A Special Point on the Median

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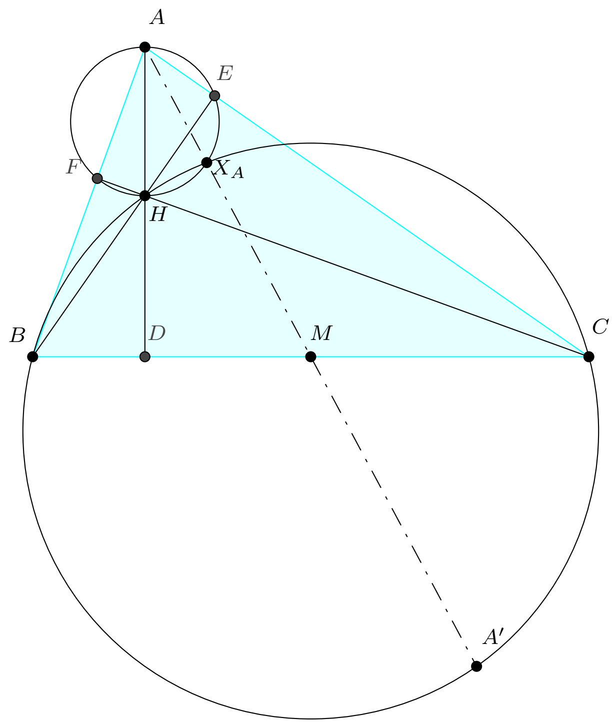
We discuss the properties of the HM point (which remains nameless until now), which seems to be well-known among the community. In addition, we provide some problems where properties of the point prove useful to find solutions.

1 Power of the Point

It is important to note that the HM point is not symmetric about all three vertices for a given triangle; rather, given each vertex, there is a corresponding HM point. As there are many existing definitions of our point, we will simply present characterizations, rather than giving a direct definition. It is left to the reader to choose which definition they like best.

Characterization 1.1. In $\triangle ABC$ with orthocenter H, the circle with diameter $\overline{AH}$ and $\odot(BHC)$ intersect again on the A-median at a point X_A.



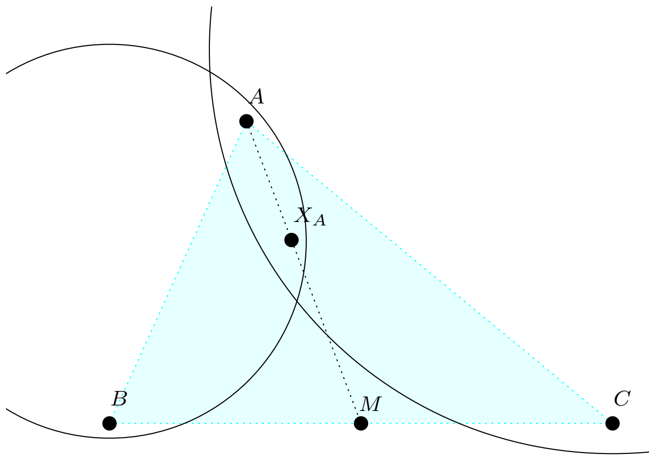
Proof 1. Let A' be the point such that $ABA'C$ is a parallelogram. Since $\angle BAC = \angle BA'C = \pi - \angle BHC$, A' lies on $\odot(BHC)$. Then $\angle HBA' = \angle HBC + \angle A'BC = \frac{\pi}{2} - \angle ACB + \angle ACB = \frac{\pi}{2}$, so A' is the antipode of H. Now we have that $\angle AX_AH = \frac{\pi}{2} = \pi - \angle HX_AA'$, implying that X_A lies on the A-median. □

Alternatively, consider the following proof using inversion.

Proof 2. Peform an inversion at A with power $r^2 = AH \cdot AD$. By Power of a Point, $AH \cdot AD = AF \cdot AB = AE \cdot AC$, which implies that $\{F, B\}, \{H, D\}$ and $\{E, C\}$ swap under this inversion. Since the circle with diameter AH maps to line BC and $\odot(BHC)$ maps to the nine-point circle, these two objects intersect at M, so the image of M lies on the A-median as well. □

Here, X_A is our HM point with respect to vertex A. **Characterization 1.1** is the most common characterization of the HM point, and now we will present others.

