# A Generalization of the Napoleon's Theorem

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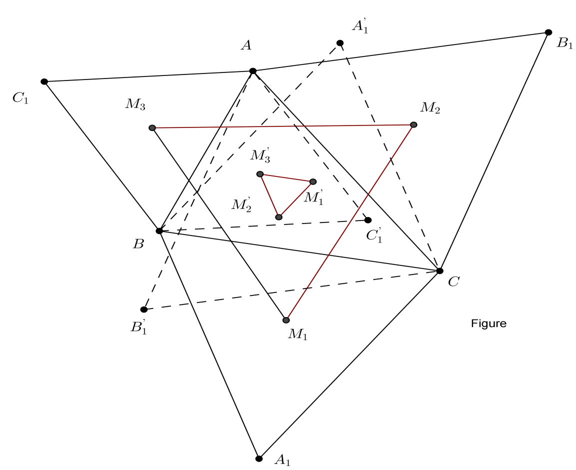
**Abstract**

In this article we present a generalization of the Napoleon's theorem. The classical Napoleon's theorem states that the centers of the equilateral triangles which were built to the sides of any triangle are the vertices of an equilateral triangle. We will finish with some solutions of classical geometry Olympiad style problems and more examples. You can solve the problems in any ways but this article illustrates the generalization of the Napoleon's theorem and you will realize that the fact is very useful.

## 1 Napoleon's Theorem

It is known that Napoleon Bonaparte was a little mathematician, especially he was interested in geometry. There are some theorems, points, facts related with name emperor of the French Napoleon Bonaparte (Napoleon I) in geometry. For example, *Napoleon's theorem, Napoleon's points, Napoleon's famous problem* and others.
Now we will show the Napoleon's theorem and Napoleon's triangle.

**Theorem 1 (Napoleon).** *Given any triangle $ABC$ and equilateral triangles $BA_1C$, $CB_1A$, $AC_1B$ are constructed on the sides of the triangle $ABC$ such that all of them are either externally or internally. If $M_1, M_2, M_3$ are centers of the triangles $BA_1C$, $CB_1A$, $AC_1B$ respectively, then $M_1M_2M_3$ is also equilateral triangle.*




The triangle thus formed is called the inner or outer Napoleon triangle. The difference in area of these two triangles equals the area of the original triangle.

There are in fact many proofs of the theorem's statement, including a trigonometric one, a symmetry-based approach and proofs using complex numbers. Its several proofs have given in books H. S. M.Coxeter, S. L. Greitzer[1], I. Sharygin[3], and in a article[2]. Now, we will move on generalization of this theorem without the its proof.

## 2 Generalization of the Napoleon's theorem

**Theorem 2 (Generalization of the Napoleon's theorem).** *Given a triangle $ABC$. The triangles $BA_1C$, $CB_1A$, $AC_1B$ are constructed (possibly degenerate) on the sides of the triangle $ABC$ such that all of the three triangles are either externally or internally and fellowing the conditions:*

$$(i) \qquad \angle BA_1C + \angle CB_1A + \angle AC_1B = 360^\circ;$$

$$(ii) \qquad AB_1 \cdot BC_1 \cdot CA_1 = BA_1 \cdot CB_1 \cdot AC_1;$$

*Then the angles of the triangle $A_1B_1C_1$ are equal to*

$$\angle B_1A_1C_1 = \angle B_1CA + \angle C_1BA,$$
$$\angle A_1C_1B_1 = \angle A_1BC + \angle B_1AC,$$
$$\angle C_1B_1A_1 = \angle C_1AB + \angle A_1CB.$$

Thus, the theorem denotes that if the conditions (i) and (ii), the angles of the triangle $A_1B_1C_1$ depend on the angles of the triangles constructed on the sides of the triangle $ABC$, but do not depend on the angles of the triangle $ABC$. If $\angle BA_1C = \angle CB_1A = \angle AC_1B = 120^\circ$ and $A_1B = A_1C$, $B_1C = B_1A$, $C_1A = C_1B$, the theorem will be the original theorem of Napoleon.

