Junior problems

J397. Find all positive integers n for which $3^4 + 3^5 + 3^6 + 3^7 + 3^n$ is a perfect square.

Proposed by Adrian Andresscu, Dallas, TX, USA

Solution by Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan

If $n < 4$ then the only solution is $n = 2$.

Let $n \geq 4$ and k, s be positive integers

$$3^4(1 + 3 + 3^2 + 3^3 + 3^{n-4}) = k^2 \Longrightarrow 1 + 3 + 3^2 + 3^3 + 3^{n-4} = s^2 \Longrightarrow 40 + 3^{n-4} = s^2.$$

There are two cases

(i) Let $n-4 = 2k_1$ so $(s-3^{k_1})(s+3^{k_1}) = 40 \Longrightarrow (s; k_1) = (7; 1), (11; 2)$ so $n = 2k_1 + 4 \Longrightarrow n = 6$ and $n = 8$.

(ii) Let $n - 4 = 2k_2 + 1$ so $40 + 3^{2k_2+1} = s^2$. Then $s^2 \equiv 0, 1 \pmod 4$ and $40 + 3^{2k_2+1} \equiv -1 \pmod 4$ is a contradiction.

Hence, the only solutions are $n = 2, 6, 8$.

Also solved by Alessandro Ventullo, Milan, Italy; Abdushukur Ahadov, Syrdarya, Uzbekistan; Adnan Ali, A.E.C.S-4, Mumbai, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramures, Romania; José Hernández Santiago, México; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nikolaos Evgenidis, M.N.Raptou High Scool, Larissa, Greece; Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua; Robert Bosch, USA; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland; Polyahedra, Polk State College, FL, USA; Soo Young Choi, Vestal Senior Highschool, NY, USA; Anderson Torres, Sao Paulo, Brazil.

J398. Let a, b, c be real numbers. Prove that

$$\left(a^2+b^2+c^2-2\right)(a+b+c)^2+(1+ab+bc+ca)^2 \geq 0.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Robert Bosch, USA

We have to prove that

$$(a^2+b^2+c^2-2)(a+b+c)^2+(1+ab+bc+ca)^2 \geq 0.$$

We know that

$$a^2+b^2+c^2-2=(a+b+c)^2-2(ab+bc+ca)-2,$$

now denote $a+b+c=x$ and $ab+bc+ca=y$. The inequality becomes

$$(x^2-2y-2)x^2+(y+1)^2 \geq 0,$$

or

$$x^4-2(y+1)x^2+(y+1)^2 \geq 0.$$

Finally the last expression is

$$\left(x^2-(y+1)\right)^2 \geq 0.$$

Also solved by Polyahedra, Polk State College, FL, USA; Soo Young Choi, Vestal Senior Highschool, NY, USA; Anderson Torres, Sao Paulo, Brazil; Alessandro Ventullo, Milan, Italy; Abdushukur Ahadov, Syrdarya, Uzbekistan; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Catalin Prajitura, College at Brockport, SUNY, NY, USA; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; Henry Ricardo, New York Math Circle; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Joel Schlosberg, Bayside, NY; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focșani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greeece; Nishant Dhankhar, New Delhi, India; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Nikos Kalapodis, Patras, Greece; Vincelot Ravoson,Lycée Henry IV, Paris, France; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA.

J399. Two nine-digit numbers m and n are called *cool* if

(a) they have the same digits but in different order,

(b) no digit appears more than once,

(c) m divides n or n divides m.

Prove that if m and n are cool, then they contain digit 8.

Proposed by Titu Andreescu, University of Texas at Dallas, TX, USA

First solution by Alessandro Ventullo, Milan, Italy

Assume by contradiction that there exist two *cool* numbers m and n not containing digit 8. Then in m and n appear the digits $0, 1, 2, 3, 4, 5, 6, 7, 9$ exactly once. Since the sum of their digits is 37, then $m, n \equiv 1 \pmod 9$. Assume without loss of generality that m divides n. Then, $n = mk$, where k is a natural number. Hence, $n - m = m(k-1)$. Reducing modulo 9 this equation, we get $k - 1 \equiv 0 \pmod 9$, i.e. $k-1$ is divisible by 9. Since m and n have the same digits in different order, then $m \neq n$, which gives $k \neq 1$. So, $k \geq 10$. But then $n \geq 10m$, i.e. n has more digits than m, contradiction.

Second solution by Polyahedra, Polk State College, FL, USA

Suppose that m and n are cool and do not contain digit 8. Then $m = a_8 10^8 + \cdots + a_1 10 + a_0$ and $n = b_8 10^8 + \cdots + b_1 10 + b_0$, with $\{a_0, \ldots, a_8\} = \{b_0, \ldots, b_8\} = \{0, 1, 2, 3, 4, 5, 6, 7, 9\}$. Note that both m and n have digit-sum 37, so neither is divisible by 3. Also, if $a_i = b_j$, then $a_i 10^i - b_j 10^j$ is divisible by 9, thus $m - n$ is divisible by 9. Now assume, without loss of generality, that $m = qn$. Since m is not divisible by 3, $q \in \{2, 4, 5, 7, 8\}$. Then $m - n = (q-1)n$, with $q - 1 \in \{1, 3, 4, 6, 7\}$. Since n is not divisible by 3, $(q-1)n$ cannot be divisible by 9, a contradiction.

The proof reveals that the digit 8 in the problem can be replaced by any digit not divisible by 3. Also, cool numbers do exist in abundance. For example, $m = 123456789$ and $n = 987654312$.

Also solved by Adamopoulos Dionysios, 4th Junior High School, Pyrgos, Greece; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Adnan Ali, A.E.C.S-4, Mumbai, India; Paraskevi-Andrianna Maroutsou, High School of Evangeliki, Athens, Greece; Shuborno Das, Ryan International School, Bangalore, India.

J400. Prove that for all real numbers a, b, c the following inequality holds:

$$\frac{|a|}{1+|b|+|c|}+\frac{|b|}{1+|c|+|a|}+\frac{|c|}{1+|a|+|b|} \geq \frac{|a+b+c|}{1+|a+b+c|}.$$

When does the equality occur?

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Zafar Ahmed, BARC, Mumbai, India and Dona Ghosh, JU, Jadavpur, India

Let us call the LHS as F. We can write

$$\frac{|a|}{1+|b|+|c|} \geq \frac{|a|}{1+|a|+|b|+|c|}$$

and so on, then we have to prove that

$$F \geq \frac{|a|+|b|+|c|}{1+|a|+|b|+|c|} \geq \frac{|a+b+c|}{1+|a+b+c|}.$$

Use

$$|a|+|b|+|c| \geq |a+b+c| \Rightarrow 1+\frac{1}{|a|+|b|+|c|} \leq 1+\frac{1}{|a+b+c|} \Rightarrow \frac{|a|+|b|+|c|}{1+|a|+|b|+|c|} \geq \frac{|a+b+c|}{1+|a+b+c|}.$$

Hence proved. The given inequality is symmetric in a, b, c so the equality holds when $a=b=c=k$ and k turns out to be zero.

Also solved by Polyahedra, Polk State College, FL, USA; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Alessandro Ventullo, Milan, Italy; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focșani, Romania; Nishant Dhankhar, New Delhi, India; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Robert Bosch, USA; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland.

J401. Find all integers n for which $n^2 + 2^n$ is a perfect square.

Proposed by Proposed by Adrian Andreescu, Dallas, TX, USA

Solution by Alessandro Ventullo, Milan, Italy

Clearly, $n \geq 0$. If $n = 0$, we get $n^2 + 2^n = 1$, which is a perfect square. Let $n > 0$. If n is even, then $n = 2k$ for some $k \in \mathbb{N}^*$. If $k \geq 7$, then

$$(2^k)^2 = 2^{2k} < 4k^2 + 2^{2k} < 2^{2k} + 2^{k+1} + 1 = (2^k + 1)^2,$$

so $n^2 + 2^n$ is not a perfect square if n is even and $n \geq 14$. So, $n \in \{2, 4, 6, 8, 10, 12\}$. An easy check gives the solution $n = 6$. If n is odd, then $n = 2k + 1$ for some $k \in \mathbb{N}$. If $k = 0$, we get no solutions, so assume $k \geq 1$. Let $m \in \mathbb{N}^*$ such that $(2k + 1)^2 + 2^{2k+1} = m^2$. Then,

$$(m - 2k - 1)(m + 2k + 1) = 2^{2k+1}.$$

Since $m - 2k - 1 < m + 2k + 1$ and the two factors have the same parity, then

$$\begin{aligned} m - 2k - 1 &= 2^a \\ m + 2k + 1 &= 2^b, \end{aligned}$$

where $a, b \in \mathbb{N}$, $1 \leq a \leq b \leq 2k$ and $a + b = 2k + 1$. If $a \geq 2$, then subtracting we get $2(2k + 1) = 2^b - 2^a = 2^a(2^{b-a} - 1)$, i.e. $2k+1 = 2^{a-1}(2^{b-a} - 1)$, contradiction. So, $a = 1$ and $b = 2k$, which gives $2k+1 = 2^{2k-1} - 1$, i.e. $k = 2^{2k-2} - 1$. If $k \geq 2$, then $k < 2^{2k-2} - 1$, so it must be $k = 1$. But if $k = 1$, we get no solutions. So, there are no solutions when n is odd. We conclude that $n \in \{0, 6\}$.

Also solved by Polyahedra, Polk State College, FL, USA; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Abdushukur Ahadov, Syrdarya, Uzbekistan; Adnan Ali, A.E.C.S-4, Mumbai, India; Alok Kumar, New Delhi, India; José Hernández Santiago, México; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focșani, Romania; Rajdeep Majumder, Durgapur, India; Robert Bosch, USA; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland.

J402. Consider a nonisosceles triangle ABC. Let I be its incenter and G its centroid. Prove that GI is perpendicular to BC if and only if $AB + AC = 3BC$.

