A Special Point on the Median

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We discuss the properties of the $HM$ point (which remains nameless until now), which seems to be well-known among the community. In addition, we provide some problems where properties of the point prove useful to find solutions.

1 Power of the Point

It is important to note that the $HM$ point is not symmetric about all three vertices for a given triangle; rather, given each vertex, there is a corresponding $HM$ point. As there are many existing definitions of our point, we will simply present characterizations, rather than giving a direct definition. It is left to the reader to choose which definition they like best.

**Characterization 1.1.** In $\triangle ABC$ with orthocenter $H$, the circle with diameter $\overline{AH}$ and $\odot(BHC)$ intersect again on the $A$-median at a point $X_A$.

![Diagram of Characterization 1.1](image)

**Proof 1.** Let $A'$ be the point such that $ABA'C$ is a parallelogram. Since $\angle BAC = \angle BAC' = \pi - \angle BHC$, $A'$ lies on $\odot(BHC)$. Then $\angle HBA' = \angle HBC + \angle A'BC = \frac{\pi}{2} - \angle ACB + \angle ACB = \frac{\pi}{2}$, so $A'$ is the antipode of $H$. Now we have that $\angle AX_AH = \frac{\pi}{2} = \pi - \angle HX_AA'$, implying that $X_A$ lies on the $A$-median.

Alternatively, consider the following proof using inversion.

**Proof 2.** Peform an inversion at $A$ with power $r^2 = AH \cdot AD$. By Power of a Point, $AH \cdot AD = AF \cdot AB = AE \cdot AC$, which implies that $\{F, B\}$, $\{H, D\}$ and $\{E, C\}$ swap under this inversion. Since the circle with diameter $AH$ maps to line $BC$ and $\odot(BHC)$ maps to the nine-point circle, these two objects intersect at $M$, so the image of $M$ lies on the $A$-median as well.

Here, $X_A$ is our $HM$ point with respect to vertex $A$. **Characterization 1.1** is the most common characterization of the $HM$ point, and now we will present others.
**Characterization 1.2.** Let $\omega_B$ be the circle through $A$ and $B$ tangent to line $BC$, and define $\omega_C$ similarly. Then $\omega_B$ and $\omega_C$ intersect again at $X_A$.

![Diagram showing circles and points](image)

**Proof.** Define $X'_A$ as the intersection of the two circles. Since $\angle AX'_A B = \pi - \angle CBA$ and $\angle AX'_A C = \pi - \angle ACB$, we get that

$$\angle BX'_A C = 2\pi - \angle AX'_A B - \angle AX'_A C = \angle ACB + \angle CBA = \pi - \angle BAC$$

whence $X'_A$ lies on $\odot(BHC)$. Since $X'_A$ lies on the $A$-median by the radical axis theorem, $X'_A \equiv X_A$. \qed

**Characterization 1.3.** $X_A$ is the isogonal conjugate of the midpoint of the $A$-symmedian chord in $\odot(ABC)$.

**Proof.** Let $P$ be the midpoint of the $A$-symmedian chord; it is well-known that $P$ is the spiral center sending segment $BA$ to segment $AC$. Then

$$\angle PBA = \angle PAC = \angle BAX_A = \angle CBX_A$$

where the last angle equality follows from **Characterization 1.2**. Since clearly $\angle BAP = \angle CAX_A$, we have proven that $P$ and $X_A$ are isogonal conjugates. \qed
Characterization 1.4. Suppose the $A$-symmedian intersects $\odot(ABC)$ again at a point $K$. Then $X_A$ is the reflection of $K$ over line $BC$.