*Proof.* Suppose that the triangles are constructed externally. There exists an angle among the angles $\angle B_1A_1C_1$, $\angle A_1C_1B_1$, $\angle C_1B_1A_1$ is not equal to 0 or $\pi$.

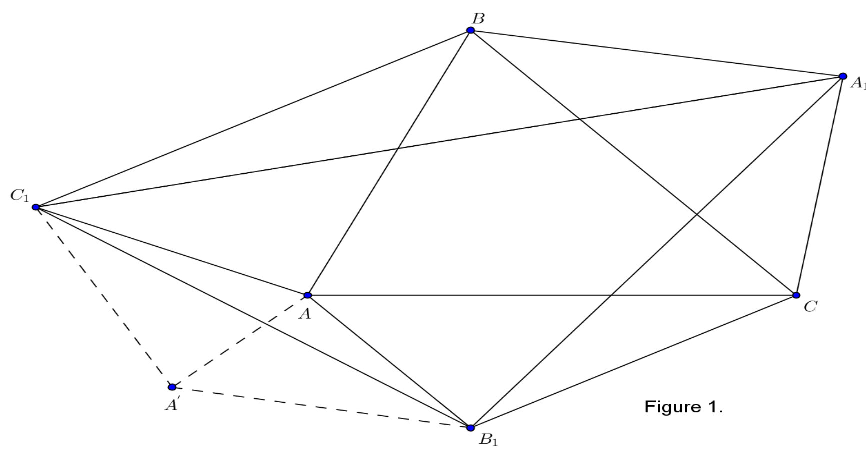



Figure 1.

Let we have a point $A'$ such that $\angle A'B_1A_1 = \angle AB_1C$ (See Figure 1) and

$$\frac{A'B_1}{A_1B_1} = \frac{AB_1}{CB_1}.$$

Hence the triangles $AB_1C$ and $A'B_1A_1$ are similar and $\angle A'AB_1 = \angle A_1CB_1$. Since $(i)$, we have that

$$\angle C_1BA_1 + \angle A_1CB_1 + \angle B_1AC_1 = 2\pi.$$

So $\angle C_1AA' = \angle C_1BA_1$ and since $(ii)$ we get that

$$\frac{A'A}{AC_1} = \frac{A_1C \cdot \frac{AB_1}{CB_1}}{AC_1} = \frac{BA_1}{BC_1}.$$

Thus the triangles $C_1AA'$ and $C_1BA_1$ are similar. Hence we get that:

(1) $A'B_1A_1$ and $AB_1C$ are similar and $\angle A'A_1B_1 = \angle ACB_1$;

(2) $A'C_1A_1$ and $AC_1B$ are similar and $\angle A'A_1C_1 = \angle ABC_1$;

Since (1) and (2), we have that $\angle B_1A_1C_1 = \angle B_1CA + \angle C_1BA$. Analogously, $\angle A_1C_1B_1 = \angle A_1BC + \angle B_1AC$ and $\angle C_1B_1A_1 = \angle C_1AB + \angle A_1CB$. This completes the proof of **Theorem 2 (Generalization of the theorem of Napoleon)**.

# 3 Examples and problems

**Problem 1.** Given a triangle $ABC$. Suppose that isosceles triangles $AKB$, $BLC$, $CMA$ are constructed on the sides $AB$, $BC$, $CA$ respectively, such that all of the three triangles are either externally or internally. Let $\angle AKB = \gamma$, $\angle BLC = \alpha$ $\angle CMA = \beta$ and $\alpha + \beta + \gamma = 2\pi$. Prove that the angles of the triangle $KLM$ are $\frac{\alpha}{2}$, $\frac{\beta}{2}$ and $\frac{\gamma}{2}$.