Proposed by Bazarbaev Sardar, National University of Uzbekistan, Uzbekistan

Solution by Polyahedra, Polk State College, FL, USA

Let M be the midpoint of BC, and let D and E be the feet of perpendiculars from A and G onto BC, respectively. Then $MD/ME = MA/MG = 3$. Also,

$$MD = BD - BM = c\cos B - \frac{a}{2} = \frac{c^2-b^2}{2a} = \frac{(c-b)(c+b)}{2a}.$$

Therefore,

$$GI \perp BC \quad \Leftrightarrow \quad BE = s-b \quad \Leftrightarrow \quad ME = s-b-\frac{a}{2} = \frac{c-b}{2} \quad \Leftrightarrow \quad \frac{c+b}{a} = 3,$$

completing the proof.

Editorial Remark: During the edition of this issue Daniel Campos, University of Chicago, suggested an alternative proof. Say,

$$\begin{aligned} BG^2 - CG^2 &= \frac{4}{9}\left(\frac{1}{4}(2a^2+2c^2-b^2) - \frac{1}{4}(2a^2+2b^2-c^2)\right) = \frac{c^2-b^2}{3}, \\ BI^2 - CI^2 &= (s-b)^2 - (s-c)^2 = a(c-b). \end{aligned}$$

So GI is perpendicular to BC if and only if $b + c = 3a$.

Also solved by Soo Young Choi, Vestal Senior Highschool, NY, USA; Anderson Torres, Sao Paulo, Brazil; Nikos Kalapodis, Patras, Greece; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Abdushukur Ahadov, Syrdarya, Uzbekistan; Adnan Ali, A.E.C.S-4, Mumbai, India; Alok Kumar, New Delhi, India; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focșani, Romania; Nikolaos Evgenidis, M.N.Raptou High Scool, Larissa, Greece; Nishant Dhankhar, New Delhi, India; Shuborno Das, Ryan International School, Bangalore, India.

Senior problems

S397. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}+\frac{3(ab+bc+ca)}{2(a+b+c)} \geq a+b+c.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, CA, USA

Note that

$$\sum_{cyc} \frac{a^2}{a+b}+\frac{3(ab+bc+ca)}{2(a+b+c)} \geq a+b+c \iff \sum_{cyc} \frac{a^2(a+b+c)}{a+b}+\frac{3(ab+bc+ca)}{2} \geq (a+b+c)^2.$$

Since

$$\sum_{cyc} \frac{a^2(a+b+c)}{a+b} = \sum_{cyc} a^2+\sum_{cyc} \frac{a^2 c}{a+b}$$

and by Cauchy-Schwarz Inequality,

$$\sum_{cyc} \frac{a^2 c}{a+b}=\sum_{cyc} \frac{c^2 a^2}{ca+bc} \geq \frac{(ca+ab+bc)^2}{\sum_{cyc}(ca+bc)}=\frac{ab+bc+ca}{2}$$

therefore

$$\sum_{cyc} \frac{a^2(a+b+c)}{a+b}+\frac{3(ab+bc+ca)}{2} \geq a^2+b^2+c^2+\frac{ab+bc+ca}{2}+\frac{3(ab+bc+ca)}{2}=$$

$$a^2+b^2+c^2+2(ab+bc+ca)=(a+b+c)^2.$$

Also solved by Adnan Ali, A.E.C.S-4, Mumbai, India; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Nikos Kalapodis, Patras, Greece; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Alessandro Ventullo, Milan, Italy; Alok Kumar, New Delhi, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Rajarshi Kanta Ghosh, Kolkata, India; Robert Bosch, USA; Albert Stadler, Herrliberg, Switzerland; Zafar Ahmed, BARC, Mumbai, India and Dona Ghosh, JU, Jadavpur, India.

S398. Let tetrahedron $ABCD$ lies inside a unit cube. Let M and N be midpoint of the side AB and CD, respectively. Prove that $AB \cdot CD \cdot MN \leq 2$

Proposed by Nairi Sedrakian, Yerevan, Armenia

First solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Denote the center of cube is O, and symmetric transformation for point O be Z_0.

$$Z_0(A) = A_1,\ Z_0(B) = B_1,\ Z_0(C) = C_1,$$
$$Z_0(D) = D_1,\ Z_0(M) = M_1,\ Z_0(N) = N_1$$

then we have $A_1,\ B_1,\ C_1,\ D_1,\ M_1,\ N_1$ lies inside a unit cube. Consider the parallelogramms D_1CDC_1, ABA_1B_1, MNM_1N_1. Then we get

$$CD^2 + NN_1^2 = CD^2 + CD_1^2 \leq \max(DD_1^2, CC_1^2) \leq 3$$
$$AB^2 + MM_1^2 = AB^2 + AB_1^2 \leq \max(AA_1^2, BB_1^2) \leq 3$$

hence we have

$$CD^2 + AB^2 + NN_1^2 + MM_1^2 \leq 6 \tag{1}$$

By the law of parallelogramms,

$$NN_1^2 + MM_1^2 = 2 \cdot MN^2 + 2MN_1^2.$$

Thus, we get

$$NN_1^2 + MM_1^2 \geq 2 \cdot MN^2 \tag{2}$$

From (1) and (2), we get

$$AB^2 + CD^2 + 2MN^2 \leq 6.$$

Using AM-GM inequality, we get

$$6 \geq AB^2 + CD^2 + 2MN^2 \geq 3 \cdot \sqrt[3]{AB^2 \cdot CD^2 \cdot (2MN^2)}.$$

Hence $AB \cdot CD \cdot MN \leq 2$.

Second solution by Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan

Let $AB = CD = a$ and $AQ = h$ then $BN = \dfrac{\sqrt{3}a}{2}$ and $MP = \dfrac{h}{2}$. On BN P, Q are chosen like that $MP \perp BN$ and $AQ \perp BN$. $BP^2 = BM^2 - MP^2 = \dfrac{a^2}{12} \Longrightarrow BP = \dfrac{\sqrt{3}a}{6}$, $PN = BN - BP = \dfrac{\sqrt{3}a}{3}$, $MN^2 = MP^2 + NP^2 \Longrightarrow MN = \dfrac{a}{\sqrt{2}}$. Let $AB \cdot CD \cdot MN = \dfrac{a^3}{\sqrt{2}}$ It is exact that if the cub whose edge is equal to b there are many tetrahedrons lies inside a cube . And among them the tetrahedron's edge whose edge is the longest will be equal to $b\sqrt{2}$. By that we gain $a \leq \sqrt{2}b = \sqrt{2}$ because $b = 1$.

$$AB \cdot CD \cdot MN = \frac{a^3}{\sqrt{2}} \leq 2.$$

Third solution by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina
Lemma:
For all $0 \leq a, b, c, d \leq 1$ following inequality holds

$$2(a-b)^2 + 2(c-d)^2 + (a+b-c-d)^2 \leq 4$$

Proof:

Since
$$f(a,b,c,d) = 2(a-b)^2 + 2(c-d)^2 + (a+b-c-d)^2$$
is convex quadratic parabola for all of its variables it reaches maximum when all of them are either 0 or 1. By checking we obtain maximum as 4 and reached when two of them are 0 and two of them are 1. Let the unit cube be with vertices $(0,0,0),(0,0,1),(1,0,1),(1,0,0),(0,1,0),(1,1,0),(1,1,1),(0,1,1)$

$$A = (x_A, y_A, z_A), B = (x_B, y_B, z_B), C = (x_C, y_C, z_C), D = (x_D, y_D, z_D)$$

obviously we have

$$0 \leq x_i, y_i, z_i \leq 1, \forall i \in \{A, B, C, D\}$$

and $M = \left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}, \frac{z_A + z_B}{2}\right)$,$N = \left(\frac{x_C + x_D}{2}, \frac{y_C + y_D}{2}, \frac{z_C + z_D}{2}\right)$

$$AB \cdot CD \cdot MN \leq 2 \Leftrightarrow AB^2 \cdot CD^2 \cdot 2MN^2 \leq 8$$

Using AM-GM inequality we get

$$AB^2 \cdot CD^2 \cdot 2MN^2 \leq \left(\frac{AB^2 + CD^2 + 2MN^2}{3}\right)^3$$

Only remains to prove

$$\left(\frac{AB^2 + CD^2 + 2MN^2}{3}\right)^3 \leq 8 \Leftrightarrow$$

$$AB^2 + CD^2 + 2MN^2 \leq 6 \Leftrightarrow$$

$$(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 + (x_C - x_D)^2 + (y_C - y_D)^2 + (z_C - z_D)^2 +$$

$$+2\left(\frac{x_A + x_B - x_C - x_D}{2}\right)^2 + 2\left(\frac{y_A + y_B - y_C - y_D}{2}\right)^2 + 2\left(\frac{z_A + z_B - z_C - z_D}{2}\right)^2 \leq 6 \Leftrightarrow$$

$$\Leftrightarrow 2(x_A - x_B)^2 + 2(y_A - y_B)^2 + 2(z_A - z_B)^2 + 2(x_C - x_D)^2 + 2(y_C - y_D)^2 + 2(z_C - z_D)^2 +$$

$$+(x_A + x_B - x_C - x_D)^2 + (y_A + y_B - y_C - y_D)^2 + (z_A + z_B - z_C - z_D)^2 \leq 12 \Leftrightarrow$$

$$2(x_A - x_B)^2 + 2(x_C - x_D)^2 + (x_A + x_B - x_C - x_D)^2 + (y_A - y_B)^2 + (y_C - y_D)^2 +$$

$$+(y_A + y_B - y_C - y_D)^2 + 2(z_A - z_B)^2 + 2(z_C - z_D)^2 + (z_A + z_B - z_C - z_D)^2 \leq 12$$

Which follows by application of lemma.Equality holds if and only if $AB = CD = \sqrt{2}, MN = 1$.

S399. Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2=1$. Prove that

$$\sqrt{2} \leq \sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}} \leq \sqrt[4]{27}.$$

When do equalities occur?

Proposed by Marcel Chirita, Bucharest, Romania

First solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

First, let us prove LHS. Let $\max(a,b,c)=a$. Using AM-GM inequality, we get

$$\begin{aligned}
\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}} &\geq 2\cdot\sqrt{\sqrt{\frac{a+b}{2}}\cdot\sqrt{\frac{c+a}{2}}}+\sqrt{\frac{b+c}{2}}\\
&= 2\sqrt[4]{\frac{1}{4}(a^2+ab+bc+ca)}+\sqrt{\frac{b+c}{2}}\\
&\geq 2\sqrt[4]{\frac{1}{4}(a^2+b^2+c^2+0)}+0\\
&= \sqrt[4]{4(a^2+b^2+c^2)}=\sqrt{2}.
\end{aligned}$$

LHS is proved. Equality holds only when $\{a,b,c\}=\{1,0,0\}$.