Characterization 1.2. Let ω_B be the circle through A and B tangent to line BC, and define ω_C similarly. Then ω_B and ω_C intersect again at X_A.



Proof. Define X'_A as the intersection of the two circles. Since $\angle AX'_AB = \pi - \angle CBA$ and $\angle AX'_AC = \pi - \angle ACB$, we get that

$$\angle BX'_AC = 2\pi - \angle AX'_AB - \angle AX'_AC = \angle ACB + \angle CBA = \pi - \angle BAC$$

whence X'_A lies on $\odot(BHC)$. Since X'_A lies on the A-median by the radical axis theorem, $X'_A \equiv X_A$. □

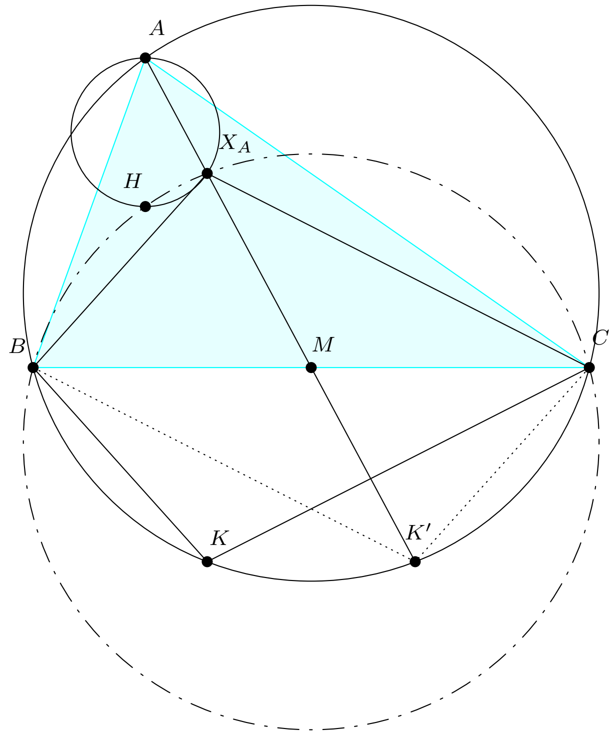
Characterization 1.3. X_A is the isogonal conjugate of the midpoint of the A-symmedian chord in $\odot(ABC)$.

Proof. Let P be the midpoint of the A-symmedian chord; it is well-known that P is the spiral center sending segment BA to segment AC. Then

$$\angle PBA = \angle PAC = \angle BAX_A = \angle CBX_A$$

where the last angle equality follows from **Characterization 1.2**. Since clearly $\angle BAP = \angle CAX_A$, we have proven that P and X_A are isogonal conjugates. □

Characterization 1.4. Suppose the A-symmedian intersects $\odot(ABC)$ again at a point K. Then X_A is the reflection of K over line BC.



Proof. Let K' be the reflection of X_A over the midpoint of $\overline{BC}$. Note that $K' \in \odot(ABC)$, since $\odot(BHC)$ is the reflection of $\odot(ABC)$ about M; additionally, $\frac{BX_A}{CX_A} = \frac{CK'}{BK'}$. Let $P_{\infty,BC}$ denote the point at infinity for line BC, then

$$-1 = (B, C; M, P_{\infty,BC}) \stackrel{A}{=} (B, C; K', A) \Longrightarrow \frac{BK'}{CK'} = \frac{AC}{AB}$$

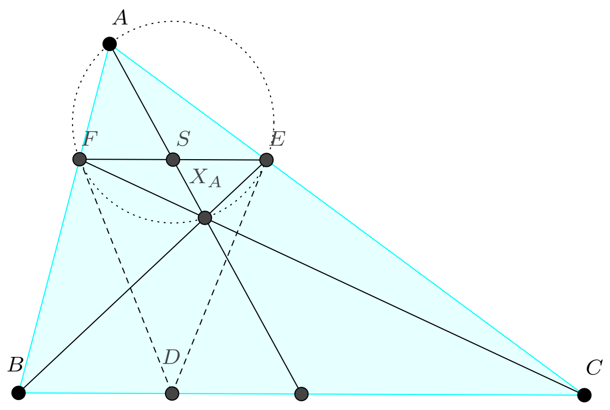
This implies that the reflection of X_A over BC forms a harmonic bundle with A, B, C, so it is precisely the point K. □

Corollary 1.0.1. X_A lies on the A-Apollonius circle.