Proof. Let $K'$ be the reflection of $X_A$ over the midpoint of $BC$. Note that $K' \in \odot(ABC)$, since $\odot(BHC)$ is the reflection of $\odot(ABC)$ about $M$; additionally, $\frac{BX_A}{CX_A} = \frac{CK'}{BK'}$. Let $P_{\infty,BC}$ denote the point at infinity for line $BC$, then

$$-1 = (B,C; M, P_{\infty,BC}) \overset{A}{=} (B,C; K', A) \implies \frac{BK'}{CK'} = \frac{AC}{AB}$$

This implies that the reflection of $X_A$ over $BC$ forms a harmonic bundle with $A, B, C$, so it is precisely the point $K$.

Corollary 1.0.1. $X_A$ lies on the $A$-Apollonius circle.

Characterization 1.5 (ELMO SL 2013 G3). In $\triangle ABC$, a point $D$ lies on line $BC$. The circumcircle of $ABD$ meets $AC$ at $F$ (other than $A$), and the circumcircle of $ADC$ meets $AB$ at $E$ (other than $A$). Prove that as $D$ varies, the circumcircle of $AEF$ always passes through a fixed point other than $A$, and that this point lies on the median from $A$ to $BC$.

Proof. Consider an inversion about $A$ of power $r^2 = AB \cdot AC$ followed by a reflection in the bisector of angle $A$. Then points $B$ and $C$ are interchanged and $D$ is sent on the circumcircle of triangle $ABC$. Now, $E' = B'D' \cap AC'$ and $F = C'D' \cap AB'$. Also, the $HM$ point opposite to $A$ is mapped to the intersection of tangents at $B', C'$ and so by applying Pascal’s Theorem on the cyclic hexagon $AB'B'D'C'C'$ we get that the images of $E, F$ and the $HM$ point are collinear meaning that originally $\odot(AEF)$ passes through the aforementioned point. This proves our characterization.

Characterization 1.6. Suppose the $A$-symmedian intersects $BC$ at $D$. The perpendicular from $D$ to line $BC$ intersects the $A$-median at $S$, and the line through $S$ parallel to $BC$ intersects $AB$ and $AC$ at $F$ and $E$, respectively. Then $X_A \equiv BE \cap CF$. 

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Proof. We will prove the reverse implication of the problem, by starting with $X_A$ and ending with the $A$-symmedian. First notice that $\angle FX_AE = \angle BX_AC = \pi - \angle A = \pi - \angle FAE$, so quadrilateral $AFX_AE$ is cyclic. Then $\angle FEX_A = \angle FAX_A = \angle BAX_A = \angle CBA$ by Characterization 2, so $FE \parallel BC$, and accordingly $S$ lies on the $A$-median.

By Brokard’s Theorem on $\odot(AFX_AE)$, it follows that line $BC$ is the polar of $S$ with respect to the circle. Therefore, the tangents to the circle at $F$ and $E$ intersect on $BC$ at $D$, and since $AD$ is an $A$-symmedian in $\triangle AFE$, it follows that $AD$ is also an $A$-symmedian in $\triangle ABC$, and we’re done. \qed

**Corollary 1.0.2.** The orthocenter of $\triangle DEF$, $X_A$, and the circumcenter of $\triangle ABC$ are collinear.

**Proof.** Let the tangents to $\odot(BX_AC)$ at $B$ and $C$ intersect at $D'$; the problem is equivalent to showing that $O$ is the orthocenter of $\triangle BD'C$. Since $D'$ clearly lies on the perpendicular bisector of $BC$, we just need that $\angle BOC = \pi - \angle BD'C$. But it is clear that both sides are equal to twice of angle $A$, so the corollary follows. \qed
2 Example Problems

In this section we will see the HM point appearing in a wide variety of problems. Notice that the HM point may not be central to the problem, but it is an important step towards obtaining the desired conclusion.

**Example 1** (ELMO 2014/5). Let $ABC$ be a triangle with circumcenter $O$ and orthocenter $H$. Let $\omega_1$ and $\omega_2$ denote the circumcircles of triangles $BOC$ and $BHC$, respectively. Suppose the circle with diameter $AO$ intersects $\omega_1$ again at $M$, and line $AM$ intersects $\omega_1$ again at $X$. Similarly, suppose the circle with diameter $AH$ intersects $\omega_2$ again at $N$, and line $AN$ intersects $\omega_2$ again at $Y$. Prove that lines $MN$ and $XY$ are parallel.