**Solution.** This is same with the generalized theorem of Napoleon

$$\angle KLM = \angle KBA + \angle MCA = 90^\circ - \frac{\gamma}{2} + 90^\circ - \frac{\beta}{2} = \frac{\alpha}{2},$$

$$\angle LMK = \angle LCB + \angle KAB = 90^\circ - \frac{\alpha}{2} + 90^\circ - \frac{\gamma}{2} = \frac{\beta}{2},$$

$$\angle MKL = \angle MAC + \angle LBC = 90^\circ - \frac{\beta}{2} + 90^\circ - \frac{\alpha}{2} = \frac{\gamma}{2}.$$

**Problem 2.** In a cyclic quadrilateral $ABCD$ the diagonals $AC$, $BD$ intersect at point $E$ and points $K$ and $M$ are midpoints of the sides $AB$ and $CD$, $L$ and $N$ are projection of the point $E$ on the sides $BC$ and $AD$. Prove that $KM \perp LN$.

**Solution.** It's nice and useful lemma. We will prove that $ML = MN$ and $KL = KN$.(See Figure 2)

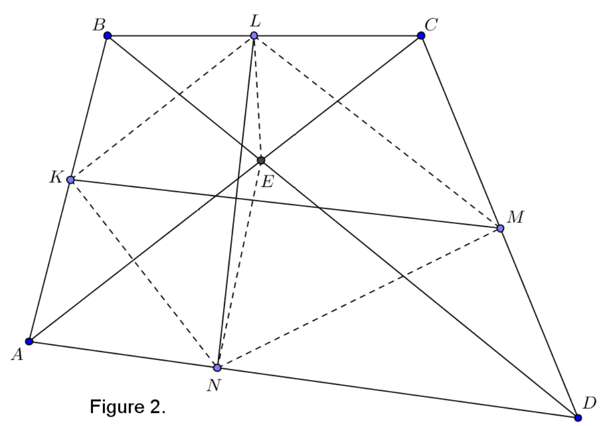



Figure 2.

We have that $\angle EBC = \angle EAD$ and triangle $EBL$ similar to $EAN$. So $\angle AMB + \angle BLE + \angle ANE = 360^\circ$ and

$$\frac{AM}{BM} \cdot \frac{BL}{LE} \cdot \frac{EN}{NA} = 1.$$

By the Generalization of the Theorem of Napoleon, we get that $\angle MLN = \angle MBA + \angle NEA = \angle MAB + \angle LEB = \angle MNL$. Hence $ML = MN$, analogously $KL = KN$. Thus, we get that $KM \perp LN$.

**Problem 3.** *(A. Zaslavsky, Geometry Olympiad I.Sharygin)* Points $A_1$,$B_1$,$C_1$ are on the circumcircle of the triangle $ABC$, such that $AA_1$,$BB_1$ and $CC_1$ are meet at one point. The reflection $A_1$,$B_1$,$C_1$ relatively to the sides $BC$,$CA$,$AB$ are obtained by points $A_2$,$B_2$,$C_2$ respectively. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar.

**Solution.** We have that $\angle BA_1C + \angle CB_1A + \angle AC_1B = 360^\circ$ and

$$AB_1 \cdot BC_1 \cdot CA_1 = BA_1 \cdot CB_1 \cdot AC_1.$$

Because from the condition $AA_1$,$BB_1$ and $CC_1$ are concurrent. We have that also $\angle BA_2C + \angle CB_2A + \angle AC_2B = 360^\circ$ and

$$AB_2 \cdot BC_2 \cdot CA_2 = BA_2 \cdot CB_2 \cdot AC_2.$$

Obviously, the triangles $BA_1C$, $CB_1A$, $AC_1B$ are constructed externally on the sides of the triangle $ABC$, the triangles $BA_2C$, $CB_2A$, $AC_2B$ are constructed internally on the sides of the triangle $ABC$. By generalized theorem of Napoleon, we get that $\angle B_2A_2C_2 = \angle B_2CA + \angle C_2BA$, $\angle A_2C_2B_2 = \angle A_2BC + \angle B_2AC$, $\angle C_2B_2A_2 = \angle C_2AB + \angle A_2CB$ and $\angle B_1A_1C_1 = \angle B_1CA + \angle C_1BA$, $\angle A_1C_1B_1 = \angle A_1BC + \angle B_1AC$, $\angle C_1B_1A_1 = \angle C_1AB + \angle A_1CB$. Hence