Now, by weighted power mean inequalities, we get

$$\begin{aligned}
&M_{1/2}\left(\frac{a+b}{2},\frac{b+c}{2},\frac{c+a}{2}\right) \leq M_1\left(\frac{a+b}{2},\frac{b+c}{2},\frac{c+a}{2}\right)\\
&\Leftrightarrow \left(\frac{1}{3}\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}\right)\right)^2 \leq \frac{1}{3}\left(\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}\right)\\
&\Leftrightarrow \sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}} \leq \sqrt{3}(a+b+c)^{\frac{1}{2}}=\sqrt{3}\sqrt[4]{(a+b+c)^2}.
\end{aligned}$$

$\big($Using, $(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)\leq 3(a^2+b^2+c^2)\big)$

$$\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}} \leq \sqrt{3}\sqrt[4]{3(a^2+b^2+c^2)}=\sqrt[4]{27}.$$

RHS is proved. Equality holds only when $a=b=c=\frac{1}{\sqrt{3}}$.

Second solution by Robert Bosch, USA

Let us prove first the right inequality.

$$\begin{aligned}\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}} &\leq 3\sqrt{\frac{a+b+c}{3}},\\ &\leq 3\sqrt{\sqrt{\frac{a^2+b^2+c^2}{3}}},\\ &= 3\sqrt[4]{\frac{1}{3}}=\sqrt[4]{27}.\end{aligned}$$

By double application of AM-QM inequality. Equality holds when $a=b=c=\frac{\sqrt{3}}{3}$.

Now we need to show that

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\geq 2,$$

with $a^2+b^2+c^2=1$. Squaring this inequality is equivalent to

$$2(a+b+c)+2\sum_{cyc}\sqrt{a^2+ab+bc+ca}\geq 4.$$

By double application of Minkowsky inequality we obtain

$$\sum_{cyc}\sqrt{a^2+ab+bc+ca}\geq\sqrt{(a+b+c)^2+9(ab+bc+ca)},$$

so remains to show

$$(a+b+c)+\sqrt{(a+b+c)^2+9(ab+bc+ca)}\geq 2,$$

but

$$(a+b+c)^2=1+2(ab+bc+ca)\geq 1,$$

thus our inequality becomes

$$\sqrt{\frac{11}{2}(a+b+c)^2-\frac{9}{2}}\geq 2-(a+b+c),$$

squaring is equivalent to $9(a+b+c)^2+8(a+b+c)-17\geq 0$, or $(9(a+b+c)+17)(a+b+c-1)\geq 0$ clearly true since $a+b+c\geq 1$. The equality holds when $a=0,b=0,c=1$ and permutations.

Also solved by Vincelot Ravoson,Lycée Henri IV, Paris, France; Abdushukur Ahadov, Syrdarya, Uzbekistan; Adnan Ali, A.E.C.S-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Ashley Case, SUNY Brockport, NY, USA; Henry Ricardo, New York Math Circle; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Rajarshi Kanta Ghosh, Kolkata, India.

S400. Find all n for which $(n-4)! + \frac{(n+3)!}{36n}$ is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, TX, USA

Solution by Robert Bosch, USA

We need to solve the equation

$$(n-4)! + \frac{1}{36n}(n+3)! = m^2.$$

Clearly $n \geq 4$. Curiously this equation becomes

$$(n-4)!n^2(n^2-7)^2 = (6m)^2,$$

so $(n-4)!$ have to be a perfect square. In general $n!$ is a perfect square if and only if $n = 0, 1$. So the answer to our problem is $n = 4$ and $n = 5$ since $1 + \frac{7!}{36 \cdot 4} = 36 = 6^2$ and $1 + \frac{8!}{36 \cdot 5} = 225 = 15^2$.

Let us prove the last statement in detail. Suppose $n!$ is a perfect square where $n \geq 4$, and let p be a prime divisor of the factorial, with $\frac{n}{2} < p \leq n$ by Bertrand's postulate. Note that $p^2 \mid n!$, thus there is m with $p < m < n$ such that $p \mid m$, therefore $\frac{m}{p} \geq 2$ and so $m \geq 2p > n$, contradiction. The cases $2! = 2$ and $3! = 6$ are clearly eliminated.

This result admit a generalization proved by Erdős and Selfridge in 1975. Say, the equation

$$(n+1)(n+2)\cdots(n+k) = x^\ell,$$

with $k, \ell \geq 2$ and $n \geq 0$ is unsolvable. See the article *The product of consecutive integers is never a power, Illinois J. Math. 19, (1975).*

Second solution by José Hernández Santiago, México

We have that

$$\begin{aligned}(n-4)! + \frac{1}{36n}(n+3)! &= (n-4)!\left(\frac{36 + (n-3)(n-2)(n-1)(n+1)(n+2)(n+3)}{36}\right) \\ &= (n-4)!\left(\frac{36 + (n^2-9)(n^2-4)(n^2-1)}{36}\right) \\ &= \frac{(n-4)!(n^6 - 14n^4 + 49n^2)}{36} \\ &= (n-4)!\left(\frac{n(n^2-7)}{6}\right)^2.\end{aligned}$$

Since $6 \mid n(n^2-7)$ for every $n \in \mathbb{N}$, it follows that $(n-4)! + \frac{1}{36n}(n+3)!$ is perfect square if and only if $(n-4)!$ is a perfect square. By resorting to Bertrand's Postulate it can easily be shown that $N!$ is a perfect square if and only if $N = 0$ or $N = 1$ (cf. the solution to problem O100 or J191). Hence, $(n-4)! + \frac{1}{36n}(n+3)!$ is a perfect square if and only if $n \in \{4, 5\}$.

Also solved by Adnan Ali, A.E.C.S-4, Mumbai, India; Anderson Torres, Sao Paulo, Brazil; Alessandro Ventullo, Milan, Italy; Abdushukur Ahadov, Syrdarya, Uzbekistan; Catalin Prajitura, College at Brockport, SUNY; NY, USA; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focșani, Romania; Robert Bosch, USA; Albert Stadler, Herrliberg, Switzerland.

S401. Let a, b, c, d be nonnegative real numbers such that $ab + ac + ad + bc + bd + cd = 6$. Prove that

$$a + b + c + d + (3\sqrt{2} - 4)abcd \geq 3\sqrt{2}.$$

Proposed by Marius StÃ„Æ'nean, ZalÃ„Æ'u, Romania

First solution by Adnan Ali, A.E.C.S-4, Mumbai, India

First we note that if any one variable is 0 (say d), then the inequality becomes $a + b + c \geq 3\sqrt{2}$ for nonnegative ab, c satisfying $ab + bc + ca = 6$; which is true since $(a + b + c)^2 \geq 3(ab + bc + ca) = 18$. Next if two variables are 0 (say $c = d = 0$) then the inequality becomes $a + b \geq 3\sqrt{2}$ for nonnegative a, b satisfying $ab = 6$; which follows from the AM-GM inequality $a + b \geq 2\sqrt{ab} = 2\sqrt{6} > 3\sqrt{2}$. Clearly not more than two variables can be 0 due to the given constraint. So from now we assume that $a, b, c, d > 0$ and use the method of Lagrange Multipliers to proceed further. First we note that $f(a, b, c, d) := a + b + c + d + (3\sqrt{2} - 4)abcd$ grows unbounded for $c, d \to$ and $ab \to 6$ and so $f(a, b, c, d)$ doesn't have a maxima. So for a minima (which must occur since f is strictly greater than 0) to occur, the following system of equations must be satisfied for some real constant λ:

$$1 + (3\sqrt{2} - 4)bcd = \lambda(b + c + d) \quad (1)$$

$$1 + (3\sqrt{2} - 4)cda = \lambda(c + d + a) \quad (2)$$

$$1 + (3\sqrt{2} - 4)dab = \lambda(d + a + b) \quad (3)$$

$$1 + (3\sqrt{2} - 4)abc = \lambda(a + b + c) \quad (4)$$

Since the inequality is symmetric in all the variables, $a = b = c \neq d$ is same as $b = c = d \neq a$ and etc. So we can have the following five cases:

(i) a, b, c, d are all equal

Clearly, in this case we get $a = b = c = d = 1$ and the inequality is trivially true.

(ii) a, b, c, d are distinct

$(2) - (1)$ yields $(3\sqrt{2} - 4)cd(a - b) = \lambda(a - b) \Rightarrow \lambda = (3\sqrt{2} - 4)cd$. Similar pairwise subtractions yield $ab = bc = cd = da = ac = bd$ which forces $a = b = c = d = 1$ (using the constraint). So a minima cannot occur for distinct a, b, c, d.

(iii) $a = b$ and b, c, d are distinct

From the pairwise subtractions $(3) - (2)$ and $(4) - (3)$, we get $ab = ad \Rightarrow b = d$, a contradiction to our initial assumption and so minima cannot occur in this case.

(iv) $a = b = c \neq d$

$(1) - (4)$ gives $\lambda = (3\sqrt{2} - 4)bc$. Putting this back into (1), we get

$$1 + (3\sqrt{2} - 4)a^2 d = (3\sqrt{2} - 4)a^2(2a + d) \Rightarrow (3\sqrt{2} - 4)a^2(2a + d - d) = 1 \Rightarrow a^3 = (2(3\sqrt{2} - 4))^{-1}.$$

Next using the sum constraint $d(a + b + c) + ab + bc + ca = 6 \Rightarrow 3a^2 + 3ad = 6 \Rightarrow a^2 + ad = 2$. Thus we get

$$f(a, b, c, d) - 3\sqrt{2} = 3a + d + (3\sqrt{2} - 4)a^3 d - 3\sqrt{2} = 3a + d + \frac{d}{2} - 3\sqrt{2} = (3/2)(2a + d - 2\sqrt{2})$$
$$= (3/2a)(2a^2 + ad - 2\sqrt{2}a) = (3/2a)(a^2 + 2 - 2\sqrt{2}a) = \frac{3(a - \sqrt{2})^2}{2a} > 0$$

Thus we showed that the inequality is true for this case (even though equality is not achieved in this case).