**Solution.** It is clear by angle chasing that $M$ is the midpoint of the symmedian chord through $A$, $N$ is the HM point of triangle $ABC$ opposite to $A$, $X$ is the intersection of the tangents to $(ABC)$ at $B,C$ and $Y$ is the point such that $ABYC$ is a parallelogram. Inverting about $A$ with power $r^2 = AB \cdot AC$ and reflecting in the $A$-angle bisector, we see that $\{M,Y\}$ and $\{N,X\}$ are swapped. Therefore,

$$AB \cdot AC = AM \cdot AY = AX \cdot AN$$

which gives the conclusion.

**Example 2** (USA TST 2005/6). Let $ABC$ be an acute scalene triangle with $O$ as its circumcenter. Point $P$ lies inside triangle $ABC$ with $\angle PAB = \angle PBC$ and $\angle PAC = \angle PCB$. Point $Q$ lies on line $BC$ with $QA = QP$. Prove that $\angle AQP = 2\angle OQB$. 
Proof. Assume without loss of generality, $AB < AC$. Let $M$ be the midpoint of $BC$. From the angle conditions, it is clear by tangency that $P$ is the HM point opposite to $A$ in triangle $ABC$. Therefore, we conclude that $P$ lies on the $A$-Apollonius circle (the circle with diameter $DK$), where $D, K$ are the feet of the internal and external bisectors of angle $BAC$. Therefore, $QA$ is tangent to the circle $\odot(ABC)$.

We have $\angle AQP = 2\angle AKP$ since $Q$ is the center of $(DK)$. Let the tangents to $(ABC)$ at $B, C$ meet at $X$ and $AK$ meet $\odot(ABC)$ again at $Y$. Let $F$ be the midpoint of the $A$ symmedian chord. Then $F$ lies on $OQ$ and $\angle AFO = \frac{\pi}{2}$.

As an inversion about $A$ with power $r^2 = AB \cdot AC$ composed with a reflection about the $A$-angle bisector sends $P$ to $X$ and $K$ to $Y$, we have $\angle AKP = \angle AXY = \angle FXM$. Since $\angle QFX = \angle QMX = 90^\circ$, points $Q, F, M, X$ lie on a circle. We have

$$\angle OQB = \angle OQM = \angle FQM = \angle FXM = \angle AKY,$$

establishing the desired result. \qed
Example 3 (Brazil National Olympiad 2015/6). Let \( \triangle ABC \) be a scalene triangle and \( X, Y \) and \( Z \) be points on the lines \( BC, AC \) and \( AB \), respectively, such that \( \angle AXB = \angle BYC = \angle CZA \). The circumcircles of \( BXZ \) and \( CXY \) intersect at \( P \). Prove that \( P \) lies on the circle with diameter \( HG \) where \( H \) and \( G \) are the orthocenter and the centroid, respectively, of triangle \( ABC \).

Solution. Note that \( \angle BXA + \angle BZC = \pi \). It follows that the intersection of lines \( AX \) and \( CZ \) lies on the circle \( \odot(BXZ) \). We conclude that \( \odot(BXZ) \) passes through the HM point \( X_B \) opposite to \( B \). Similarly the circle \( \odot(CXY) \) passes through the HM point \( X_C \) opposite \( C \). In triangle \( BGC \), the circles \( \odot(BXX_B) \) and \( \odot(CXX_C) \) meet at \( P \neq X \) where \( X_B, X_C \) lie on the lines \( BG \) and \( CG \), respectively. By Miquel’s Theorem, it follows that \( P \) lies on the circumcircle of triangle \( GX_BX_C \). As \( \angle HX_BG = \angle HX_CG = \frac{\pi}{2} \), we conclude that \( \angle HPG = \frac{\pi}{2} \). \( \square \)