$$\angle B_1A_1C_1 = \angle B_2A_2C_2,$$

$$\angle A_1C_1B_1 = \angle A_2C_2B_2,$$

$$\angle C_1B_1A_1 = \angle C_2B_2A_2$$

and the triangles $A_1B_1C_1$, $A_2B_2C_2$ are similar.

**Problem 4.** *(China Team Selection Test-2013, day 1, problem 1)* The quadrilateral $ABCD$ is inscribed in a circle $\omega$. Suppose that $F$ is the intersection point of the diagonals $AC$ and $BD$, $E$ is the intersection point of the lines $BA$ and $CD$. Let the projection of $F$ on the lines $AB$ and $CD$ be points $G$ and $H$, respectively. Let $M$ and $N$ be the midpoints of the $BC$ and $EF$, respectively. If the circumcircle of the triangle $MNG$ meets the segment $BF$ at only one point $P$, and the circumcircle of the $MNH$ meets the segment $CF$ at only one point $Q$, then prove that $PQ$ is parallel to $BC$.

**Solution.** In this problem we can use from the nice lemma(see the Problem 2). We will prove the points $P$ and $Q$ are midpoints of the segments $BF$ and $CF$, respectively(See Figure 3).

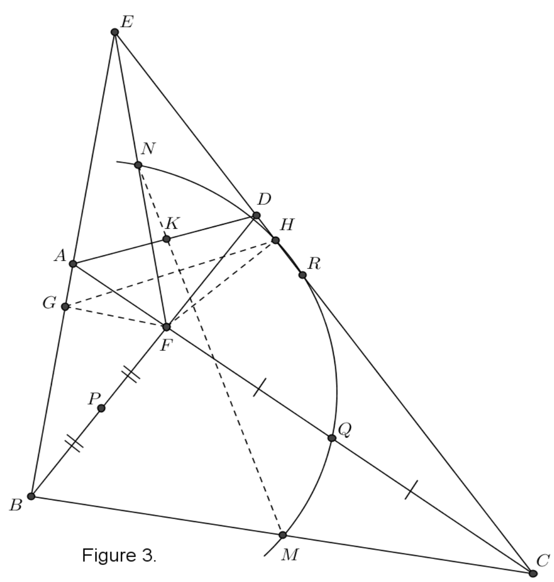


Figure 3.

Let the midpoint of the segment $AD$ is $K$ and the midpoint of the segment $EC$ is $R$. From the Problem 2, we have that $KM$ bisects the segment $GH$ and the line $KM$ is perpendicular to the line $GH$. So, since

$$NG = NF = NH = NE,$$

we get that the line $KM$ passes through the point $N$.

Further out, we have that $\angle GNH = 2\angle GEH = 2\angle BEC = 2\angle MRC$ and $\angle MNH = \angle MRC$. Hence $MNHR$ is cyclic quadrilateral. In the triangle $CFE$, we have that the circumcircle of the triangle $MNH$ passes through the points $N$, $R$ and $H(.)$ which the midpoints of the sides $FE$, $EC$ and the base of the altitude from the $F$ to the side $EC$. So, the circumcircle of the triangle $MNH$ is Euler circle(nine point circle) of the triangle $CFE$ and the circle passes through the point $Q$, which the midpoint of the segment $CF$. Therefore, the point $Q$ is midpoint of the segment $CF$. Analogously, the point $P$ is midpoint of the segment $BF$. Hence $PQ \parallel BC$.

Now, we will see two well-known and similar problems.