(v) $a = b \neq c = d$

Note that we have $ac = ad = bd = bc$. Thus from the given constraint we get $a^2 + 4ac + c^2 = 6 \Rightarrow (a + c)^2 = 6 - 2ac$. So

$$f(a, b, c, d) = 2(a + c) + (3\sqrt{2} - 4)a^2c^2 = 2\sqrt{6 - 2ac} + (3\sqrt{2} - 4)a^2c^2.$$

Since $0 \le 6ac \le a^2+4ac+c^2=6$, the above equation can be rewritten as $f(a,b,c,d)=g(x)=2\sqrt{6-2x}+(3\sqrt{2}-4)x^2$ (where $x=ac$). We need to show that $g(x)\ge 3\sqrt{2}$ for $x\in[0,1]$. Clearly

$$g'(x)=2(3\sqrt{2}-4)x-\frac{2}{\sqrt{6-2x}}<0$$

for all $0\le x\le 1$ since $x\sqrt{6-2x}=\sqrt{x^2(6-2x)}\le\left(\frac{6-2x+x+x}{3}\right)^{3/2}=2\sqrt{2}\le\frac{1}{3\sqrt{2}-4}$. Thus g is decreasing over $[0,1]$ and so the minima of g is at $x=1$ which gives the minima as $g(1)=3\sqrt{2}$ (although equality cannot hold in this case since $ac=1\Rightarrow a=c=1$, which contradicts our assumption for this case).

Finally we conclude that the minima of $f(a,b,c,d)$ is $3\sqrt{2}$ and is achieved for $(a,b,c,d)=(1,1,1,1),(\sqrt{2},\sqrt{2},\sqrt{2},0)$ and permutations.

Second solution by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina

Using AM-GM inequality we have

$$6=ab+ac+ad+bc+bd+cd\ge 6\sqrt[6]{a^3b^3c^3d^3}\Rightarrow abcd\le 1\Rightarrow$$

$$(3\sqrt{2}-4)abcd\le 3\sqrt{2}-4<3\sqrt{2}\wedge a^2b^2c^2d^2\le abcd$$

Now we have

$$a+b+c+d+(3\sqrt{2}-4)abcd\ge 3\sqrt{2}\Leftrightarrow$$

$$a+b+c+d\ge 3\sqrt{2}-(3\sqrt{2}-4)abcd\Leftrightarrow$$

$$a^2+b^2+c^2+d^2+12\ge 18+(34-24\sqrt{2})a^2b^2c^2d^2-(36-24\sqrt{2})abcd$$

Since we have

$$(34-24\sqrt{2})a^2b^2c^2d^2\Rightarrow$$

$$18+(34-24\sqrt{2})a^2b^2c^2d^2-(36-24\sqrt{2})abcd\le 18-2abcd$$

It is sufficies to prove

$$a^2+b^2+c^2+d^2+12\ge 18+2abcd\Leftrightarrow$$

$$a^2+b^2+c^2+d^2+2abcd\ge 6\Leftrightarrow$$

$$2abcd\ge ab+ac+ad+bc+bd+cd-(a^2+b^2+c^2+d^2)\Leftrightarrow$$

$$12abcd\ge(ab+ac+ad+bc+bd+cd)(ab+ac+ad+bc+bd-a^2-b^2-c^2-d^2)\Leftrightarrow$$

$$\sum_{cyc}a(b^3+c^3+d^3)+6abcd\ge\frac{1}{2}\sum_{cyc}a^2(b^2+c^2+d^2)+\sum_{cyc}abc(a+b+c)$$

Where the cyclic sum runs throught the set $\{a,b,c,d\}$ e.g

$$\sum_{cyc} a(b^3+c^3+d^3) = a(b^3+c^3+d^3)+b(c^3+d^3+a^3)+c(d^3+a^3+b^3)+d(a^3+b^3+c^3)$$

Using Schurs inequality we have

$$a(b^3+c^3+d^3+3bcd) \geq a[bc(b+c)+cd(c+d)+db(d+b)] \Rightarrow$$

$$\sum_{cyc} a(b^3+c^3+d^3)+12abcd \geq 2\sum_{cyc} abc(a+b+c) \Rightarrow$$

$$\frac{1}{2}\sum_{cyc} a(b^3+c^3+d^3)+6abcd \geq \sum_{cyc} abc(a+b+c)$$

Hence it is sufficies to prove

$$\frac{1}{2}\sum_{cyc} a(b^3+c^3+d^3) \geq \frac{1}{2}\sum_{cyc} a^2(b^2+c^2+d^2)$$

Since we have $\{3,1,0\} \succ \{2,2,0\}$ using Muirhead inequality

$$a^3b+ab^3+b^3c+bc^3+c^3a+ca^3 \geq 2(a^2b^2+b^2c^2+c^2a^2)$$

$$a^3b+ab^3+b^3d+bd^3+d^3a+da^3 \geq 2(a^2b^2+b^2d^2+d^2a^2)$$

$$b^3c+bc^3+c^3d+cd^3+d^3b+db^3 \geq 2(b^2c^2+c^2d^2+d^2b^2)$$

$$c^3d+cd^3+d^3a+da^3+a^3c+ac^3 \geq 2(c^2d^2+d^2a^2+a^2c^2)$$

Adding these we obtain

$$\sum_{cyc} a(b^3+c^3+d^3) \geq \sum_{cyc} a^2(b^2+c^2+d^2) \Rightarrow$$

$$\frac{1}{2}\sum_{cyc} a(b^3+c^3+d^3) \geq \frac{1}{2}\sum_{cyc} a^2(b^2+c^2+d^2)$$

Equality holds if and only if $a=b=c=d=1$ or $d=0, a=b=c=\sqrt{2}$ and cyclic permutations.

Third solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

Let's define

$$\sum ab = ab + bc + ca + ad + bd + cd, \quad \sum a = a + b + c + d$$

Upon homogenizing we come to

$$\sum a \left(\frac{\sum ab}{6}\right)^{\frac{3}{2}} \geq 3\sqrt{2}\left(\frac{\sum ab}{6}\right)^2 - (3\sqrt{2}-4)abcd \tag{1}$$

Since

$$\sum ab = (ab + dc) + (ca + bd) + (da + cb) \geq 3 \cdot 2\sqrt{abcd}$$

we have

$$3\sqrt{2}\left(\frac{\sum ab}{6}\right)^2 - (3\sqrt{2}-4)abcd \geq 3\sqrt{2}(abcd) - (3\sqrt{2}-4)abcd \geq 0$$

so we can square (1) getting

$$\frac{1}{216}\left(\sum a\right)^2\left(\sum ab\right)^3 \geq \left(\frac{\sqrt{2}}{12}\left(\sum ab\right)^2 - (3\sqrt{2}-4)abcd\right)^2$$

Let $d = 0$. The inequality becomes

$$\frac{1}{216}(a+b+c)^2(ab+bc+ca)^3 \geq \frac{1}{72}(ab+bc+ca)^4$$

that is

$$(a+b+c)^2(ab+bc+ca)^3 \geq 3(ab+bc+ca)^4$$

and this follows by $(a+b+c)^2 \geq 3(ab+bc+ca)$. The equality case is $(a,b,c,d)(=\sqrt{2},\sqrt{2},\sqrt{2},0)$ and cyclic.

Now let's suppose $d \neq 0$ and because of homogeneity, we can set $d = 1$ and moreover $\sum ab = 6$ yields

$$1 = d = \frac{6 - ab - bc - ac}{a+b+c}$$

The inequality becomes

$$(7 - ab - bc - ca)^2 - (3\sqrt{2} - (3\sqrt{2}-4)abc)^2 \geq 0 \tag{2}$$

This is a a symmetric inequality. Let $a+b+c = 3u$, $ab+bc+ca = 3v^2$, $abc = w^3$. (2) becomes

$$P(w^3) \doteq -(3\sqrt{2}-4)^2(w^3)^2 + 6\sqrt{2}(3\sqrt{2}-4)w^3 + (7-3v^2)^2 - 18 \geq 0$$

This is a concave parabola and $P(w^3) \geq 0$ if and only if it holds for the extreme value of the w^3. The minimum value of w^3 occurs when at least one variable out of a,b,c is zero or when at least two variables out of a,b,c are equal.
The
maximum value of w^3 occurs when at least two variables
out of a,b,c are equal.
Let's set $c = 0$ in (2). We get

$$(7-ab)^2 \geq 18 \iff (ab)^2 - 14(ab) + 31 \geq 0 \tag{3}$$

that is

$$ab \leq 7 - 3\sqrt{2} \ \vee \ ab \geq 7 + 3\sqrt{2}$$

Now remember that $d = 1$ and $c = 0$ in $\sum ab = 6$ yield $a + b + ab = 6$ and then

$$6 = a + b + ab \geq ab + 2\sqrt{ab} \iff \sqrt{ab} \leq \sqrt{7} - 1$$

Since

$$(\sqrt{7} - 1)^2 < 7 - 3\sqrt{2}$$

(3) holds.
Now let's set $b = c$ in (2). Moreover we have from $\sum ab = 6$ and $d = 1$

$$a + b + c = 6 - ab - bc - ac, \quad a = \frac{6 - c^2 - 2c}{1 + 2c} \underbrace{\Longrightarrow}_{a \geq 0} c \leq \sqrt{7} - 1$$

(2) becomes

$$\frac{1}{2(1+2c)^2}(17 - 12\sqrt{2})(2c^4 + 4c^3 + 9\sqrt{2}c^2 + 44c + 30\sqrt{2}c + 46 + 33\sqrt{2}) \times$$
$$\times(-2c^2 - 8c + 10 + 9\sqrt{2})(c - 1)^2 \geq 0$$

and

$$-2c^2 - 8c + 10 + 9\sqrt{2} \geq 0 \iff c \leq \frac{-4 + \sqrt{36 + 18\sqrt{2}}}{2}$$

The last step is to observe that

$$1.6 \sim \sqrt{7} - 1 < \frac{-4 + \sqrt{36 + 18\sqrt{2}}}{2} \sim 1.92$$

and this concludes the proof.

