Junior Problems

J403. In triangle ABC, $\angle B = 15^\circ$ and $\angle C = 30^\circ$. Let D be the point on side BC such that $BD = 2AC$. Prove that AD is perpendicular to AB.

Proposed by Adrian Andreescu, Dallas, Texas, USA

J404. Let a, b, x, y be real numbers such that $0 < x < a$, $0 < y < b$ and $a^2+y^2 = b^2+x^2 = 2(ax+by)$. Prove that $ab + xy = 2(ay + bx)$.

Proposed by Mircea Becheanu, Bucharest, România

J405. Solve in prime numbers the equation

$$x^2 + y^2 + z^2 = 3xyz - 4.$$

Proposed by Adrian Andreescu, Dallas, Texas, USA

J406. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$a\sqrt{a+3} + b\sqrt{b+3} + c\sqrt{c+3} \geq 6.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J407. Solve in positive real numbers the equation

$$\sqrt{x^4 - 4x} + \frac{1}{x^2} = 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J408. Let a and b be nonnegative real numbers such that $a + b = 1$. Prove that

$$\frac{289}{256} \leq (1 + a^4)(1 + b^4) \leq 2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Senior Problems

S403. Find all primes p and q such that
$$\frac{2^{p^2-q^2}-1}{pq}$$
is a product of two primes.

Proposed by Adrian Andreescu, Dallas, Texas, USA

S404. Let $ABCD$ be a regular tetrahedron and let M and N be arbitrary points in the space. Prove that
$$MA \cdot NA + MB \cdot NB + MC \cdot NC \geq MD \cdot ND.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

S405. Find all triangles with integer side-lengths a, b, c such that $a^2 - 3a + b + c$, $b^2 - 3b + c + a$, $c^2 - 3c + a + b$ are all perfect squares.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S406. Let ABC be a triangle with side-lengths a, b, c and let
$$m^2 = \min\left\{(a-b)^2,\ (b-c)^2,\ (c-a)^2\right\}.$$

(a) Prove that
$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \geq \frac{1}{2}m^2(a+b+c);$$

(b) prove that if ABC is acute then
$$a^2(a-b)(a-c) + b^2(b-c)(b-a) + c^2(c-a)(c-b) \geq \frac{1}{2}m^2\left(a^2+b^2+c^2\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S407. Let $f(x) = x^3 + x^2 - 1$. Prove that for any positive real numbers a, b, c, d satisfying
$$a + b + c + d > \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d},$$
at least one of the numbers $af(b)$, $bf(c)$, $cf(d)$, $df(a)$ is different from 1.

Proposed by Adrian Andreescu, Dallas, Texas, USA

S408. Let ABC be a triangle with area S and let a, b, c be the lengths of its sides. Prove that
$$a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \geq 4S\sqrt{3\left(1 + \frac{R-2r}{4R}\right)}.$$

Proposed by Marius Stănean, Zalău, România

Undergraduate Problems

U403. Find all cubic polynomials $P(x) \in \mathbb{R}[x]$ such that

$$P\left(1-\frac{x(3x+1)}{2}\right)-P(x)^2+P\left(\frac{x(3x-1)}{2}-1\right)=1$$

for all $x \in \mathbb{R}$.

Proposed by Alessandro Ventullo, Milan, Italy

U404. Find the coefficient of x^2 after expanding the following product as a polynomial:

$$(1+x)(1+2x)^2 \dots (1+nx)^n.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U405. Let $a_1 = 1$ and

$$a_n = 1+\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}}$$

for all $n > 1$. Find

$$\lim_{n\to\infty}(a_n-\sqrt{2n}).$$

Proposed by Robert Bosch,USA

U406. Evaluate

$$\lim_{x\to 0}\frac{\cos{(n+1)x}\cdot\sin{nx}-n\sin{x}}{x^3}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U407. Prove that for every $\varepsilon > 0$

$$\int_2^{2+\varepsilon} e^{2x-x^2}dx < \frac{\varepsilon}{1+\varepsilon}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U408. Prove that if A and B are square matrices satisfying

$$A = AB - BA + ABA - BA^2 + A^2BA - ABA^2,$$

then $\det(A) = 0$.

Proposed by Mircea Becheanu, Bucharest, România

Olympiad Problems

O403. Let a, b, c be real numbers such that $a+b+c>0$. Prove that

$$\frac{a^2+b^2+c^2-2ab-2bc-2ca}{a+b+c}+\frac{6abc}{a^2+b^2+c^2+ab+bc+ca}\geq 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O404. Let a, b, c be positive numbers such that $abc=1$. Prove that

$$(a+b+c)^2\left(\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\right)\geq 9.$$

Proposed by An Zhenping, Xianyang Normal University, China

O405. Prove that for each positive integer n there is an integer m such that 11^n divides 3^m+5^m-1.

Proposed by Navid Safaei, Tehran, Iran

O406. Solve in prime numbers the equation

$$x^3-y^3-z^3+w^3+\frac{yz}{2}(2xw+1)^2=2017.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O407. Let ABC be a triangle, O a point in the plane and ω a circle of center O passing through B and C such that it intersects AC in D and AB in E. Let H be the intersection of BD and CE and D_1 and E_1 be the intersection points of the tangent lines to ω at C and B with BD and CE respectively. Prove that AH and the perpendiculars from B and C to OE_1 and OD_1 respectively, are concurrent.

Proposed by Marius Stănean, Zalău, România.

O408. Prove that in any triangle ABC

$$\frac{a}{m_a}+\frac{b}{m_b}+\frac{c}{m_c}\geq 2\sqrt{3}.$$

Proposed by Dragoliub Milosević, Gornji Milanovac, Serbia