**Problem 5.** *(Mathematical Reflections Issue 4(2012), J240)* Let $ABC$ be an acute triangle with orthocenter $H$. Points $H_a$,$H_b$, and $H_c$ have defined in its interior, satisfying

$$\angle BH_aC = 180^\circ - \angle A, \qquad \angle CH_aA = 180^\circ - \angle C, \qquad \angle AH_aB = 180^\circ - \angle B,$$

$$\angle CH_bA = 180^\circ - \angle B, \qquad \angle AH_bB = 180^\circ - \angle A, \qquad \angle BH_bC = 180^\circ - \angle C,$$

$$\angle AH_cB = 180^\circ - \angle C, \qquad \angle BH_cC = 180^\circ - \angle B, \qquad \angle CH_cA = 180^\circ - \angle A.$$

Prove that the points $H$,$H_a$,$H_b$,$H_c$ are concyclic.

**Solution.** Since $BHH_aC$, $CH_bHA$, $AH_cHB$ are cyclic quadrilaterals we can find the lines $AH_a$, $BH_b$, $CH_c$ be the median lines of the trinagle $ABC$ and

$$\frac{BH_a}{CH_a} = \frac{\sin C}{\sin B}, \quad \frac{CH_b}{AH_b} = \frac{\sin A}{\sin C}, \quad \frac{AH_c}{BH_c} = \frac{\sin B}{\sin A}$$

and

$$AH_c \cdot BH_a \cdot CH_b = BH_c \cdot CH_a \cdot AH_b.$$

So $\angle BH_aC + \angle CH_bA + \angle AH_cB = 360^\circ$ and the triangle $H_aH_bH_c$ is a Napoleon's triangle(See Figure 4).

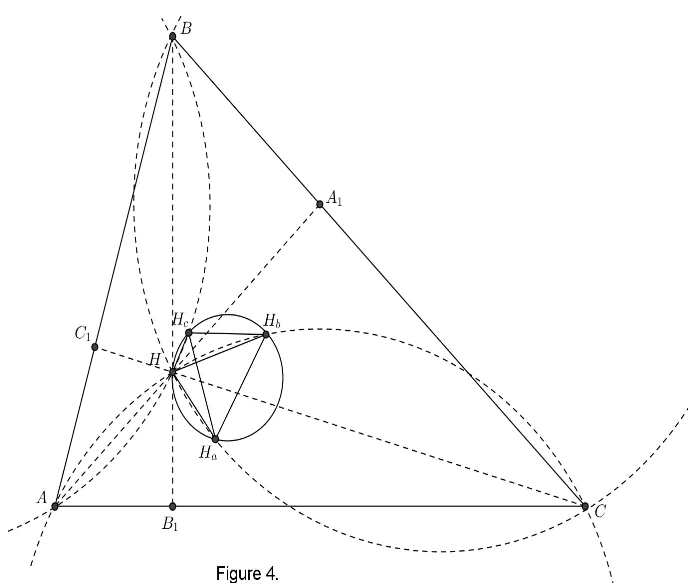


Figure 4.

Hence, we have that

$$\angle H_aH_bH_c = \angle H_aCB + \angle H_cAB = \angle(H_aH \cap H_cH),$$

$$\angle H_bH_cH_a = \angle H_bAC + \angle H_aBC = \angle(H_bH \cap H_aH),$$

$$\angle H_cH_aH_b = \angle H_cBA + \angle H_bCA = \angle(H_cH \cap H_bH)$$

and the points $H$,$H_a$,$H_b$,$H_c$ are concyclic. The completes the proof of Problem 5.

**Problem 6.** *(Mathematical Reflections Issue 6(2012), J252)* Let $ABC$ be an acute triangle and let $O_a$ be a point in its plane such that

$$\angle BO_aC = 2\angle A, \qquad \angle CO_aA = 180° - \angle A, \qquad \angle AO_aB = 180° - \angle A.$$

Similarly, define points $O_b$ and $O_c$. Prove that the circumcircle of triangle $O_aO_bO_c$ passes through the circumcenter of the triangle $ABC$.