Also solved by Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan.

S402. Prove that
$$\sum_{k=1}^{31} \frac{k}{(k-1)^{4/5}+k^{4/5}+(k+1)^{4/5}} < \frac{3}{2} + \sum_{k=1}^{31}(k-1)^{1/5}.$$

Titu Andreescu, University of Texas at Dallas, TX, USA

First solution by Adnan Ali, A.E.C.S-4, Mumbai, India

We first show that
$$\frac{x}{(x-1)^{4/5}+x^{4/5}+(x+1)^{4/5}} < (x-1)^{1/5} \tag{1}$$
for all $x \geq 2$. Straightforward cross-multiplication yields the equivalent inequality $(x^4(x-1))^{1/5} + ((x-1)(x+1)^4)^{1/5} > 1$, which is true since
$$(x^4(x-1))^{1/5} + ((x-1)(x+1)^4)^{1/5} \geq 2^{4/5} + 3^{4/5} > 1 + 1 > 1.$$

Moreover
$$\frac{1}{1+2^{4/5}} < \frac{1}{2/3} = 3/2 \tag{2}$$
as $1 + 2^{4/5} > 1 + 1 > 2/3$. Clearly, from (1) and (2), the proposed inequality follows.

Second solution by Alessandro Ventullo, Milan, Italy

Observe that
$$\frac{k}{(k-1)^{4/5}+k^{4/5}+(k+1)^{4/5}} < \frac{k}{3(k-1)^{4/5}},$$
so
$$\begin{aligned}\sum_{k=1}^{31}\left(\frac{k}{(k-1)^{4/5}+k^{4/5}+(k+1)^{4/5}} - (k-1)^{1/5}\right) &< \frac{1}{1+2^{4/5}} - \frac{1}{3}\sum_{k=2}^{31}\frac{2k-3}{(k-1)^{4/5}} \\ &< \frac{1}{1+2^{4/5}} - \frac{1}{3}\sum_{k=2}^{31}\frac{2k-3}{k-1} \\ &< \frac{1}{1+2^{4/5}} - \frac{1}{3}\left(1+\frac{3}{2}\right) < 0.\end{aligned}$$

So,
$$\sum_{k=1}^{31}\left(\frac{k}{(k-1)^{4/5}+k^{4/5}+(k+1)^{4/5}} - (k-1)^{1/5}\right) < 0 < \frac{3}{2}.$$

Also solved by Catalin Prajitura, College at Brockport, SUNY, NY, USA; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina

Undergraduate problems

U397. Let T_n be the n-the triangular number. Evaluate

$$\sum_{n\geq 1} \frac{1}{(8T_n-3)(8T_{n+1}-3)}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Adnan Ali, AECS-4, Mumbai, India

By partial fraction decomposition,

$$\begin{aligned}\frac{1}{(8T_n-3)(8T_{n+1}-3)} &= \frac{1}{(4n(n+1)-3)(4(n+1)(n+2)-3)}\\ &= \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)}\\ &= \frac{1}{16}\left(\frac{1}{2n+3}-\frac{1}{2n+1}\right)+\frac{1}{48}\left(\frac{1}{2n-1}-\frac{1}{2n+5}\right)\end{aligned}$$

Thus,

$$\sum_{n\geq 1} \frac{1}{(8T_n-3)(8T_{n+1}-3)} = \frac{1}{16}\left(-\frac{1}{3}\right)+\frac{1}{48}\left(1+\frac{1}{3}+\frac{1}{5}\right)=\frac{1}{90}.$$

Second solution by Robert Bosch, USA

$$\sum_{n=1}^{\infty} \frac{1}{(8T_n-3)(8T_{n+1}-3)} = \sum_{n=1}^{\infty} \frac{1}{(4n^2+4n-3)(4n^2+12n+5)} = \frac{1}{90}.$$

Let us prove this result.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(4n^2+4n-3)(4n^2+12n+5)} &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)},\\ &= \sum_{n=1}^{\infty} \frac{8n(n+1)(n+2)(2n-2)!}{(2n+5)!},\\ &= \sum_{n=1}^{\infty} \frac{8}{\Gamma(7)}\cdot\frac{n(n+1)(n+2)\Gamma(2n-1)\Gamma(7)}{\Gamma(2n+6)},\\ &= \frac{1}{90}\sum_{n=1}^{\infty} n(n+1)(n+2)\beta(2n-1,7),\\ &= \frac{1}{90}\sum_{n=1}^{\infty} n(n+1)(n+2)\int_0^1 x^{2n-2}(1-x)^6dx,\\ &= \frac{1}{90}\int_0^1 (1-x)^6\sum_{n=1}^{\infty} n(n+1)(n+2)x^{2n-2}dx.\end{aligned}$$

With the aid of *Maple* we obtain

$$\sum_{n=1}^{\infty} n(n+1)(n+2)x^{2n-2} = \frac{6}{(1-x^2)^4},$$

thus the next step is to calculate the following integral

$$\begin{aligned}
\int_0^1 \frac{6(1-x)^6}{(1-x^2)^4}dx &= \int_0^1 \frac{6(1-x)^2}{(x+1)^4}dx, \\
&= \int_0^1 \frac{6}{(x+1)^2}dx - \int_0^1 \frac{24}{(x+1)^3}dx + \int_0^1 \frac{24}{(x+1)^4}dx, \\
&= -\frac{6}{x+1}\Big|_0^1 + \frac{12}{(x+1)^2}\Big|_0^1 - \frac{8}{(x+1)^3}\Big|_0^1, \\
&= 1.
\end{aligned}$$

Also solved by Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Zafar Ahmed, BARC, Mumbai, India and Dona Ghosh, JU, Jadavpur, India; Alessandro Ventullo, Milan, Italy; Alok Kumar, New Delhi, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Arpon Basu, AECS-4, Mumbai, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Joel Schlosberg, Bayside, NY, USA; Julio Cesar Mohnsam, IF Sul - Campus Pelotas -RS, Brazil; Khakimboy Egamberganov, National University of Uzbekistan, Tashkent, Uzbekistan; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Albert Stadler, Herrliberg, Switzerland; Li Zhou, Polk State College, Winter Haven, FL, USA.

U398. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq 3\left(\frac{1}{3a+b}+\frac{1}{3b+c}+\frac{1}{3c+a}\right)$$

Proposed by Sardor Bazarbaev, National University of Uzbekistan, Tashkent, Uzbekistan

Solution by Khakimboy Egamberganov, National University of Uzbekistan, Tashkent, Uzbekistan

We will use the following lemma.

Lemma: If x, y, z are real numbers, then

$$(x^2+y^2+z^2)^2 \geq 3(x^3y+y^3z+z^3x).$$

Proof: From

$$(x^2+y^2+z^2)^2-3(x^3y+y^3z+z^3x)=\frac{1}{2}\Big(\sum_{cyc}(x^2-y^2-xy-zx+2yz)^2\Big)\geq 0$$

the inequality is true for all $x, y, z \in \mathbb{R}$.

Now if $x=t^a$, $y=t^b$ and $z=t^c$ with $t \in (0,1]$, from the lemma we get

$$t^{4a-1}+t^{4b-1}+t^{4c-1}+2t^{2a+2b-1}+2t^{2b+2c-1}+2t^{2c+2a-1}\geq 3(t^{3a+b-1}+t^{3b+c-1}+t^{3c+a-1}).$$

So

$$\int_0^1(t^{4a-1}+t^{4b-1}+t^{4c-1}+2t^{2a+2b-1}+2t^{2b+2c-1}+2t^{2c+2a-1})\,dt\geq\int_0^1 3(t^{3a+b-1}+t^{3b+c-1}+t^{3c+a-1})\,dt$$

and

$$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq 3\Big(\frac{1}{3a+b}+\frac{1}{3b+c}+\frac{1}{3c+a}\Big).$$

Also solved by Li Zhou, Polk State College, FL, USA; Adnan Ali, AECS-4, Mumbai, India; Arkady Alt, San Jose, CA, USA; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy.

U399. Consider the functional equation $f(f(x)) = f(x)^2$, where $f : \mathbb{R} \to \mathbb{R}$.

(a) Find all real analytic solutions of the equation.

(b) Prove that there exist infinitely many differentiable solutions of the equation.

(c) Do only finitely many infinitely differentiable solutions exist?

Proposed by David Rose and Li Zhou, Polk State College, Florida, USA

Solution by the authors

Let $y = f(x)$ be any number in the range of f. Then, $f(y) = f(f(x)) = (f(x))^2 = y^2$. If f is a constant function, say $f(x) = c$. Then, $c = f(c) = c^2$ so that either $c = 0$ or $c = 1$. If f is nonconstant and continuous, then there is an open interval I in the range and thus, $f(x) = x^2$ for $x \in I$.

(a) The constant functions $f(x) = 0$ and $f(x) = 1$ are real analytic on $(-\infty, \infty)$. If f is a nonconstant real analytic function on $(-\infty, \infty)$ then being uniquely determined by its values on a nonempty open interval, we have $f(x) = x^2$ for all real x.

(b) For each $a > 0$, the function f_a defined by

$$f_a(x) = \begin{cases} ax^2 & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x \end{cases}$$

satisfies $f_a \circ f_a = f_a^2$ and f_a is differentiable at each real x. However, f_a is not twice differentiable at $x = 0$ for $a \neq 1$. So, uncountably many differentiable solutions have been found.

(c) For each $c > 0$, let f_c be defined by

$$f_c(x) = \begin{cases} x^2 + e^{c/x} & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x \end{cases}.$$

Then f_c is infinitely differentiable on $(-\infty, \infty)$ and satisfies $f_c \circ f_c = f_c^2$. Thus, uncountably many infinitely differentiable solutions have been exhibited.

U400. Let A and B be 3×3 matrices with integer entries such that $AB = BA$, $\det(B) = 0$, and $\det(A^3 + B^3) = 1$. Find all possible polynomials $f(x) = \det(A + xB)$.