Characterization 1.5 (ELMO SL 2013 G3)**.** In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.

Proof. Consider an inversion about A of power $r^2 = AB \cdot AC$ followed by a reflection in the bisector of angle A. Then points B and C are interchanged and D is sent on the circumcircle of triangle ABC. Now, $E' = B'D' \cap AC'$ and $F = C'D' \cap AB'$. Also, the HM point opposite to A is mapped to the intersection of tangents at B', C' and so by applying Pascal's Theorem on the cyclic hexagon $AB'B'D'C'C'$ we get that the images of E, F and the HM point are collinear meaning that originally $\odot(AEF)$ passes through the aforementioned point. This proves our characterization. □

Characterization 1.6. Suppose the A-symmedian intersects BC at D. The perpendicular from D to line BC intersects the A-median at S, and the line through S parallel to BC intersects AB and AC at F and E, respectively. Then $X_A \equiv BE \cap CF$.



Proof. We will prove the reverse implication of the problem, by starting with X_A and ending with the A-symmedian. First notice that $\angle FX_AE = \angle BX_AC = \pi - \angle A = \pi - \angle FAE$, so quadrilateral AFX_AE is cyclic. Then $\angle FEX_A = \angle FAX_A = \angle BAX_A = \angle CBX_A$ by **Characterization 2**, so $FE \parallel BC$, and accordingly S lies on the A-median.

By Brokard's Theorem on $\odot(AFX_AE)$, it follows that line BC is the polar of S with respect to the circle. Therefore, the tangents to the circle at F and E intersect on BC at D, and since AD is an A-symmedian in $\triangle AFE$, it follows that AD is also an A-symmedian in $\triangle ABC$, and we're done. □

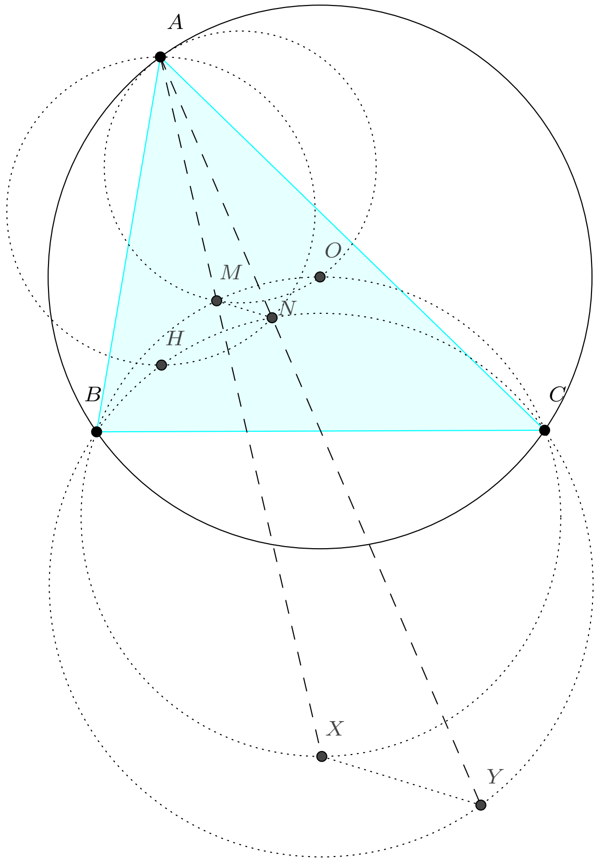
Corollary 1.0.2. The orthocenter of $\triangle DEF$, X_A, and the circumcenter of $\triangle ABC$ are collinear.

Proof. Let the tangents to $\odot(BX_AC)$ at B and C intersect at D'; the problem is equivalent to showing that O is the orthocenter of $\triangle BD'C$. Since D' clearly lies on the perpendicular bisector of $\overline{BC}$, we just need that $\angle BOC = \pi - \angle BD'C$. But it is clear that both sides are equal to twice of angle A, so the corollary follows. □

2 Example Problems

In this section we will see the HM point appearing in a wide variety of problems. Notice that the HM point may not be central to the problem, but it is an important step towards obtaining the desired conclusion.

Example 1 (ELMO 2014/5). Let ABC be a triangle with circumcenter O and orthocenter H. Let ω_1 and ω_2 denote the circumcircles of triangles BOC and BHC, respectively. Suppose the circle with diameter $\overline{AO}$ intersects ω_1 again at M, and line AM intersects ω_1 again at X. Similarly, suppose the circle with diameter $\overline{AH}$ intersects ω_2 again at N, and line AN intersects ω_2 again at Y. Prove that lines MN and XY are parallel.

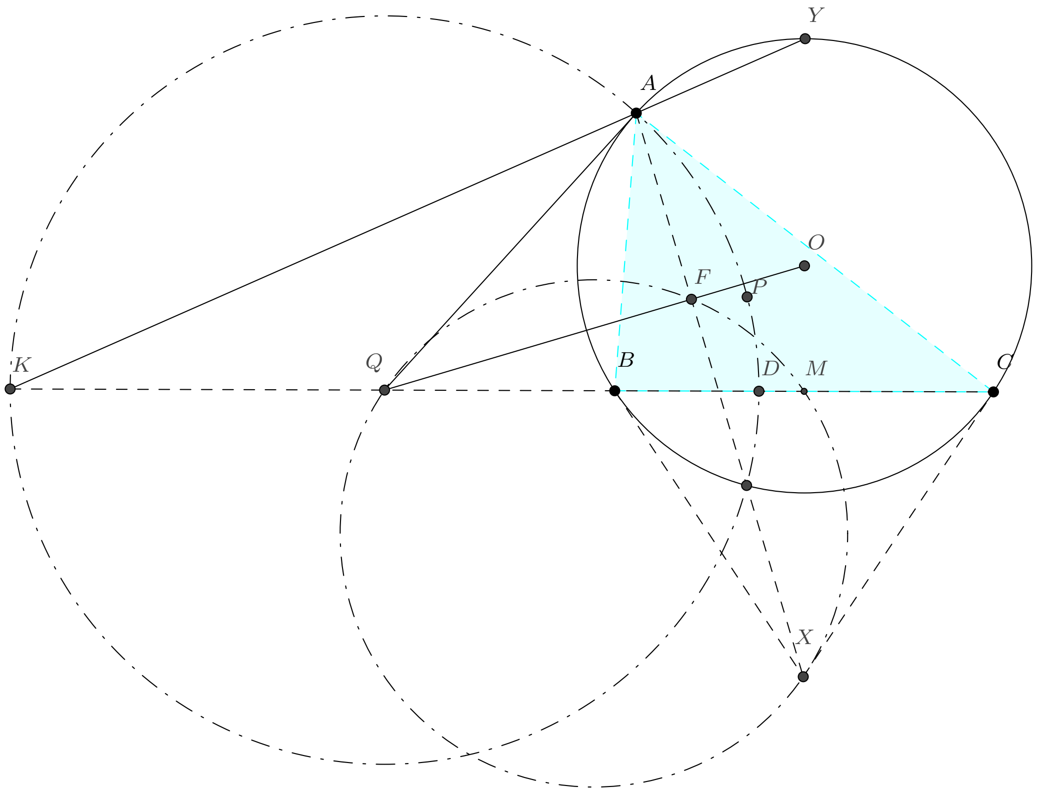


Solution. It is clear by angle chasing that M is the midpoint of the symmedian chord through A, N is the HM point of triangle ABC opposite to A, X is the intersection of the tangents to (ABC) at B, C and Y is the point such that $ABYC$ is a parallelogram. Inverting about A with power $r^2 = AB \cdot AC$ and reflecting in the A-angle bisector, we see that $\{M, Y\}$ and $\{N, X\}$ are swapped. Therefore,

$$AB \cdot AC = AM \cdot AY = AX \cdot AN$$

which gives the conclusion. □

Example 2 (USA TST 2005/6). Let ABC be an acute scalene triangle with O as its circumcenter. Point P lies inside triangle ABC with $\angle PAB = \angle PBC$ and $\angle PAC = \angle PCB$. Point Q lies on line BC with $QA = QP$. Prove that $\angle AQP = 2\angle OQB$.



Proof. Assume without loss of generality, $AB < AC$. Let M be the midpoint of BC. From the angle conditions, it is clear by tangency that P is the HM point opposite to A in triangle ABC. Therefore, we conclude that P lies on the A-Apollonius circle (the circle with diameter $\overline{DK}$), where D, K are the feet of the internal and external bisectors of angle BAC. Therefore, QA is tangent to the circle $\odot(ABC)$.

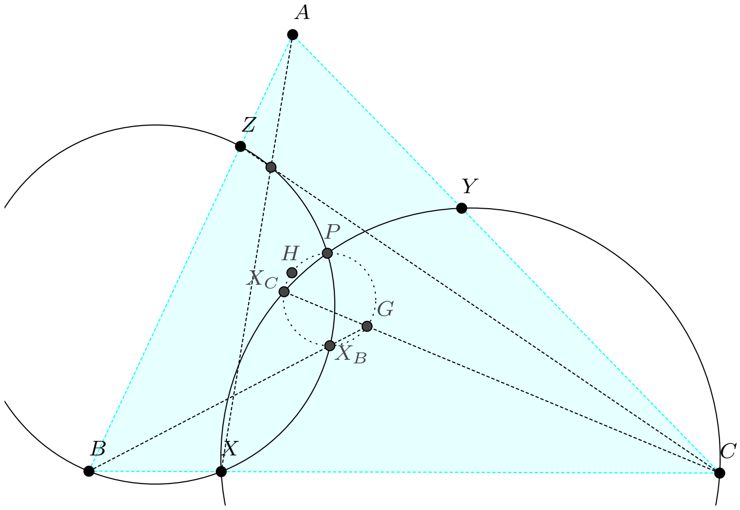
We have $\angle AQP = 2\angle AKP$ since Q is the center of (DK). Let the tangents to (ABC) at B, C meet at X and AK meet $\odot(ABC)$ again at Y. Let F be the midpoint of the A symmedian chord. Then F lies on OQ and $\angle AFO = \frac{\pi}{2}$.

As an inversion about A with power $r^2 = AB \cdot AC$ composed with a reflection about the A-angle bisector sends P to X and K to Y, we have $\angle AKP = \angle AXY = \angle FXM$. Since $\angle QFX = \angle QMX = 90°$, points Q, F, M, X lie on a circle. We have

$$\angle OQB = \angle OQM = \angle FQM = \angle FXM = \angle AKY,$$

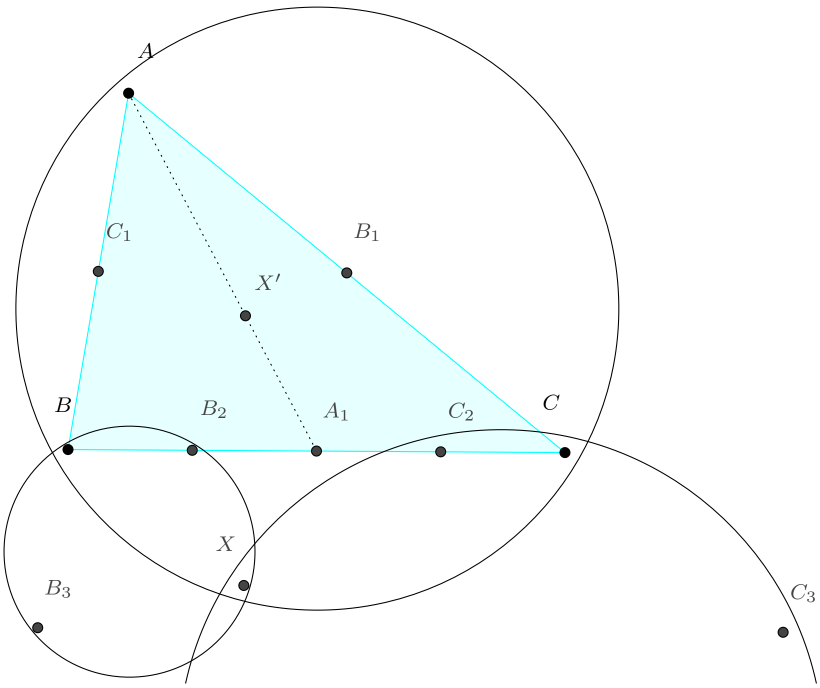
establishing the desired result. $\square$

Example 3 (Brazil National Olympiad 2015/6). Let $\triangle ABC$ be a scalene triangle and X, Y and Z be points on the lines BC, AC and AB, respectively, such that $\measuredangle AXB = \measuredangle BYC = \measuredangle CZA$. The circumcircles of BXZ and CXY intersect at P. Prove that P lies on the circle with diameter HG where H and G are the orthocenter and the centroid, respectively, of triangle ABC.