**Solution.** Let $O(.)$ is circumcenter of the triangle $ABC$. From the conditions, $O_a$,$O_b$,$O_c$ are lies in the triangle $ABC$ and $BO_aOC$, $CO_bOA$, $AO_cOB$ are cyclic quadrilaterals. We have also the lines $AO_a$, $BO_b$, $CO_c$ be the symmedian lines of the triangle $ABC$ and

$$\frac{BO_a}{CO_a} = \frac{\sin^2 C}{\sin^2 B}, \quad \frac{CO_b}{AO_b} = \frac{\sin^2 A}{\sin^2 C}, \quad \frac{AO_c}{BO_c} = \frac{\sin^2 B}{\sin^2 A}.$$

So $\angle BO_aC + \angle CO_bA + \angle AO_cB = 360°$ and

$$AO_c \cdot BO_a \cdot CO_b = BO_c \cdot CO_a \cdot AO_b,$$

$O_aO_bO_c$ is Napoleon's triangle. Hence

$$\angle O_aO_bO_c = \angle O_aCB + \angle O_cAB = \angle(O_aO \cap O_cO),$$

$$\angle O_bO_cO_a = \angle O_bAC + \angle O_aBC = \angle(O_bO \cap O_aO),$$

$$\angle O_cO_aO_b = \angle O_cBA + \angle O_bCA = \angle(O_cO \cap O_bO)$$

and the circumcircle of triangle $O_aO_bO_c$ passes through the circumcenter of the triangle $ABC$. The completes the proof of Problem 6.

Here, another beatiful problem and now, we will see a nice solution by Generalization of The Theorem of Napoleon.

**Problem 7.** *(Indian IMOTC 2013, Team Selection Test 3, Problem 2)* In a triangle $ABC$, let $I$ be the incenter and points $D$,$E$,$F$ are chosen on the segments $BC$,$CA$,$AB$, respectively, such that $BD + BF = AC$ and $CD + CE = AB$. The circumcircles of triangles $AEF$,$BFD$,$CDE$ intersect lines $AI$,$BI$,$CI$, respectively, at points $K$,$L$,$M$(different from $A$,$B$,$C$), respectively. Prove that the points $K$,$L$,$M$,$I$ are concyclic.

**Solution.** The points $D$,$E$,$F$ are tangent points of excircles $(I_a)$,$(I_b)$,$(I_c)$ of the triangle $ABC$ to the sides $BC$,$CA$,$AB$ respectively. Look at the triangle $DEF$, we have the triangles $FKE$,$DLF$ and $EMD$ are constructed inside of the triangle $DEF$ and since the circles $(AEF)$,$(BFD)$,$(CDE)$ we get that

$$KF = KE, \qquad LD = LF, \qquad ME = MD.$$

We have also

$$\angle FKE + \angle DLF + \angle EMD = (180^\circ - \angle A) + (180^\circ - \angle B) + (180^\circ - \angle C) = 360^\circ$$

and

$$KF \cdot ME \cdot LD = KE \cdot MD \cdot LF.$$

So, by the Generalization of Napoleon theorem, we have

$$\angle LKM = \frac{\angle B + \angle C}{2} = \angle(LI \cap MI), \quad \angle KLM = \frac{\angle A + \angle C}{2} = \angle(KI \cap MI),$$

$$\angle KML = \frac{\angle B + \angle A}{2} = \angle(LI \cap KI)$$

and $K$,$L$,$M$,$I$ are concyclic.

# References


[1] H. S. M. Coxeter, S. L. Greitzer, Geometry Rivisited (Russian translation), Moscow (1978), pp.76-82.

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[3] I. Sharygin, Problems in Plane Geometry (Russian translation), Moscow 1996.

[4] Mathematical Reflections Issue 6(2012).

[5] Mathematical Reflections Issue 4(2012).

[6] https://en.wikipedia.org/wiki/Napoleonstheorem



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