Proposed by Florin Stanescu, Gaesti, Romania

First solution by Li Zhou, Polk State College, USA

If $A = I$ and $B = 0$, then $f(x) = 1$. If $A = \text{diag}[0,1,1]$ and $B = \text{diag}[1,0,0]$, then $f(x) = x$. If $A = \text{diag}[0,0,1]$ and $B = \text{diag}[1,1,0]$, then $f(x) = x^2$. We show that these are the only possibilities of $f(x)$.

Since the coefficient of x^3 in $\det(A+xB)$ is $\det(B)$, $f(x) = ax^2 + bx + c$, with $a, b, c \in \mathbb{Z}$. Let $\omega = e^{2\pi i/3}$, then $A^3 + B^3 = (A+B)(A+\omega B)(A+\omega^2 B)$, so

$$1 = \det(A^3 + B^3) = f(1)f(\omega)f(\omega^2) = f(1)|f(\omega)|^2.$$

Therefore, $1 = f(1) = a + b + c$ and

$$1 = |f(\omega)|^2 = a^2 + b^2 + c^2 - ab - bc - ca = \frac{3}{2}\left(a^2 + b^2 + c^2\right) - \frac{1}{2}(a+b+c)^2.$$

Thus $a^2 + b^2 + c^2 = 1$ as well, which forces $(a, b, c) = (0,0,1)$, $(0,1,0)$, or $(1,0,0)$.

Second solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Let w be $w^3 = 1$. Then we have $\overline{w} = w^2$,

$$\begin{aligned} A^3 + B^3 &= (A+B)(A+wB)(A+\overline{w}B) \\ &= (A+B)(A+wB)(\overline{A+wB}). \end{aligned}$$

Using $\det(A^3 + B^3) = 1$, we get

$$\begin{aligned} 1 &= \det(A+B) \cdot \det(A+wB) \cdot \det(A+\overline{w}B) \\ &= \det(A+B) \cdot \det(A+wB) \cdot \det(\overline{A+wB}) \\ &= \det(A+B) \cdot |\det(A+wB)|^2 \end{aligned}$$

Hence we have

$$\det(A+B) = 1 \text{ and } \det(A+wB) \cdot \det(A+\overline{w}B) = 1.$$

Using $\det(B) = 0$, we get

$$f(x) = \det(A+xB) = ax^2 + bx + c,$$

where $c = \det(A)$.

Choosing $x = 0$, we get $a + b + c = 1$

Choosing $x = w$, we get $aw^2 + bw + c = \det(A+wB)$

Choosing $x = \overline{w}$, we get $a\overline{w}^2 + b\overline{w} + c = \det(A+\overline{w}B)$

Hence we have

$$\begin{aligned} 1 &= (aw^2 + bw + c)(a\overline{w}^2 + b\overline{w} + c) \\ &= a^2 + b^2 + c^2 - ab - bc - ca. \end{aligned}$$

$$\left.\begin{aligned} a^2 + b^2 + c^2 - ab - bc - ca = 1 \\ a + b + c = 1 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} ab + bc + ca = 0 \\ a^2 + b^2 + c^2 = 1 \end{aligned}\right\}.$$

$c^2 \le 1$ and $c \in \mathbb{Z}$, hence we check only the following numbers $c \in \{-1, 0, 1\}$.
1) Let $c = 0$. Then $ab = 0$ and $a + b = 1$. Hence we get

$$a = 0, b = 1 \text{ or } a = 1,\ b = 0.$$

2) Let $c = 1$. Then $a = b = 0$.
3) Let $c = -1$. Then $a = b = 0$. That is contradicts the $a + b + c = 1$.

Hence $f(x) = x^2$ or $f(x) = x$ or $f(x) = 1$.

a) $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow f(x) = x^2$

b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow f(x) = x$

c) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow f(x) = 1$

U401. Let P be a polynomial of degree n such that $P(k) = \frac{1}{k^2}$ for all $k = 1, 2, \cdots, n+1$. Determine $P(n+2)$.

Proposed by Dorin Andrica, Babeș-Bolyai University, Cluj-Napoca, Romania

First solution by Robert Bosch, USA

The idea is to use Lagrange's interpolation formula. Say, given that $p(x_i) = a_i$ for $i \in \{0, 1, ..., n\}$ the polynomial $p(x)$ is

$$p(x) = \sum_{i=0}^{n} a_i \prod_{0 \le j \ne i \le n} \frac{x - x_j}{x_i - x_j}.$$

In our problem $p(k) = \frac{1}{k^2}$ for $k = 1, 2, ..., n+1$, and we need to find the value of $p(n+2)$. We have that

$$p(x) = \sum_{i=1}^{n+1} \frac{1}{i^2} \prod_{1 \le j \ne i \le n+1} \frac{x - j}{i - j},$$

so

$$p(n+2) = \sum_{k=0}^{n} \frac{1}{(k+1)^2} \binom{n+1}{k} (-1)^{n-k}.$$

Second solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Consider the $Q(x) = x^2 P(x) - 1$. From the given conditions, we get

$$\forall k \in \{1, 2, \ldots, n+1\} : \; k^2 P(k) - 1 = 0.$$

Hence we have

$$Q(x) = x^2 P(x) - 1 = a_0 (x-1)(x-2) \cdot \ldots \cdot (x - (n+1))(x - a).$$

Substituting $x = 0:$

$$-1 = a_0(-1)(-2) \cdot \ldots \cdot (-(n+1)) \cdot (-a) \;\Leftrightarrow\; 1 = a_0 (-1)^{n+1} (n+1)! \cdot a$$

If $x \ne 1, 2, \ldots, n+1, a$, we get

$$\frac{Q'(x)}{Q(x)} = \frac{1}{x-1} + \frac{1}{x-2} + \ldots + \frac{1}{x-(n+1)} + \frac{1}{x-a}$$

and

$$Q'(x) = 2x \cdot P(x) + x^2 P'(x).$$

Substituting $x = 0:$

$$0 = -\left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n+1} + \frac{1}{a}\right) \;\Leftrightarrow\; \frac{1}{a} = -\left(1 + \frac{1}{2} + \ldots + \frac{1}{n+1}\right) = -H_{n+1},$$

$$a_0 = \frac{(-1)^{n+1}}{(n+1)!} \cdot \frac{1}{a} = \frac{(-1)^n}{(n+1)!} \cdot H_{n+1}.$$

Hence, we have

$$\begin{aligned} (n+2)^2 P(n+2) - 1 &= a_0 (n+1)!(n+2-a) \\ \Leftrightarrow\; P(n+2) &= \frac{a_0 (n+1)!(n+2-a) + 1}{(n+2)^2} \\ &= \frac{(-1)^n \cdot H_{n+1} \cdot (n+2+H_{n+1}) + 1}{(n+2)^2}. \end{aligned}$$

Also solved by Adnan Ali, AECS-4, Mumbai, India; Zafar Ahmed, BARC, Mumbai, India and Dona Ghosh, JU, Jadavpur, India; Li Zhou, Polk State College, Winter Haven, FL, USA; Alessandro Ventullo, Milan, Italy; Arkady Alt, San Jose, CA, USA; Arpon Basu, AECS-4, Mumbai, India; Khakimboy Egamberganov, National University of Uzbekistan, Tashkent, Uzbekistan; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Albert Stadler, Herrliberg, Switzerland.

U402. Let n be a positive integer and let $P(x)$ be a polynomial of degree at most n such that $|P(x)| \le x+1$ for all $x \in [0,n]$. Prove that

$$|P(n+1)| + |P(-1)| \le (n+2)(2^{n+1}-1).$$

Alessandro Ventulo, Milan, Italy

First solution by Li Zhou, Polk State College, USA

Since $P(x)$ is of degree at most n, its $(n+1)$-th finite difference vanishes. That is, for all x,

$$\binom{n+1}{0}P(x) - \binom{n+1}{1}P(x-1) + \binom{n+1}{2}P(x-2) - \cdots + (-1)^{n+1}\binom{n+1}{n+1}P(x-n-1) = 0.$$

In particular, letting $x = n+1$ and $x = n$ we get

$$|P(n+1)| \le \binom{n+1}{1}|P(n)| + \binom{n+1}{2}|P(n-1)| + \cdots + \binom{n+1}{n+1}|P(0)|,$$

$$|P(-1)| \le \binom{n+1}{0}|P(n)| + \binom{n+1}{1}|P(n-1)| + \cdots + \binom{n+1}{n}|P(0)|.$$

Hence,

$$\begin{aligned} |P(n+1)| + |P(-1)| &\le \binom{n+2}{1}|P(n)| + \binom{n+2}{2}|P(n-1)| + \cdots + \binom{n+2}{n+1}|P(0)| \\ &\le (n+1)\binom{n+2}{1} + n\binom{n+2}{2} + \cdots + \binom{n+2}{n+1}. \end{aligned}$$

Now by differentiating the binomial series of $(t+1)^{n+2}$ we get

$$(n+2)(t+1)^{n+1} = (n+2)\binom{n+2}{0}t^{n+1} + (n+1)\binom{n+2}{1}t^{n} + \cdots + \binom{n+2}{n+1}.$$

Letting $t = 1$ yields

$$(n+1)\binom{n+2}{1} + n\binom{n+2}{2} + \cdots + \binom{n+2}{n+1} = (n+2)2^{n+1} - (n+2),$$

which completes the proof.