Example 4 (Sharygin Geometry Olympiad 2015). Let \( A_1, B_1, C_1 \) be the midpoints of the sides opposite \( A, B, C \) in triangle \( ABC \). Let \( B_2 \) and \( C_2 \) denote the midpoints of segments \( BA_1 \) and \( CA_1 \), respectively. Let \( B_3 \) and \( C_3 \) denote the reflections of \( C_1 \) in \( B \) and \( B_1 \) in \( C \), respectively. Prove that the circumcircles of triangles \( BB_2B_3 \) and \( CC_2C_3 \) meet on the circumcircle of triangle \( ABC \).
Solution. Let $X$ be a point on the circle $\odot(ABC)$ such that $AX$ is a symmedian in triangle $ABC$. We will show that $X$ is a common point of circles $\odot(BB_2B_3)$ and $\odot(CC_2C_3)$. Clearly, the reflection of $X$ in $BC$ lies on the median $AA_1$. Thus,

$$\angle XA_1B = \angle AA_1B = \angle ACX = \angle XB_3B,$$

showing that the line $AB$ is tangent to the circle $\odot(BXA_1)$. The spiral similarity centered at $X$ which takes $CA_1$ to $AB$ will send $B_2$ to $B_3$ as they divide $CA_1$ and $AB$ in the same ratio of negative half. Hence, $X$ lies on $\odot(BB_2B_3)$ and repeating the argument for the circle $\odot(CC_2C_3)$ yields the result.

We end our discussion with a nice problem from a recent IMO ShortList. It is a perfect example of the properties of the HM point we have discussed!

**Example 5 (ISL 2014/G6).** Let $ABC$ be a fixed acute-angled triangle. Consider some points $E$ and $F$ lying on the sides $AC$ and $AB$, respectively, and let $M$ be the midpoint of $EF$. Let the perpendicular bisector of $EF$ intersect the line $BC$ at $K$, and let the perpendicular bisector of $MK$ intersect the lines $AC$ and $AB$ at $S$ and $T$, respectively. We call the pair $(E, F)$ interesting, if the quadrilateral $KSAT$ is cyclic. Suppose that the pairs $(E_1, F_1)$ and $(E_2, F_2)$ are interesting. Prove that

$$\frac{E_1E_2}{AB} = \frac{F_1F_2}{AC}.$$

Solution. Note that $EF \parallel ST$ implying that $M$ lies on the median from $A$ of triangle $AST$. As the reflection of $M$ in $ST$ lies on the circle $(AST)$, we conclude that $AK$ is a symmedian in triangle $AST$, and hence in triangle $AEF$ as well. Since $K$ lies on the perpendicular bisector of $EF$, we see that $KE, KF$ are tangents to the circle $\odot(AEF)$.

Let $BE$ meet $\odot(AEF)$ at $P \neq E$. Applying Pascal’s Theorem on the cyclic hexagon $(AEFPFF)$ we observe that $AE \cap PF, T$, and $B$, are collinear. This shows that $P$ lies on the line $CF$. Therefore, as $A, E, F,$ and the intersection of $BE$ and $CF$ lie on a circle, the circle $\odot(AEF)$ passes through the point $X_A$.

Let $B_1$ and $C_1$ be the feet of altitudes from $B, C$ to sides $AC, AB$ respectively. Evidently, $X_A$ lies on $(AB_1C_1)$. Thus, $X_A$ is the center of a spiral similarity sending $B_1C_1$ to $EF$. So, for any pair of interesting points $\{E_1, F_1\}$ and $\{E_2, F_2\}$, we have

$$\frac{E_1E_2}{F_1F_2} = \frac{B_1E_2 - B_1E_1}{C_1F_2 - C_1F_1} = \frac{B_1E}{C_1F} = \frac{X_AB_1}{X_AC_1} = \frac{B_1A}{C_1A} = \frac{AB}{AC},$$

as desired.
Note that in the following problems, the HM point may not be directly involved, but it provides appropriate motivation or intuition to solve the problem.