Solution. Note that $\angle BXA + \angle BZC = \pi$. It follows that the intersection of lines AX and CZ lies on the circle $\odot(BXZ)$. We conclude that $\odot(BXZ)$ passes through the HM point X_B opposite to B. Similarly the circle $\odot(CXY)$ passes through the HM point X_C opposite C. In triangle BGC, the circles $\odot(BXX_B)$ and $\odot(CXX_C)$ meet at $P \neq X$ where X_B, X_C lie on the lines BG and CG, respectively. By Miquel's Theorem, it follows that P lies on the circumcircle of triangle GX_BX_C. As $\angle HX_BG = \angle HX_CG = \frac{\pi}{2}$, we conclude that $\angle HPG = \frac{\pi}{2}$. □

Example 4 (Sharygin Geometry Olympiad 2015). Let A_1, B_1, C_1 be the midpoints of the sides opposite A, B, C in triangle ABC. Let B_2 and C_2 denote the midpoints of segments BA_1 and CA_1, respectively. Let B_3 and C_3 denote the reflections of C_1 in B and B_1 in C, respectively. Prove that the circumcircles of triangles BB_2B_3 and CC_2C_3 meet on the circumcircle of triangle ABC.



Solution. Let X be a point on the circle $\odot(ABC)$ such that AX is a symmedian in triangle ABC. We will show that X is a common point of circles $\odot(BB_2B_3)$ and (CC_2C_3). Clearly, the reflection of X in BC lies on the median AA_1. Thus,

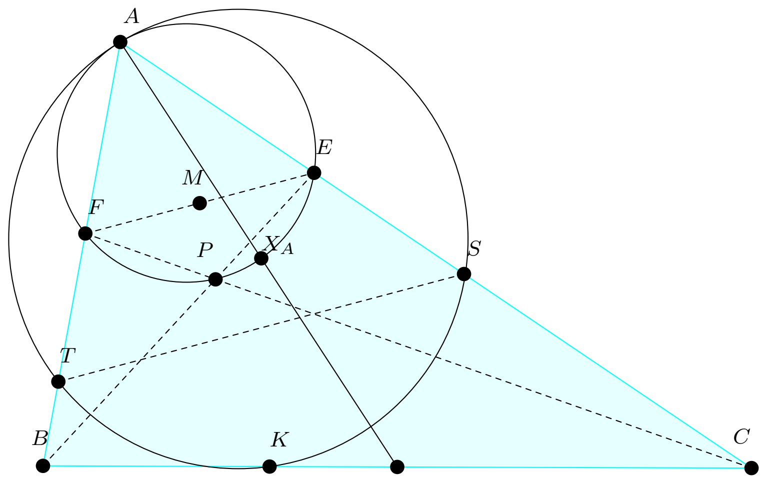
$$\angle XA_1B = \angle AA_1B = \angle ACX = \angle XBB_3,$$

showing that the line AB is tangent to the circle $\odot(BXA_1)$. The spiral similarity centered at X which takes CA_1 to AB will send B_2 to B_3 as they divide CA_1 and AB in the same ratio of negative half. Hence, X lies on $\odot(BB_2B_3)$ and repeating the argument for the circle $\odot(CC_2C_3)$ yields the result. □

We end our discussion with a nice problem from a recent IMO ShortList. It is a perfect example of the properties of the HM point we have discussed!

Example 5 (ISL 2014/G6)**.** Let ABC be a fixed acute-angled triangle. Consider some points E and F lying on the sides AC and AB, respectively, and let M be the midpoint of EF . Let the perpendicular bisector of EF intersect the line BC at K, and let the perpendicular bisector of MK intersect the lines AC and AB at S and T , respectively. We call the pair (E, F) *interesting*, if the quadrilateral $KSAT$ is cyclic. Suppose that the pairs (E_1, F_1) and (E_2, F_2) are interesting. Prove that

$$\frac{E_1E_2}{AB} = \frac{F_1F_2}{AC}.$$



Solution. Note that $EF \parallel ST$ implying that M lies on the median from A of triangle AST. As the reflection of M in ST lies on the circle (AST), we conclude that AK is a symmedian in triangle AST, and hence in triangle AEF as well. Since K lies on the perpendicular bisector of EF, we see that KE, KF are tangents to the circle $\odot(AEF)$.

Let BE meet $\odot(AEF)$ at $P \neq E$. Applying Pascal's Theorem on the cyclic hexagon $(AEEPFF)$ we observe that $AE \cap PF, T$, and B, are collinear. This shows that P lies on the line CF. Therefore, as A, E, F, and the intersection of BE and CF lie on a circle, the circle $\odot(AEF)$ passes through the point X_A.

Let B_1 and C_1 be the feet of altitudes from B, C to sides AC, AB respectively. Evidently, X_A lies on (AB_1C_1). Thus, X_A is the center of a spiral similarity sending B_1C_1 to EF. So, for any pair of interesting points $\{E_1, F_1\}$ and $\{E_2, F_2\}$, we have

$$\frac{E_1E_2}{F_1F_2} = \frac{B_1E_2 - B_1E_1}{C_1F_2 - C_1F_1} = \frac{B_1E}{C_1F} = \frac{X_AB_1}{X_AC_1} = \frac{B_1A}{C_1A} = \frac{AB}{AC}$$

as desired. □

3 Exercises

Note that in the following problems, the HM point may not be directly involved, but it provides appropriate motivation or intuition to solve the problem.

Exercise 3.1 (USA TSTST 2015/2)**.** Let ABC be a scalene triangle. Let K_a, L_a and M_a be the respective intersections with BC of the internal angle bisector, external angle bisector, and the median from A. The circumcircle of AK_aL_a intersects AM_a a second time at point X_a different from A. Define X_b and X_c analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of ABC.

Exercise 3.2 (WOOT 2013 Practice Olympiad 3/5)**.** A semicircle has center O and diameter AB. Let M be a point on AB extended past B. A line through M intersects the semicircle at C and D, so that D is closer to M than C. The circumcircles of triangles AOC and DOB intersect at O and K. Show that $\angle MKO = 90°$.

Exercise 3.3 (IMO 2010/4, Modified)**.** In $\triangle ABC$ with orthocenter H, suppose P is the projection of H onto the C-median; let the second intersections of AP, BP, CP with $\odot(ABC)$ be K, L, M respectively. Show that $MK = ML$.

Exercise 3.4 (EGMO 2016/4)**.** Two circles ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω.

Exercise 3.5 (USA TST 2008)**.** Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.

Exercise 3.6 (Mathematical Reflections O 371)**.** Let ABC be a triangle $AB < AC$. Let D, E be the feet of altitudes from B, C to sides AC, AB respectively. Let M, N, P be the midpoints of the segments BC, MD, ME respectively. Let NP intersect BC again at a point S and let the line through A parallel to BC intersect DE again at point T. Prove that ST is tangent to the circumcircle of triangle ADE.

Exercise 3.7 (ELMO Shortlist 2012/G7)**.** Let ABC be an acute triangle with circumcenter O such that $AB < AC$, let Q be the intersection of the external bisector of $\angle A$ with BC, and let P be a point in the interior of $\triangle ABC$ such that $\triangle BPA$ is similar to $\triangle APC$. Show that $\angle QPA + \angle OQB = 90°$.

Exercise 3.8 (Iranian Geometry Olympiad 2014)**.** The tangent to the circumcircle of an acute triangle ABC (with $AB < AC$) at A meets BC at P. Let X be a point on line OP such that $\angle AXP = 90°$. Points E and F lie on sides AB and AC, respectively, and are on the same side of line OP such that $\measuredangle EXP = \measuredangle ACX$ and $\measuredangle FXO = \measuredangle ABX$. Let EF meet the circumcircle of triangle ABC at points K, L. Prove that the line OP is tangent to the circumcircle of triangle KLX.