Second solution by Khakimboy Egamberganov, National University of Uzbekistan, Tashkent, Uzbekistan

Since the condition $\deg(P(x)) \le n$, we can find by Lagrange Interpolation Formula

$$P(x) = \sum_{i=0}^{n} \frac{(-1)^{n-i}\prod_{j=\overline{0,n},j\ne i}(x-j)}{i!(n-i)!}P(i).$$

After that we can easily find

$$|P(n+1)| = \left|(n+1)!\Big(\sum_{i=0}^{n} \frac{(-1)^{n-i}P(i)}{i!(n+1-i)!}\Big)\right| \le (n+1)!\Big(\sum_{i=0}^{n} \frac{i+1}{i!(n+1-i)!}\Big) = \sum_{i=0}^{n}\binom{n+1}{i}\cdot(i+1)$$

and

$$|P(-1)| = \left|(-1)^{n+1}(n+1)!\Big(\sum_{i=0}^{n} \frac{(-1)^{n-i}P(i)}{-(i+1)!(n-i)!}\Big)\right| \leq (n+1)!\Big(\sum_{i=0}^{n} \frac{i+1}{(i+1)!(n-i)!}\Big) = \sum_{i=0}^{n} \binom{n+1}{i+1} \cdot (i+1).$$

So

$$|P(n+1)| + |P(-1)| \leq \sum_{i=0}^{n} \binom{n+1}{i} \cdot (i+1) + \sum_{i=0}^{n} \binom{n+1}{i+1} \cdot (i+1)$$

and since $\binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1}$ we get

$$|P(n+1)| + |P(-1)| \leq \sum_{i=0}^{n} \binom{n+2}{i+1} \cdot (i+1) = (n+2) \sum_{i=0}^{n} \binom{n+1}{i} = (n+2)(2^{n+1} - 1).$$

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Robert Bosch, USA.

Olympiad problems

O397. Solve in integers the equation:

$$(x^3-1)(y^3-1)=3(x^2y^2+2).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Alessandro Ventullo, Milan, Italy

The given equation can be written as

$$x^3y^3-(x^3+y^3)-3x^2y^2=5.$$

Let $s=x+y$ and $t=xy$. Observe that $x^3+y^3=(x+y)^3-3xy(x+y)=s^3-3st$, so the given equation becomes

$$(t^3-s^3)-3t(t-s)=5,$$

i.e.

$$(t-s)(t^2+ts+s^2-3t)=5.$$

We obtain the four systems of equations:

$$\begin{aligned} t-s &= \pm 1 \\ t^2+ts+s^2-3t &= \pm 5, \end{aligned} \qquad \begin{aligned} t-s &= \pm 5 \\ t^2+ts+s^2-3t &= \pm 1 \end{aligned}$$

If $t-s=\pm 1$, then $t^2-2ts+s^2=1$ and subtracting this equation to the second equation, we get $3ts-3t=4$ or $3ts-3t=-6$. The first equation is impossible, the second gives $t(s-1)=-2$. So, $(s,t)\in\{(-1,1),(0,2),(2,-2),(3,-1)\}$. It's easy to see that none of these pairs satisfies $t-s=\pm 1$, so there are no solutions in this case. If $t-s=\pm 5$, then $t^2-2ts+s^2=25$ and subtracting this equation to the second equation, we get $3ts-3t=-24$ or $3ts-3t=-26$. The second equation is impossible, the first gives $t(s-1)=-8$. So, $(s,t)\in\{(-7,1),(-3,2),(-1,4),(0,8),(2,-8),(3,-4),(5,-2),(9,-1)\}$. As $t-s=\pm 5$, we obtain $(s,t)\in\{(-3,2),(-1,4)\}$. Since s and t also satisfy the condition $s^2-4t=n^2$ for some $n\in\mathbb{Z}$, we obtain $(s,t)=(-3,2)$. So, $x+y=-3$ and $xy=2$, which gives $(x,y)\in\{(-1,-2),(-2,-1)\}$.

Second solution by José Hernández Santiago, México

The equation is equivalent to

$$x^3y^3-x^3-y^3-3x^2y^2=5. \tag{1}$$

Then, applying the well-known identity $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ to the expression in the left-hand side of (1), we can further rewrite the original equation as

$$(xy-x-y)((xy)^2+x^2+y^2+x^2y-xy+xy^2)=5.$$

Since 5 is a prime number, it follows that $xy-x-y=\pm 1$ or $xy-x-y=\pm 5$:

If $xy-x-y=1$, then $(x-1)(y-1)=2$. There are four possibilities for the pair (x,y) in this case: $(x=2,y=3),(x=0,y=-1),(x=3,y=2)$, and $(x=-1,y=0)$. None of these pairs is a solution of the original equation.

If $xy-x-y=-1$, then $(x-1)(y-1)=0$. Since $3(x^2y^2+2)>0$, no solution of $(x-1)(y-1)=0$ yields a solution of the original equation.

If $xy - x - y = 5$, then $(x-1)(y-1) = 6$. There are eight possibilities for the pair (x, y) in this case: $(x = 2, y = 7), (x = 0, y = -5), (x = 3, y = 4), (x = -1, y = -2), (x = 4, y = 3), (x = -2, y = -1), (x = 7, y = 2)$, and $(x = -5, y = 0)$. Within these pairs, there are only two which satisfy the original equation: $(x = -1, y = -2)$ and $(x = -2, y = -1)$.

If $xy - x - y = -5$, then $(x-1)(y-1) = -4$. There are six possibilities for the pair (x, y) in this case: $(x = 2, y = -3), (x = 3, y = -1), (x = 5, y = 0), (x = 0, y = 5), (x = -1, y = 3)$, and $(x = -3, y = 2)$. We see that none of these six pairs yields a solution of the original equation, either.

The previous analysis allows us to conclude that there are only two solutions to the given equation: $(x = -1, y = -2)$ and $(x = -2, y = -1)$.

Also solved by Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Adnan Ali, AECS-4, Mumbai, India; Arpon Basu, AECS-4, Mumbai, India; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nikos Kalapodis, Patras, Greece; Ricardo Largaespada, Universidad Nacional de IngenierÃa, Managua, Nicaragua; Robert Bosch, USA; Tolibjon Ismoilov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan.

O398. Let a, b, c, d be positive real numbers such that $abcd \geq 1$. Prove that

$$\frac{1}{a+b^5+c^5+d^5}+\frac{1}{b+c^5+d^5+a^5}+\frac{1}{c+d^5+a^5+b^5}+\frac{1}{d+a^5+b^5+c^5} \leq 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

Since

$$(3b^5+c^5+d^5)/5 \geq b^3cd$$

and $bcd \geq 1/a$, we have

$$a+b^5+c^5+d^5 \geq a+bcd(b^2+c^2+d^2) \geq a+\frac{1}{a}(b^2+c^2+d^2)=\frac{a^2+b^2+c^2+d^2}{a}$$

hence

$$\sum_{\text{cyc}} \frac{1}{a+b^5+c^5+d^5} \leq \sum_{\text{cyc}} \frac{a}{a^2+b^2+c^2+d^2}=\frac{a+b+c+d}{a^2+b^2+c^2+d^2}$$

Moreover $a^2+b^2+c^2+d^2 \geq \sqrt[4]{abcd}(a+b+c+d)$
whence

$$\frac{a+b+c+d}{a^2+b^2+c^2+d^2} \leq \frac{a+b+c+d}{\sqrt[4]{abcd}(a+b+c+d)}=\frac{1}{\sqrt[4]{abcd}} \leq 1$$

$\sqrt[4]{abcd}(a+b+c+d)$ follows by (AGM)

$$(5a^2+b^2+c^2+d^2)/8 \geq \sqrt[4]{abcd}(a+b+c+d).$$

Also solved by Adnan Ali, AECS-4, Mumbai, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Tolibjon Ismoilov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan.

O399. Let a, b, c be positive real numbers. Prove that

$$\frac{a^5+b^5+c^5}{a^2+b^2+c^2} \geq \frac{1}{2}(a^3+b^3+c^3-abc).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Adnan Ali, AECS-4, Mumbai, India

From Schur's and MÃ$\frac{1}{4}$irhead's inequalities (applied in succession), we have

$$\sum_{cyc} a^3(a-b)(a-c) \geq 0$$

$$\Rightarrow a^5+b^5+c^5+abc(a^2+b^2+c^2) \geq \sum_{cyc}(a^4b+ab^4) \geq \sum_{cyc}(a^3b^2+a^2b^3) = (a^2+b^2+c^2)(a^3+b^3+c^3)-(a^5+b^5+c^5).$$

Thus we get

$$2(a^5+b^5+c^5) \geq (a^2+b^2+c^2)(a^3+b^3+c^3-abc) \Rightarrow \frac{a^5+b^5+c^5}{a^2+b^2+c^2} \geq \frac{1}{2}(a^3+b^3+c^3-abc),$$

and the problem follows. Equality holds iff $a=b=c$.

Second solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Given inequality equivalent to following inequality

$$a^5+b^5+c^5+abc(a^2+b^2+c^2) \geq a^3(b^2+c^2)+b^3(c^2+a^2)+c^3(a^2+b^2) \quad (1)$$

By the Schur's inequality, we get

$$\sum a^3(a-b)(a-c) \geq 0 \;\Leftrightarrow\; \sum a^5 + \sum a^3bc \geq \sum a^4(b+c) \quad (2)$$

Using AM-GM inequality, we have

$$\begin{aligned} a^4(b+c)+b^4(c+a)+c^4(a+b) &= \frac{4a^4b+2b^4a}{6}+\frac{4a^4c+2c^4a}{6}+\frac{4b^4a+2a^4b}{6} \\ &\quad+\frac{4b^4c+2c^4b}{2}+\frac{4c^4a+2a^4c}{6}+\frac{4c^4b+2b^4c}{6} \\ &\geq a^3b^2+a^3c^2+b^3a^2+b^3c^2+c^3a^2+c^3b^2 \\ &= a^3(b^2+c^2)+b^3(c^2+a^2)+c^3(a^2+b^2) \end{aligned} \quad (3)$$

From the (2) and (3), get the (1). Equality holds only when $a=b=c$.

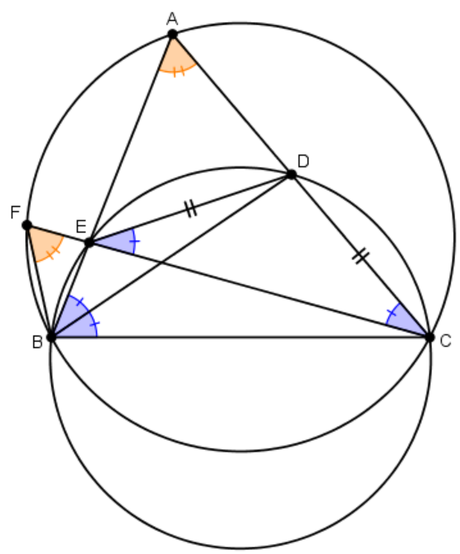
Also solved by Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Nikos Kalapodis, Patras, Greece; Arkady Alt, San Jose, CA, USA; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Rajdeep Majumder, Durgapur, India; Robert Bosch, USA; Albert Stadler, Herrliberg, Switzerland; Tolibjon Ismoilov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan.

O400. Let ABC be a triangle and let BD be the angle bisector of $\angle ABC$. The circumcircle of triangle BCD intersects the side AB at E such that E lies between A and B. The circumcircle of triangle ABC intersects the line CE at F. Prove that

$$\frac{BC}{BD}+\frac{BF}{BA}=\frac{CE}{CD}.$$

Proposed by Florin Stanescu, Caesti, România

Solution by Nikos Kalapodis, Patras, Greece



Since quadrilateral $EDCB$ is cyclic by the law of sines in triangle DEC we have

$$\frac{CE}{CD}=\frac{\sin\angle EDC}{\sin\angle CED}=\frac{\sin(180^\circ-B)}{\sin\angle CBD}=\frac{\sin B}{\sin\dfrac{B}{2}}=2\cos\frac{B}{2}. \ (1)$$

By the law of sines in trianles DBC, FBC and ABC we have

$$\frac{BC}{BD}+\frac{BF}{BA}=\frac{\sin\angle BDC}{\sin\angle DCB}+\frac{2R\sin\angle FCB}{2R\sin\angle ACB}=\frac{\sin\left(180^\circ-C-\dfrac{B}{2}\right)}{\sin C}+\frac{\sin(C-\angle DCE)}{\sin C}=$$

$$\frac{\sin\left(C+\dfrac{B}{2}\right)}{\sin C}+\frac{\sin\left(C-\dfrac{B}{2}\right)}{\sin C}=\frac{2\sin C\cos\dfrac{B}{2}}{\sin C}=2\cos\frac{B}{2}. \ (2)$$

From (1) and (2) we obtain $\dfrac{BC}{BD}+\dfrac{BF}{BA}=\dfrac{CE}{CD}$.

Also solved by Soo Young Choi, Vestal Senior Highschool, NY, USA; Anderson Torres, Sao Paulo, Brazil; Adnan Ali, AECS-4, Mumbai, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Robert Bosch, USA; Tolibjon Ismoilov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan.

O401. Let a, b, c be positive real numbers. Prove that

$$\sqrt{\frac{9a+b}{9b+a}}+\sqrt{\frac{9b+c}{9c+b}}+\sqrt{\frac{9c+a}{9a+c}}\geq 3.$$

Proposed by An Zhenping, Xianyang Normal University, China

First solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

The inequality is

$$\frac{9+\frac{b}{a}}{9\frac{b}{a}+1}+\frac{9+\frac{c}{b}}{9\frac{b}{c}+1}+\frac{9+\frac{a}{c}}{9\frac{c}{a}+1}\geq 3$$

Now by defining $x=b/a$, $y=c/b$, $z=a/c$ we get

$$\sqrt{\frac{9+x}{9x+1}}+\sqrt{\frac{9+y}{9y+1}}+\sqrt{\frac{9+z}{9z+1}}\geq 3, \quad xyz=1$$

Then by setting $x=e^r$, $y=e^s$, $z=e^t$ we get

$$\sqrt{\frac{9+e^r}{9e^r+1}}+\sqrt{\frac{9+e^s}{9e^s+1}}+\sqrt{\frac{9+e^t}{9e^t+1}}\geq 3, \quad r+s+t=0$$

$$\left(\sqrt{\frac{9+e^r}{9e^r+1}}\right)''=40\frac{(9e^{2r}+40e^r-9)e^r}{(9+e^r)(9e^r+1)^3\sqrt{\frac{(9+r^r)}{(9r^r+1)}}}\geq 0$$

$$\iff \quad e^r\leq\frac{-20-\sqrt{481}}{9} \;\wedge\; e^r\geq\frac{-20+\sqrt{481}}{9}$$

It follows that the function changes concavity one time and the standard theory states that its minimum is achieved when $r=s$ (or cyclic). So we set $t=s$ or $y=x$ ($z=1/x^2$) an get

$$2\sqrt{\frac{9+x}{9x+1}}+\sqrt{\frac{9+\frac{1}{x^2}}{9\frac{1}{x^2}+1}}\geq 3 \iff \sqrt{\frac{9x^2+1}{9+x^2}}\geq 3-2\sqrt{\frac{9+x}{9x+1}} \tag{1}$$

$$x\leq 27/77 \implies 3-2\sqrt{\frac{9+x}{9x+1}}\leq 0$$

so (1) is satisfied. If $x>27/77>1/3$ we square

$$12\sqrt{\frac{9+x}{9x+1}}\geq 9+4\frac{9+x}{9x+1}+\frac{9x^2+1}{9+x^2}$$

Squaring again we get

$$640\frac{(2x^4+22x^3+69x^2+358x-91)(x-1)^2}{(9x+1)^2(9+x^2)^2}\geq 0, \quad x\geq 27/77$$

and clearly the inequality holds.

Second solution by Adnan Ali, AECS-4, Mumbai, India

Let $x = b/a, y = c/b$ and $z = a/c$. Then the inequality is equivalent to

$$\sqrt{\frac{9+x}{9x+1}} + \sqrt{\frac{9+y}{9y+1}} + \sqrt{\frac{9+z}{9z+1}} \geq 3,$$

for positive real numbers x, y, z such that $xyz = 1$. Define $f(x) = \sqrt{\frac{9+x}{9x+1}} + \frac{2}{5}\ln x - 1$. Then $f(x) = 0$ has exactly two roots for $x > 0$. They are $x = 1$ and $x = x_0 \approx 0.009$. Moreover for $0 < r < x_0$, $f(r) < 0$ while for $r \geq x_0$, $f(r) \geq 0$. So if $x, y, z \in [x_0, \infty)$, we get

$$\sum\nolimits_{cyc} \sqrt{\frac{9+x}{9x+1}} \geq \sum\nolimits_{cyc} \left(1 - \frac{2\ln x}{5}\right) = 3.$$

Next if $x < x_0$, then from $9(9+t) - 1(9t+1) > 0$ we get

$$\sum\nolimits_{cyc} \sqrt{\frac{9+x}{9x+1}} > \sqrt{\frac{9+x_0}{9x_0+1}} + 2\sqrt{\frac{1}{9}} > 3,$$

and the problem follows. Equality holds for $a = b = c$.

Also solved by Arpon Basu, AECS-4, Mumbai, India; Jae Woo Lee, Hamyang-gun, South Korea; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Tolibjon Ismoilov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan.

O402. Prove that in any triangle ABC, the following inequality holds

$$\sin^2 2A + \sin^2 2B + \sin^2 2C \geq 2\sqrt{3} \sin 2A \sin 2B \sin 2C.$$

Proposed by Dorin Andrica, Babeș-Bolyai University, Cluj-Napoca, Romania

First solution by Robert Bosch, USA

Lemma:

$$\sin^2 2A + \sin^2 2B + \sin^2 2C \geq \frac{16}{3} \sin^2 A \sin^2 B \sin^2 C.$$

Proof:

$$(\sin 2A - \sin 2B)^2 + (\sin 2B - \sin 2C)^2 + (\sin 2C - \sin 2A)^2 \geq 0,$$
$$\Leftrightarrow 3\left(\sin^2 2A + \sin^2 2B + \sin^2 2C\right) \geq (\sin 2A + \sin 2B + \sin 2C)^2,$$
$$\Leftrightarrow \sin^2 2A + \sin^2 2B + \sin^2 2C \geq \frac{16}{3} \sin^2 A \sin^2 B \sin^2 C,$$

by the well-known identity

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

Now we shall prove that

$$\frac{16}{3} \sin^2 A \sin^2 B \sin^2 C \geq 2\sqrt{3} \sin 2A \sin 2B \sin 2C.$$

By the formula for the sine of double angle this inequality is

$$\sin A \sin B \sin C \geq 3\sqrt{3} \cos A \cos B \cos C,$$

since clearly $\sin A, \sin B, \sin C$ are positive, by the same reason the left hand side is always positive, but the right one is negative for obtuse triangle, zero for the right triangle, and positive for the acute triangle. Thus our inequality becomes

$$\tan A \tan B \tan C \geq 3\sqrt{3},$$

but consider the well-known identity

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C,$$

so by AM-GM inequality we have

$$S \geq 3\sqrt[3]{S},$$

where $S = \tan A + \tan B + \tan C$, cubing is equivalent to $S \geq 3\sqrt{3}$.

Second solution by Arkady Alt, San Jose, CA, USA

Since for nonacute triangle this inequality is obvious (because then $LHS > 0$ and $RHS \leq 0$)

So further we can assume that ABC is acute triangle, that is $A, B, C < \pi/2$. Let

$$\alpha := \pi - 2A, \beta := \pi - 2B, \gamma := \pi - 2C.$$

Then $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = \pi$ and original inequality

inequality can be equivalently rewritten as

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma \geq 2\sqrt{3} \sin \alpha \sin \beta \sin \gamma. \tag{1}$$

Let a, b, c, R and F be sidelengths circumradius and area of a triangle with angles α, β, γ.

Then multiplying inequality (1) by $4R^2$ we obtain

$$(1) \iff a^2 + b^2 + c^2 \geq \frac{\sqrt{3}abc}{R}.$$

Since $abc = 4FR$ then $\iff$ (1) $\iff$ $a^2 + b^2 + c^2 \geq 4\sqrt{3}F$ where latter inequality is Weitzenböck's inequality.

Just in case proof of inequality $a^2 + b^2 + c^2 \geq 4\sqrt{3}F$:

Since

$$16F^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$$

then

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}F \iff \left(a^2 + b^2 + c^2\right)^2 \geq 3\left(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\right) \iff$$

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2.$$

Also solved by Adnan Ali, AECS-4, Mumbai, India; Arpon Basu, AECS-4, Mumbai, India; Khurshid Turgunboyev, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Tolibjon Ismoilov, Academic Lyceum S.H.Sirojiddinov, Tashkent, Uzbekistan.