**Exercise 3.1** (USA TSTST 2015/2). Let $ABC$ be a scalene triangle. Let $K_a$, $L_a$ and $M_a$ be the respective intersections with $BC$ of the internal angle bisector, external angle bisector, and the median from $A$. The circumcircle of $AK_aL_a$ intersects $AM_a$ a second time at point $X_a$ different from $A$. Define $X_b$ and $X_c$ analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of $ABC$.

**Exercise 3.2** (WOOT 2013 Practice Olympiad 3/5). A semicircle has center $O$ and diameter $AB$. Let $M$ be a point on $AB$ extended past $B$. A line through $M$ intersects the semicircle at $C$ and $D$, so that $D$ is closer to $M$ than $C$. The circumcircles of triangles $AOC$ and $DOB$ intersect at $O$ and $K$. Show that $\angle MKO = 90^\circ$.

**Exercise 3.3** (IMO 2010/4, Modified). In $\triangle ABC$ with orthocenter $H$, suppose $P$ is the projection of $H$ onto the $C$-median; let the second intersections of $AP, BP, CP$ with $\odot(ABC)$ be $K, L, M$ respectively. Show that $MK = ML$.

**Exercise 3.4** (EGMO 2016/4). Two circles $\omega_1$ and $\omega_2$, of equal radius intersect at different points $X_1$ and $X_2$. Consider a circle $\omega$ externally tangent to $\omega_1$ at $T_1$ and internally tangent to $\omega_2$ at point $T_2$. Prove that lines $X_1T_1$ and $X_2T_2$ intersect at a point lying on $\omega$.

**Exercise 3.5** (USA TST 2008). Let $ABC$ be a triangle with $G$ as its centroid. Let $P$ be a variable point on segment $BC$. Points $Q$ and $R$ lie on sides $AC$ and $AB$ respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as $P$ varies along segment $BC$, the circumcircle of triangle $AQR$ passes through a fixed point $X$ such that $\angle BAG = \angle CAX$.

**Exercise 3.6** (Mathematical Reflections O 371). Let $ABC$ be a triangle $AB < AC$. Let $D, E$ be the feet of altitudes from $B, C$ to sides $AC, AB$ respectively. Let $M, N, P$ be the midpoints of the segments $BC, MD, ME$ respectively. Let $NP$ intersect $BC$ again at a point $S$ and let the line through $A$ parallel to $BC$ intersect $DE$ again at point $T$. Prove that $ST$ is tangent to the circumcircle of triangle $ADE$.

**Exercise 3.7** (ELMO Shortlist 2012/G7). Let $ABC$ be an acute triangle with circumcenter $O$ such that $AB < AC$, let $Q$ be the intersection of the external bisector of $\angle A$ with $BC$, and let $P$ be a point in the interior of $\triangle ABC$ such that $\triangle BPA$ is similar to $\triangle APC$. Show that $\angle QPA + \angle OQB = 90^\circ$.

**Exercise 3.8** (Iranian Geometry Olympiad 2014). The tangent to the circumcircle of an acute triangle $ABC$ (with $AB < AC$) at $A$ meets $BC$ at $P$. Let $X$ be a point on line $OP$ such that $\angle AXP = 90^\circ$. Points $E$ and $F$ lie on sides $AB$ and $AC$, respectively, and are on the same side of line $OP$ such that $\angle EXP = \angle ACX$ and $\angle FXO = \angle ABX$. Let $EF$ meet the circumcircle of triangle $ABC$ at points $K, L$. Prove that the line $OP$ is tangent to the circumcircle of triangle $KLX$. 