes p for which a is a quadratic nonresidue. (See K. Ireland & M. Rosen, *A Classical Intro. to Modern Number Theory*, 2nd ed., Springer, 1990, p.57.)

Theorem 2. If $f(X) \in \mathbb{Z}[X]$ such that $\sqrt{f(n)} \in \mathbb{Z}$ for all positive integers n, then there exists a polynomial $g(X) \in \mathbb{Z}[X]$ such that $f = g^2$. (See T. Andreescu & G. Dospinescu, *Problems from the Book*, 2nd ed., XYZ Press, 2010, p.224.)

Now suppose that $P(X) = a_dX^d + \cdots + a_1X + a_0$ has the property. Since any $n = m^2$ is a quadratic residue modulo all primes p, Theorem 1 implies that $f(m) = P(m^2)$ must be perfect squares for all positive integers m. Then Theorem 2 implies that $P(X^2) = f(X) = g(X)^2$ for some $g(X) \in \mathbb{Z}[X]$. Write $g(X) = b_dX^d + \cdots + b_1X + b_0$. Comparing coefficients in $P(X^2) = g(X)^2$, we see that if $d = 2k$, then $b_{d-1} = b_{d-3} = \cdots = b_1 = 0$ and

$$P(X) = g\left(\sqrt{X}\right)^2 = \left(b_{2k}X^k + b_{2k-2}X^{k-1} + \cdots + b_2X + b_0\right)^2.$$

Similarly, if $d = 2k+1$, then $b_{d-1} = b_{d-3} = \cdots = b_0 = 0$ and

$$P(X) = X\left(b_{2k+1}X^k + b_{2k-1}X^{k-1} + \cdots + b_3X + b_1\right)^2.$$

The proof is now complete.

Also solved by Navid Safaei, Sharif University of Technlogy, Tehran, Iran; Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland.