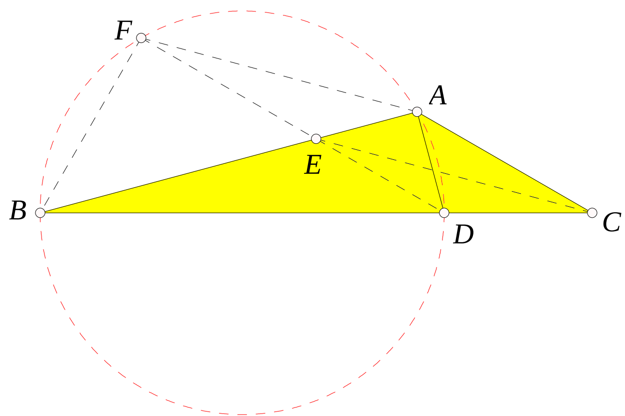
Junior problems

J403. In triangle ABC, $\angle B = 15^\circ$ and $\angle C = 30^\circ$. Let D be the point on side BC such that $BD = 2AC$. Prove that AD is perpendicular to AB.

Proposed by Adrian Andreescu, Dallas, Texas

Solution by Polyahedra, Polk State College, FL, USA



Let E be the point on AB such that $DE \parallel CA$, and let F be the foot of perpendicular from B onto DE. We may assume that $AC = 1$. Then $BD = 2$ and $EF = BF = 1$. Thus, $AFEC$ is a parallelogram and

$$\frac{BC}{2} = \frac{BC}{BD} = \frac{AC}{ED} = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2}.$$

Hence, $DC = BC - BD = \sqrt{3} - 1 = DE$. Therefore, $\angle AFD = \angle CED = 15^\circ = \angle ABD$, which implies that A, F, B, D lie on a circle. Thus $AD \perp AB$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Adam Krause, College at Brockport, SUNY, NY, USA; Aditya Ghosh, Kolkata, India; Arkady Alt, San Jose, CA, USA; Jamal Gadirov, Istanbul, Turkey; Carlos Yeddiel, Mexico; Nandansai Dasireddy, Hyderabad, India; Jiwon Park, St. Andrew's School, DE, USA; Joel Schlosberg, Bayside, NY, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paul Revenant, Lycée du Parc, Lyon, France; Shuborno Das, Ryan International School, Bangalore, India; Soumava Pal, Indian Statistical Institute, India; Albert Stadler, Herrliberg, Switzerland; Tamoghno Kandar, Mumbai, India; Michael Tang, MN, USA; Jang Hun Choi, Jericho, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Anderson Torres, Sao Paulo, Brazil; Celine Lee, Chinese International School, Hong Kong; Chris Lee, Seoul International School, Seoul, Republic of Korea; Nikos Kalapodis, Patras, Greece; Stephanie Li; Soo Young Choi, Vestal Senior High School, NY, USA; Matthew Li; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Jio Jeong, Seoul International School, Seoul, Korea.

J404. Let a, b, x, y be real numbers such that $0 < x < a$, $0 < y < b$ and $a^2 + y^2 = b^2 + x^2 = 2(ax + by)$. Prove that $ab + xy = 2(ay + bx)$.

Proposed by Mircea Becheanu, Bucharest, Romania

Solution by Michael Tang, MN, USA

Let $a^2 + y^2 = b^2 + x^2 = 2(ax + by) = r^2$, for some $r > 0$. Since $0 < x < a$ and $0 < y < b$, there exist $0 < \theta < \alpha \leq \frac{\pi}{2}$ such that $a = r\cos\theta$, $y = r\sin\theta$, $b = r\sin\alpha$, and $x = r\cos\alpha$. This gives $ax + by = r^2(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = r^2\cos(\alpha - \theta)$, so $\cos(\alpha - \theta) = \frac{1}{2}$. Then

$$\begin{aligned}
2(ay + bx) = r^2(2\cos\theta\sin\theta + 2\sin\alpha\cos\alpha) &= r^2(\sin 2\theta + \sin 2\alpha) \\
&= r^2 \cdot 2\sin(\alpha + \theta)\cos(\alpha - \theta) \\
&= r^2\sin(\alpha + \theta) \\
&= r^2(\sin\alpha\cos\theta + \sin\theta\cos\alpha) \\
&= (ab + xy)
\end{aligned}$$

by the sum-to-product formulas. Thus $ab + xy = 2(ay + bx)$, as requested.

Also solved by Daniel Lasaosa, Pamplona, Spain; Jang Hun Choi, Jericho, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Anderson Torres, Sao Paulo, Brazil; Celine Lee, Chinese International School, Hong Kong; Chris Lee, Seoul International School, Seoul, Republic of Korea; Matthew Li; Polyahedra, Polk State College, FL, USA; Jamal Gadirov, Istanbul, Turkey; Jiwon Park, St. Andrew's School, DE, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Paul Revenant, Lycée du Parc, Lyon, France; Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua; Albert Stadler, Herrliberg, Switzerland; Gheorghe Rotariu, Dorohoi, Romania; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Jio Jeong, Seoul International School, Seoul, Korea.

J405. Solve in prime numbers the equation

$$x^2 + y^2 + z^2 = 3xyz - 4.$$

Proposed by Adrian Andreescu, Dallas, Texas, USA

Solution by David E. Manes, Oneonta, NY, USA

Rewrite the equation as $x^2 + y^2 + z^2 + 2^2 = 3xyz$ so that 3 is a divisor of the sum of primes squared. If x is a prime different from 3, then $x \equiv 1$ or $2 \pmod 3$ and $x^2 \equiv 1 \pmod 3$. Therefore, if x, y, z are primes different from 3, then $x^2 + y^2 + z^2 + 2^2 \equiv 1 \pmod 3$ which implies that exactly one of the primes is 3. Using this fact, one quickly obtains

$$3^2 + 2^2 + 17^2 + 4 = 306 = 3 \cdot 3 \cdot 2 \cdot 17.$$

This solution is not unique since

$$3^2 + 17^2 + 151^2 + 4 = 23103 = 3 \cdot 3 \cdot 17 \cdot 151.$$

Also solved by Michael Tang, MN, USA; Jio Jeong, Seoul International School, Seoul, Korea.

J406. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$a\sqrt{a+3} + b\sqrt{b+3} + c\sqrt{c+3} \geq 6.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain
Using the AM-QM inequality we have

$$\sqrt{\frac{a+1+1+1}{4}} \geq \frac{\sqrt{a}+1+1+1}{4}, \qquad \sqrt{a+3} \geq \frac{\sqrt{a}+3}{2},$$

with equality iff $a = 1$, and similarly for b, c. It then suffices to show that

$$\frac{a\sqrt{a} + b\sqrt{b} + c\sqrt{c}}{2} \geq 6 - \frac{3(a+b+c)}{2} = \frac{3}{2},$$

which is equivalent to

$$\left(\frac{a\sqrt{a} + b\sqrt{b} + c\sqrt{c}}{3}\right)^{\frac{2}{3}} \geq 1 = \frac{a+b+c}{3},$$

and which is in turn clearly true by the power mean inequality, and with equality iff $a = b = c$. The conclusion follows, equality holds iff $a = b = c = 1$.

Also solved by Michael Tang, MN, USA; Jang Hun Choi, Jericho, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Anderson Torres, Sao Paulo, Brazil; Nikos Kalapodis, Patras, Greece; Soo Young Choi, Vestal Senior High School, NY, USA; Matthew Li; Polyahedra, Polk State College, FL, USA; Gheorghe Rotatiu, Dorohoi, Romania; Alessandro Ventullo, Milan, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Arkady Alt, San Jose, CA, USA; Jamal Gadirov, Istanbul, Turkey; Christos Karaoglanis, Evangeliki Gymnasiium, Athens, Greece; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nandansai Dasireddy, Hyderabad, India; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; Andrew Rowley, Ashley Case, Khanh Tran, Stewart Negron, College at Brockport, SUNY, NY, USA; Henry Ricardo, Westchester Area Math Circle; Jiwon Park, St. Andrew's School, DE, USA; Joel Schlosberg, Bayside, NY, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nguyen Ngoc Tu, Ha Giang, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Paul Revenant, Lycée du Parc, Lyon, France; Rajarshi Kanta Ghosh, Kolkata, India; Rajdeep Majumder, Drgapur, India; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland; Tamoghno Kandar, Mumbai, India; Vincelot Ravoson, Lycée Henri IV, Paris, France; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Fong Ho Leung, Hoi Ping Chamber of Commerce Secondary School; Niyanth Rao, Redwood Middle School, Saratoga, CA, USA; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Zafar Ahmed, BARC, Mumbai, India and Dhruv Sharma, NIT, Rourkela, India.

J407. Solve in positive real numbers the equation

$$\sqrt{x^4-4x}+\frac{1}{x^2}=1$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Albert Stadler, Herrliberg, Switzerland

The given equation implies $x^4-4x-\left(1-\frac{1}{x^2}\right)^2=0$ or equivalently

$$\frac{(x^2-x-1)(x^2+x+1)(x^4+x^2-2x+1)}{x^4}=0.$$

However $x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}>0$ and $x^4+x^2-2x+1=x^4+(x-1)^2>0$. $x^2-x-1=0$ has two roots. Since x is required to be positive the only root of the given equation is $\frac{1+\sqrt{5}}{2}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Michael Tang, MN, USA; Jang Hun Choi, Jericho, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Celine Lee, Chinese International School, Hong Kong; Chris Lee, Seoul International School, Seoul, Republic of Korea; Stephanie Li; Matthew Li; Polyahedra, Polk State College, FL, USA; Gheorghe Rotariu, Dorohoi, Romania; Vincelot Ravoson, Lycée Henri IV, Paris, France; Alessandro Ventullo, Milan, Italy; Arkady Alt, San Jose, CA, USA; Jamal Gadirov, Istanbul, Turkey; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; Jiwon Park, St. Andrew's School, DE, USA; Joel Schlosberg, Bayside, NY, USA; Julio Cesar Mohnsam, IF Sul - Pelotas-RS, Brazil; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Paul Revenant, Lycée du Parc, Lyon, France; Paul Vanborre-Jamin, Lycée Henri IV, Paris, France; Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Jio Jeong, Seoul International School, Seoul, Korea; Pedro Acosta, West Morris Mendham High School, Mendham, NJ, USA.

J408. Let a and b be nonnegative real numbers such that $a+b=1$. Prove that

$$\frac{289}{256} \leq \left(1+a^4\right)\left(1+b^4\right) \leq 2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

First solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Using several times Cauchy-Schwarz inequality we get

$$\begin{aligned}(1+a^4)(1+b^4) &= \Bigg(\underbrace{\frac{1}{2^4}+\ldots+\frac{1}{2^4}}_{15}+(a^2)^2+\frac{1}{2^4}\Bigg)\Bigg(\underbrace{\frac{1}{2^4}+\ldots+\frac{1}{2^4}}_{15}+\frac{1}{2^4}+(b^2)^2\Bigg)\\ &\geq \Bigg(\underbrace{\frac{1}{2^4}+\ldots+\frac{1}{2^4}}_{15}+\frac{1}{2^2}\cdot a^2+\frac{1}{2^2}\cdot b^2\Bigg)^2\\ &= \Bigg(\underbrace{\frac{1}{2^4}+\ldots+\frac{1}{2^4}}_{15}+\frac{1}{4}(a^2+b^2)\Bigg)^2\\ &\geq \Bigg(\underbrace{\frac{1}{2^4}+\ldots+\frac{1}{2^4}}_{15}+\frac{(a+b)^2}{8}\Bigg)^2=\frac{289}{256}.\end{aligned}$$

LHS is proved. Equality holds only when $a=b=\frac{1}{2}$.

Now we prove that RHS of given inequality. From the given conditions we get

$$ab \leq \frac{1}{4} \tag{1}$$

Using (1) and AM-GM inequality we get,

$$\begin{aligned}(1+a^4)(1+b^4) &= 1+a^4b^4+a^4+b^4 = 1+a^4+b^4+(ab)^2(a^2b^2)\\ &\leq 1+a^4+b^4+\frac{1}{16}\cdot(a^2b^2) \leq 1+a^4+b^4+14a^2b^2\\ &= 1+a^4+b^4+14\cdot\sqrt[14]{(a^3b)^4\cdot(a^2b^2)^6\cdot(ab^3)^4}\\ &\leq 1+a^4+b^4+4a^3b+6a^2b^2+4ab^3 = 1+(a+b)^4 = 2.\end{aligned}$$

RHS is proved. Equality holds only when $\{a,b\}=\{0,1\}$.

Second solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Since $a+b=1$, we may consider funtion $f(x)=\left(1+x^4\right)\left(1+(1-x)^4\right)$ for $x\in[0,1]$. $f''(x)=12(1-x)^2\left(x^4+1\right)-32(1-x)^3x^3+12\left((1-x)^4+1\right)x^2$, which may be written by doing $x=a$ and $1-x=b$ as $f''(a,b)=12a^2+12b^2+12a^4b^2-32a^3b^3+12a^2b^4$. Now, since $a+b=1$ by normalization, we have

$$\begin{aligned}f''(a,b) &= -32a^3b^3+12\left(a^2+b^2\right)(a+b)^4+12\left(a^4b^2+a^2b^4\right)\\ &= 12a^6+48a^5b+96a^4b^2+64a^3b^3+96a^2b^4+48ab^5+12b^6>0\end{aligned}$$

for a and b nonnegative real numbers such that $a+b=1$. This proves that $f(x)$ is extrictly convex for $x\in[0,1]$. Since $f(x)=f(1-x)$ its minimum value is attained at $x=1/2$ and it maximum at $x=0$ or $x=1$: $f(1/2)=\dfrac{289}{256}$ and $f(0)=f(1)=2$. That is

$$\frac{289}{256}\leq(1+a)^4(1+b)^4\leq 2.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Jang Hun Choi, Jericho, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Nikos Kalapodis, Patras, Greece; Matthew Li; Polyahedra, Polk State College, FL, USA; Gheorghe Rotariu, Dorohoi, Romania; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Niyanth Rao, Redwood Middle School, Saratoga, CA, USA; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Pedro Acosta, West Morris Mendham High School, Mendham, NJ, USA; Alessandro Ventullo, Milan, Italy; Arkady Alt, San Jose, CA, USA; Jamal Gadirov, Istanbul, Turkey; Andrew Rowley, Khanh Tran, College at Brockport, SUNY, NY, USA; Jiwon Park, St. Andrew's School, DE, USA; Joel Schlosberg, Bayside, NY, USA; Kevin Soto Palacios, Huarmey, Perú; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nguyen Ngoc Tu, Ha Giang Specialized High School, Ha Giang, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Rajarshi Kanta Ghosh, Kolkata, India; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland.

Senior problems

S403. Find all primes p and q such that

$$\frac{2^{p^2-q^2}-1}{pq}$$

is a product of two primes.

Proposed by Adrian Andreescu, Dallas, Texas, USA

Solution by Alessandro Ventullo, Milan, Italy

Since $2^{p^2-q^2}-1$ is odd and

$$\frac{2^{p^2-q^2}-1}{pq}$$

is an integer, then p and q are odd primes and clearly $p>q$. So, $p^2-q^2\equiv 0 \pmod 8$. If $q>3$, then $p^2-q^2\equiv 0 \pmod 3$, so p^2-q^2 is divisible by 24. Then

$$2^{p^2-q^2}-1=2^{24k}-1,$$

where $k\in\mathbb{N}^*$. Since $2^{24k}-1$ is divisible by $2^{24}-1=16777215=3^2\cdot 5\cdot 7\cdot 13\cdot 17\cdot 241$, then $\frac{2^{p^2-q^2}-1}{pq}$ cannot be the product of two primes. So, $q=3$ and

$$\frac{2^{p^2-q^2}-1}{pq}=\frac{2^{p^2-9}-1}{3p}.$$

Since p^2-9 is divisible by 8, then $p^2-9=8k$ for some $k\in\mathbb{N}^*$, so

$$\frac{2^{p^2-9}-1}{3p}=\frac{2^{8k}-1}{3p}=\frac{(2^k-1)(2^k+1)(2^{2k}+1)(2^{4k}+1)}{3p}.$$

An easy check shows that it must be $k\geq 2$. If k is even, then $3\mid(2^k-1)$ and if $k>2$ we have $2^k-1=3n$ for some $n\in\mathbb{N}$ and $n>1$. But then

$$\frac{n(2^k+1)(2^{2k}+1)(2^{4k}+1)}{p}$$

is the product of at least three primes, contradiction. So, $k=2$ and we get $p=5$, which satisfies the condition. If k is odd, then $3\mid(2^k+1)$ and since $k\geq 3$, then $2^k+1=3n$ for some $n\in\mathbb{N}$ and $n>1$. But then, we have that

$$\frac{n(2^k-1)(2^{2k}+1)(2^{4k}+1)}{p}$$

is the product of at least three primes, contradiction. In conclusion, $(p,q)=(5,3)$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Jang Hun Choi, Jericho, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Celine Lee, Chinese International School, Hong Kong; Chris Lee, Seoul International School, Seoul, Republic of Korea; Matthew Li; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Jamal Gadirov, Istanbul, Turkey; Jiwon Park, St. Andrew's School, DE, USA; Joel Schlosberg, Bayside, NY, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Albert Stadler, Herrliberg, Switzerland.

S404. Let $ABCD$ be a regular tetrahedron and let M and N be arbitrary points in the space. Prove that

$$MA \cdot NA + MB \cdot NB + MC \cdot NC \geq MD \cdot ND.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by the author

Lemma 1: For any arbitrary points A, B, C and D the following inequality holds:

$$AB \cdot CD + BC \cdot AD \geq AC \cdot BD.$$

Proof: Pick a point A_1 laying on the ray DA such that $DA_1 = \frac{1}{DA}$. In a similar way, pick points B_1 and C_1 on the rays DB and DC, respectively. Since $\frac{DA_1}{DB} = \frac{DB_1}{DA} = \frac{1}{DA \cdot DB}$, from the similarity of triangles DAB and DB_1A_1 we get $A_1B_1 = \frac{AB}{DA \cdot DB}$. Similarly,

$$B_1C_1 = \frac{BC}{DB \cdot DC} \text{ and } C_1A_1 = \frac{CA}{DC \cdot DA} \tag{1}$$

and by plugging the results into the triangle inequality $A_1B_1 + B_1C_1 \geq A_1C_1$, get

$$AB \cdot CD + BC \cdot AD \geq AC \cdot BD.$$

Lemma 2: Given points M, N and a triangle ABC laying on a plane. Then

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} \geq 1. \tag{$*$}$$

Proof: Consider a point K, coplanar with the triangle, such that $\angle ABM = \angle KBC, \angle MAB = \angle CKB$. Notice, that

$$\frac{CK}{BK} = \frac{AM}{AB}, \frac{AK}{BK} = \frac{CM}{BC}, \frac{BC}{BK} = \frac{BM}{AB}. \tag{2}$$

For the points A, N, C, K according to the Lemma 1 we have: $AN \cdot CK + CN \cdot AK \geq AC \cdot NK$. From the triangle inequality: $NK \geq BK - BN$, therefore $AN \cdot CK + CN \cdot AK \geq AC \cdot (BK - BN)$. Hence,

$$\frac{AN \cdot CK}{AC \cdot BK} + \frac{CN \cdot AK}{AC \cdot BK} + \frac{BN}{BK} \geq 1. \tag{3}$$

From (3) and (2) follows that $\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM\dot{C}N}{CA \cdot CB} \geq 1.$

Corollary: Inequality $(*)$ holds if points M and/or N are not coplanar with the triangle ABC. This follows from the Lemma 2 if instead of point M and N we consider their projections on the plane of the triangle ABC.

Now, let's return to the main problem. On the ray DA pick a point A_1 such that $DA_1 = \frac{1}{DA}$. By analogy, pick points B_1, C_1, M_1 and N_1 on the rays DB, DC,DM and DN, respectively. From the corollary of Lemma 2 for the points M_1 and N_1 and triangle $A_1B_1C_1$ we conclude that:

$$A_1M_1 \cdot A_1N_1 + B_1M_1 \cdot B_1N_1 + C_1M_1 \cdot C_1N_1 \geq A_1B_1^2;$$

using inequalities similar to (1), we get

$$\frac{AM}{DA \cdot DM} \cdot \frac{AN}{DA \cdot DN} + \frac{BM}{DB \cdot DM} \cdot \frac{BN}{DB \cdot DN} + \frac{CM}{DC \cdot DM} \cdot \frac{CN}{DC \cdot DN} \geq \left(\frac{AB}{DA \cdot DB}\right)^2,$$

and the conclusion follows.

S405. Find all triangles with integer side-lengths a, b, c such that $a^2-3a+b+c$, $b^2-3b+c+a$, $c^2-3c+a+b$ are all perfect squares.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that by the triangular inequality, $b+c>a$, or $b+c\geq a+1$ since a, b, c are all integers, hence

$$a^2-3a+b+c\geq a^2-2a+1=(a-1)^2.$$

It follows that if wlog $a\geq b, c$, then

$$(a-1)^2\leq a^2-3a+b+c\leq a^2-a<a^2,$$

and $a^2-3a+b+c$ can only be a perfect square iff $b+c=a+1$. Assuming now that $b\geq c$, it follows that $b^2-2b+2c-1=(b-1)^2+2(c-1)$ must be a perfect square, or it must be at least $(b-1)^2$, while at the same time $b^2-2b+2c-1\leq b^2-1<b^2$, or $c=1$ and $a=b$. Note that these are always sides of a triangle, whereas the three proposed expressions take values $(a-1)^2$, $(b-1)^2$, $2(a-1)$, or for the last one to be a perfect square, a non-negative integer u must exist such that $a-1=2u^2$. We conclude that (a, b, c) must be a permutation of $\left(2u^2+1, 2u^2+1, 1\right)$, where u may take any non-negative integral value.

Note: When $u=0$, $a=b=c=1$, yielding all three proposed expressions equal to $0=0^2$. If this is not considered to be a perfect square, it then suffices to have u take any positive integer value.

Also solved by Jang Hun Choi, Jericho, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Celine Lee, Chinese International School, Hong Kong; Chris Lee, Seoul International School, Seoul, Republic of Korea; Soo Young Choi, Vestal Senior High School, NY, USA; Charalampos Platanos, Anvaryta Experimental Junior High School, Gerakas, Greece; Jiwon Park, St. Andrew's School, DE, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina.

S406. Let ABC be a triangle with side-lengths a, b, c and let

$$m^2 = \min\left\{(a-b)^2, (b-c)^2, (c-a)^2\right\}.$$

(a) Prove that

$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \geq \frac{1}{2}m^2(a+b+c);$$

(b) prove that if ABC is acute then

$$a^2(a-b)(a-c) + b^2(b-c)(b-a) + c^2(c-a)(c-b) \geq \frac{1}{2}m^2\left(a^2+b^2+c^2\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina

a) Due to triangle inequality we have

$$b+c-a > 0, c+a-b > 0, a+b-c > 0 \quad (1)$$

$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) = 3abc + \sum_{cyc} a^3 - \sum_{cyc} ab(a+b) =$$

$$= \frac{1}{2}[\sum_{cyc}(b^3+c^3-b^2c-bc^2) - \sum_{cyc} a(b^2+c^2-2bc)] = \frac{1}{2}\sum_{cyc}(b+c-a)(b-c)^2 \geq^{(1)}$$

$$\geq \frac{1}{2}\sum_{cyc}(b+c-a)m^2 = \frac{1}{2}m^2(a+b+c)$$

Equality holds if and only if $a = b = c$

b) Since ABC is acute then

$$b^2+c^2-a^2 > 0, c^2+a^2-b^2 > 0, a^2+b^2-c^2 > 0 \quad (2)$$

$$a^2(a-b)(a-c) + b^2(b-c)(b-a) + c^2(c-a)(c-b) = \sum_{cyc} a^4 - \sum_{cyc} ab(a^2+b^2) + \sum_{cyc} a^2bc =$$

$$= \frac{1}{2}[\sum_{cyc}(a^4+b^4-a^3b-ab^3) - \sum_{cyc}(a^3b+ab^3-2a^2b^2) - \sum_{cyc}(a^2c^2+b^2c^2-2abc^2)] =$$

$$= \frac{1}{2}[\sum_{cyc}(a^2+ab+b^2)(a-b)^2 - \sum_{cyc} ab(a-b)^2 - \sum_{cyc} c^2(a-b)^2] =$$

$$= \frac{1}{2}\sum_{cyc}(a^2+b^2-c^2)(a-b)^2 \geq^{(2)} \frac{1}{2}\sum_{cyc}(a^2+b^2-c^2)m^2 = \frac{1}{2}m^2(a^2+b^2+c^2)$$

Equality holds if and only if $a = b = c$.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Daniel Lasaosa, Pamplona, Spain; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA.

S407. Let $f(x) = x^3 + x^2 - 1$. Prove that for any positive real numbers a, b, c, d satisfying

$$a + b + c + d > \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d},$$

at least one of the numbers $af(b)$, $bf(c)$, $cf(d)$, $df(a)$ is different from 1.

Proposed by Adrian Andreescu, Dallas, Texas, USA

Solution by the author

Assume the contrary that there are $a, b, c, d > 0$ such that

$$a + b + c + d > \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

and

$$af(b) = bf(c) = cf(d) = df(a) = 1.$$

We obtain

$$\frac{1}{a} = f(b),\ \frac{1}{b} = f(c),\ \frac{1}{c} = f(d),\ \frac{1}{d} = f(a).$$

So, $\frac{1}{a} = f(b)$, which means $\frac{1}{a} = b^3 + b^2 - 1$, so $ab^2 = a + 1 - ab^3 = \frac{a+1}{b+1}$. Similar for bc^2, cd^2, da^2. Multiplying the result is $(abcd)^3 = 1$, and then $abcd = 1$. Thus $b^2 + b - \frac{1}{b} = cd$. But

$$0 \leq a^2 + b^2 + c^2 + d^2 - (ab + bc + cd + da) = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) - (a + b + c + d) < 0.$$

Contradiction. So, at least one of $af(b), bf(c), cf(d), df(a)$ is different from 1.

Also solved by Daniel Lasaosa, Pamplona, Spain; Jang Hun Choi, Jericho, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Alessandro Ventullo, Milan, Italy; Jamal Gadirov, Istanbul, Turkey; Jiwon Park, St. Andrew's School, DE, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Albert Stadler, Herrliberg, Switzerland.

S408. Let ABC be a triangle with area S and let a, b, c be the lengths of its sides. Prove that

$$a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \geq 4S\sqrt{3\left(1 + \frac{R - 2r}{4R}\right)}.$$

Proposed by Marius Stanean, Zalau, Romania

Solution by Daniel Lasaosa, Pamplona, Spain
Note first that by the AM-GM inequality, we have

$$a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} = \sqrt{abc}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \geq 3\sqrt{abc}\sqrt[6]{abc} = 3\sqrt[3]{a^2b^2c^2}.$$

Squaring both sides of the proposed inequality, using that $abc = 4RS$ and that $2Rr = \frac{abc}{a+b+c}$ yields that it suffices to show that

$$12R^2\sqrt[3]{abc} \geq 4S(5R - 2r), \qquad 4\frac{3R^2}{\sqrt[3]{a^2b^2c^2}} + \frac{abc}{R^2(a+b+c)} \geq 5.$$

Now, by the weighted AM-GM inequality, it suffices to show that

$$\left(\frac{3R^2}{\sqrt[3]{a^2b^2c^2}}\right)^4 \cdot \frac{abc}{R^2(a+b+c)} \geq 1,$$

or equivalently that

$$3^4R^6 \geq (a+b+c)\sqrt[3]{a^5b^5c^5}.$$

But again by the AM-GM inequality, $3\sqrt[3]{abc} \leq a + b + c$, or it suffices to show that $a + b + c \leq 3\sqrt{3}R$. Now, equality is known to occur in this last inequality when ABC is equilateral, which is also well-known to be the case for maximum perimeter of a triangle given its circumradius. The conclusion follows, and ABC being equilateral is clearly a necessary condition. Substitution in the proposed inequality yields that this condition is also sufficient.

Also solved by Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy.

U403. Find all cubic polynomials $P(x) \in \mathbb{R}[x]$ such that

$$P\left(1-\frac{x(3x+1)}{2}\right)-P(x)^2+P\left(\frac{x(3x-1)}{2}-1\right)=1$$

for all $x \in \mathbb{R}$.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Daniel Lasaosa, Pamplona, Spain

Taking respectively $x=-1,0,1$, we obtain

$$P(0)+P(1)=1+P(-1)^2 \geq 2\,|P(-1)| \geq 2P(-1),$$

$$P(1)+P(-1)=1+P(0)^2 \geq 2\,|P(0)| \geq 2P(0),$$

$$P(-1)+P(0)=1+P(1)^2 \geq 2\,|P(1)| \geq 2P(1),$$

with equality respectively iff $P(-1)=1$, $P(0)=1$ and $P(1)=1$. Adding the three inequalities, we conclude that equality must hold in all of them, or $P(-1)=P(0)=P(1)=1$. Now, since $P(x)=ax^3+bx^2+cx+d$ for some reals a,b,c,d, we conclude that $a+b+c+d=d=-a+b-c+d=1$, for $d=1$, $a+c=0$ and $b=0$, ie $P(x)=1+cx-cx^3$ for some real c. Taking now $x=2$ and $x=-2$ respectively yields

$$1=P(-6)-P(2)^2+P(4)=1+210c-(1-6c)^2+1-60c=1+162c-36c^2,$$

$$1=P(-4)-P(-2)^2+P(6)=1+60c-(1+6c)^2+1-210c=1-162c-36c^2,$$

for $162c=0$, or $c=0$. We conclude that $P(x)=1$ is the only possible solution.

Also solved by Jang Hun Choi, Jericho, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Jamal Gadirov, Istanbul, Turkey; Jiwon Park, St. Andrew's School, DE, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Albert Stadler, Herrliberg, Switzerland; Mihail Sarantis, National and Kapodistrian University of Athens, Athens, Greece.

U404. Find the coefficient of x^2 after expanding the following product as a polynomial:

$$(1+x)(1+2x)^2\cdots(1+nx)^n.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Denote the b_n, a_n coefficient of x, x^2. It is obvious that $a_1 = 0$, $b_1 = 1$. Then we have

$$\begin{aligned}\ldots + a_n x^2 + b_n x + 1 &= (1+x)(1+2x)^2\ldots(1+nx)^n \\ &= (\ldots + a_{n-1}x^2 + b_{n-1}x + 1)(1+nx)^n \\ &= (\ldots + a_{n-1}x^2 + b_{n-1}x + 1)\left(\ldots + n^2\binom{n}{2}x^2 + n\binom{n}{1}x + 1\right) \\ &= \ldots + \left(a_{n-1} + n^2 b_{n-1} + \frac{n^3(n-1)}{2}\right)x^2 + (b_{n-1}+n^2)x + 1.\end{aligned}$$

Equality conditions of two polynomials, we get

$$\begin{cases} a_n = a_{n-1} + n^2 b_{n-1} + \frac{n^3(n-1)}{2} \\ b_n = b_{n-1} + n^2 \end{cases}$$

Thus, we have

$$\begin{aligned} b_n = b_1 + 2^2 + 3^2 + \ldots + n^2 &= 1^2 + 2^2 + 3^2 + \ldots + n^2 \\ &= \frac{n(n+1)(2n+1)}{6}, \end{aligned}$$

$$\begin{aligned} a_n &= a_{n-1} + n^2 \cdot \frac{(n-1)n(2n-1)}{6} + \frac{n^3(n-1)}{2} \\ &= a_{n-1} + \frac{1}{3}(n-1)n^3(n+1). \end{aligned}$$

Hence we get

$$\begin{aligned} a_n &= a_1 + \frac{1}{3}\sum_{n=2}^{n}(k-1)k^3(k+1) \\ &= \frac{1}{3}\cdot\sum_{k=1}^{n-1}k(k+1)^3(k+2). \end{aligned}$$

Using following identity

$$\begin{aligned}
k(k+1)^3(k+2) &= \frac{1}{6}k\left((k+1)^2(k+2)((k+2)(k+3)-(k-1)k\right) \\
&= \frac{1}{6}\left(k(k+1)^2(k+2)^2(k+3)-(k-1)k^2(k+1)^2(k+3)\right)
\end{aligned}$$

we get

$$\begin{aligned}
a_n &= \frac{1}{18}\left(\sum_{k=1}^{n-1} k(k+1)^2(k+2)^2(k+3) - \sum_{k=1}^{n-1}(k-1)k^2(k+1)^2(k+2)\right) \\
&= \frac{(n-1)n^2(n+1)^2(n+2)}{18}.
\end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Jang Hun Choi, Jericho, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Anderson Torres, Sao Paulo, Brazil; Matthew Li; Vincelot Ravoson, Lycée Henri IV, Paris, France; Fong Ho Leung, Hoi Ping Chamber of Commerce Secondary School; Mihail Sarantis, National and Kapodistrian University of Athens, Athens, Greece; Problem Solving Group of the Department of Finanacial and Management Engineering of the University of the Aegean, Greece; Adam Krause, College at Brockport, SUNY, NY, USA; Arkady Alt, San Jose, CA, USA; Henry Ricardo, Westchester Area Math Circle; Jiwon Park, St. Andrew's School, DE, USA; Kousik Sett, Hooghly, India; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Soumava Pal, Indian Statistical Institute, India; Albert Stadler, Herrliberg, Switzerland; Mohammed Kharbach, Abu Dhabi National Oil Company, Abu Dhabi, UAE.

U405. Let $a_1 = 1$ and

$$a_n = 1 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}}$$

for all $n > 1$. Find

$$\lim_{n\to\infty} \left(a_n - \sqrt{2n}\right).$$

Proposed by Robert Bosch, USA

Solution by Vincelot Ravoson, Lycée Henri IV, Paris, France

First, by induction we easily get that : $\forall n \in \mathbb{N}, \qquad a_{n+1} - a_n > 0.$

And also :

$$\forall n \in \mathbb{N}, \qquad a_{n+1} - a_n = \left(1 + \sum_{k=1}^{n} \frac{1}{a_k}\right) - \left(1 + \sum_{k=1}^{n-1} \frac{1}{a_k}\right) = \frac{1}{a_n} > 0$$

$$\Leftrightarrow a_{n+1} = a_n + \frac{1}{a_n}$$

Let's suppose that (a_n) converges to a real limit l. Hence we have : $l = l + \frac{1}{l} \Leftrightarrow \frac{1}{l} = 0$, impossible.

Hence :

$$\lim_{n\to+\infty} a_n = +\infty.$$

Now, we notice that :

$$\forall n \in \mathbb{N}, \qquad a_{n+1}^2 = a_n^2 + \frac{1}{a_n^2} + 2,$$

and such that $\lim\limits_{n\to+\infty} \dfrac{1}{a_n^2} = 0$:

$$a_{n+1}^2 - a_n^2 = 2 + \frac{1}{a_n^2} \sim 2,$$

and by Stolz-Cesàro theorem, we have :

$$\sum_{k=2}^{n} (a_k^2 - a_{k-1}^2) \sim 2(n-1) \sim 2n \Leftrightarrow a_n^2 - a_1^2 \sim 2n$$

Hence :

$$a_n^2 \sim a_n^2 - a_1^2 \sim 2n \Leftrightarrow a_n \sim \sqrt{2n}.$$

Thus, we have :

$$a_n^2 - 2n \sim a_n^2 - 1 - 2(n-1) = \sum_{k=1}^{n-1} (a_{k+1}^2 - a_k^2 - 2) = \sum_{k=1}^{n-1} \frac{1}{a_k^2} \sim \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{k} \sim \frac{\ln(n-1)}{2} \sim \frac{\ln(n)}{2}$$

Therefore :

$$\begin{aligned} a_n &= \sqrt{2n + \frac{\ln(n)}{2} + o(\ln(n))} \\ &= \sqrt{2n}\sqrt{1 + \frac{\ln(n)}{4n} + o\left(\frac{\ln(n)}{n}\right)} \\ &= \sqrt{2n}\left(1 + \frac{\ln(n)}{8n} + o\left(\frac{\ln(n)}{n}\right)\right) \\ &= \sqrt{2n} + \frac{\sqrt{2}\ln(n)}{8\sqrt{n}} + o\left(\frac{\ln(n)}{\sqrt{n}}\right) \end{aligned}$$

So :

$$a_n - \sqrt{2n} = \frac{\sqrt{2}\ln(n)}{8\sqrt{n}} + o\left(\frac{\ln(n)}{\sqrt{n}}\right)$$

And finally :

$$\lim_{n\to+\infty}(a_n - \sqrt{2n}) = 0.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Mohammed Kharbach, Abu Dhabi National Oil Company, Abu Dhabi, UAE; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Jamal Gadirov, Istanbul, Turkey; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Albert Stadler, Herrliberg, Switzerland.

U406. Evaluate

$$\lim_{x\to 0}\frac{\cos(n+1)x\cdot\sin nx - n\sin x}{x^3}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Alessandro Ventullo, Milan, Italy

Observe that

$$\begin{aligned}\cos(n+1)x &= 1-\frac{1}{2}(n+1)^2x^2+o(x^3)\\ \sin nx &= nx-\frac{1}{6}n^3x^3+o(x^3)\\ n\sin x &= nx-\frac{1}{6}nx^3+o(x^3).\end{aligned}$$

So,

$$\frac{\left(1-\frac{1}{2}(n+1)^2x^2+o(x^3)\right)\left(nx-\frac{1}{6}n^3x^3+o(x^3)\right)-\left(nx-\frac{1}{6}nx^3+o(x^3)\right)}{x^3}=-\frac{\frac{x^3}{3}n(n+1)(2n+1)+o(x^3)}{x^3}.$$

So,

$$\lim_{x\to 0}\frac{\cos(n+1)x\cdot\sin nx-n\sin x}{x^3}=-\frac{n(n+1)(2n+1)}{3}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Michael Tang, MN, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Stephanie Li; Vincelot Ravoson, Lycée Henri IV, Paris, France; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Fong Ho Leung, Hoi Ping Chamber of Commerce Secondary School; Mihail Sarantis, National and Kapodistrian University of Athens, Athens, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Arkady Alt, San Jose, CA, USA; Nandansai Dasireddy, Hyderabad, India; Henry Ricardo, Westchester Area Math Circle; Jiwon Park, St. Andrew's School, DE, USA; Joel Schlosberg, Bayside, NY, USA; Julio Cesar Mohnsam, IF Sul - Pelotas-RS, Brazil; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Soumava Pal, Indian Statistical Institute, India; Albert Stadler, Herrliberg, Switzerland; Zafar Ahmed, BARC, Mumbai, India and Dhruv Sharma, NIT, Rourkela, India.

U407. Prove that for every $\epsilon > 0$

$$\int_2^{2+\epsilon} e^{2x-x^2} dx < \frac{\epsilon}{1+\epsilon}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Vincelot Ravoson, Lycée Henri IV, Paris, France

For all real number X, we have :

$$e^X \geq X+1,$$

with equality if and only if $X=0$.

Now, for every $\varepsilon > 0$, and $x \in (2, 2+\varepsilon)$, let $X = x^2 - 2x > 0$. Then we have :

$$e^{x^2-2x} > x^2 - 2x + 1 = (x-1)^2.$$

$$\Leftrightarrow e^{2x-x^2} < \frac{1}{(x-1)^2}$$

$$\Rightarrow \int_2^{2+\varepsilon} e^{2x-x^2} dx < \int_2^{2+\varepsilon} \frac{dx}{(1-x)^2} = \left[\frac{1}{1-x}\right]_2^{2+\varepsilon} = \frac{\varepsilon}{1+\varepsilon}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Pedro Acosta, West Morris Mendham High School, Mendham, NJ, USA; Mihail Sarantis, National and Kapodistrian University of Athens, Athens, Greece; Zafar Ahmed, BARC, Mumbai, India and Dhruv Sharma, NIT, Rourkela, India; Alessandro Ventullo, Milan, Italy; Jamal Gadirov, Istanbul, Turkey; Jiwon Park, St. Andrew's School, DE, USA; Joel Schlosberg, Bayside, NY, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Simon Pellicer, Paris, France; Soumava Pal, Indian Statistical Institute, India; Albert Stadler, Herrliberg, Switzerland; Prajnanaswaroopa S., Bangalore, Karnataka, India.

U408. Prove that if A and B are square matrices satisfying

$$A = AB - BA + ABA - BA^2 + A^2BA - ABA^2,$$

then $\det(A) = 0$.

Proposed by Mircea Becheanu, Bucharest, Romania

Solution by Moubinool Omarjee, Lycée Henri IV, Paris, France

We have

$$A^k = A^kB - A^{k-1}BA + A^kBA - A^{k-1}BA^2 + A^{k+1}BA - A^kBA^2.$$

Takinge the trace, with $tr(MN)tr(NM)$ we deduce

$$tr(A^k) = tr(A^kB) - tr((A^{k-1}B)A) + tr((A^kB)A) - tr((A^{k-1}B)A^2) + tr((A^{k+1}B)A) - tr((A^kB)A^2) = 0.$$

For any $k \geq 1, tr(A^k) = 0 \Rightarrow A$ is nipoltent. Therefore $\det(A) = 0$.

Also solved by Mihail Sarantis, National and Kapodistrian University of Athens, Athens, Greece; Prajnanaswaroopa S., Bangalore, Karnataka, India.

O403. Let a, b, c be real numbers such that $a+b+c>0$. Prove that

$$\frac{a^2+b^2+c^2-2ab-2bc-2ca}{a+b+c}+\frac{6abc}{a^2+b^2+c^2+ab+bc+ca}\geq 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that by the scalar product inquality, $-(ab+bc+ca)\leq |ab+bc+ca|\leq a^2+b^2+c^2$, with equality in the last inequality iff $a=b=c$, and in the first iff $ab+bc+ca<0$, which are mutually exclusive. It follows that $a^2+b^2+c^2+ab+bc+ca>0$. We may then multiply both sides of the proposed inequality by the product of denominators, yielding after rearranging terms the equivalent inequality

$$\left(a^2+b^2+c^2\right)\left(a^2+b^2+c^2-ab-bc-ca\right)\geq a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2.$$

Now,

$$a^2+b^2+c^2-ab-bc-ca-(b-c)^2=a^2-ab-ca+bc=(a-b)(a-c),$$

and similarly after cyclic permutation of a, b, c, or the proposed inequality is equivalent to

$$a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b)\geq 0,$$

which is Schur's inequality. The conclusion follows, equality holds iff either $a=b=c$ or (a, b, c) is a permutation of $(k, k, 0)$ where k is any positive real.

Also solved by Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; Lee Jae Woo, Hamyang-gun, South Korea; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Joel Schlosberg, Bayside, NY, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Rajdeep Majumder, Drgapur, India; Albert Stadler, Herrliberg, Switzerland.

O404. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$(a+b+c)^2\left(\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\right)\geq 9$$

.

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Arkady Alt, San Jose, CA, USA

Since by AM-GM Inequality

$$ab+bc+ca\geq 3\sqrt[3]{a^2b^2c^2}=3$$

then

$$(a+b+c)^2=a^2+b^2+c^2+2\left(ab+bc+ca\right)\geq a^2+b^2+c^2+6=\sum_{cyc}\left(a^2+2\right)$$

and by Cauchy-Schwarz inequality

$$\sum_{cyc}\left(a^2+2\right)\cdot\sum_{cyc}\frac{1}{a^2+2}\geq 9\ .$$

Therefore,

$$(a+b+c)^2\sum_{cyc}\frac{1}{a^2+2}\geq\sum_{cyc}\left(a^2+2\right)\cdot\sum_{cyc}\frac{1}{a^2+2}\geq 9.$$

Also solved by Michael Tang, MN, USA; Nikos Kalapodis, Patras, Greece; Soo Young Choi, Vestal Senior High School, NY, USA; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Konstantinos Metaxas, 1st High School Ag. Dimitrios, Athens, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Jamal Gadirov, Istanbul, Turkey; Daniel Lasaosa, Pamplona, Spain; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Rajarshi Kanta Ghosh, Kolkata, India; Rajdeep Majumder, Drgapur, India; Albert Stadler, Herrliberg, Switzerland; Tran Tien Manh, High School for gifted students of Vinh University, Nghe An, Vietnam; Mohammed Kharbach, Abu Dhabi National Oil Company, Abu Dhabi, UAE; Prajnanaswaroopa S., Bangalore, Karnataka, India; Nguyen Ngoc Tu, Ha Giang, Vietnam.

O405. Prove that for each positive integer n there is an integer m such that 11^n divides $3^m + 5^m - 1$.

Proposed by Navid Safaei, Tehran, Iran

Solution by the author

We prove this by induction on n. The case $n = 1$ is indeed trivial for $m = 2$. Assume that the statement of the problem holds true for n , and we have $3^m + 2^m - 2 = 11^n \cdot l$ for some positive integer l which is not divisible by 11. Since $3^5 \equiv 1 \pmod{121}$ and $2^1 0 \equiv 1 \pmod{11}$, we conlude that

$$3^{5\cdot 11^{n-2}} \equiv \pmod{11^n}, 2^{10\cdot 11^{n-1}} \equiv 1 \pmod{11^n}$$

Since $\nu_{11}(3^{5\cdot 11^{n-2}-1}) = \nu_{11}(3^5 - 1) + \nu_{11}(11^{n-2}) = n$ and $\nu_{11}(2^{10\cdot 11^{n-1}} - 1) = \nu_{11}(2^{10} - 1) + \nu_{11}(11^{n-1}) = n$. Thus we can say that $3^{5\cdot 11^{n-2}} = 1 + 11^n r, 2^{10\cdot 11^{n-1}} = 1 + 11^n s$, for some positive integers r, s which are both coprime to 11. Therefore, by use of the Binomial theorem, we can easily find that

$$3^{5t\cdot 11^{n-2}} \equiv 1 + 11^n rt \pmod{11^{n+1}}, 2^{10t\cdot 11^{n-1}} = 1 + 11^n st \pmod{11^{n+1}},$$

for all positive integers t.

Now, take $m + 10t \cdot 11^{n-1}$ instead of m:

$$3^{m+10t\cdot 11^{n-1}} + 2^{m+10t\cdot 11^{n-1}} - 2 = 3^m \cdot 3^{10t\cdot 11^{n-1}} + 2^m \cdot 2^{10t\cdot 11^{n-1}} - 2$$

Taking modulo $11n + 1$ we can find that the above expression is reduced to

$$3^m(1 + 2 \cdot 11^{n+1} rt) + 2^m(1 + 11^n st) - 2 \equiv 3^m + 2^m - 2 + 2^m \cdot 11^n st \equiv 11^n(l + 2^m st) \pmod{11^{n+1}}$$

Hence, the problem is reduced to finding a positive integer t such that $l + 2^m st \equiv 0 \pmod{11}$ since $\gcd(2^m s, 11) = 1$ and such number exists.

Also solved by Rajdeep Majumder, Drgapur, India.

O406. Solve in prime numbers the equation

$$x^3 - y^3 - z^3 + w^3 + \frac{yz}{2}(2xw+1)^2 = 2017.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

Since $x^3 - y^3 - z^3 + w^3$ and 2017 are integers, then also $\frac{yz}{2}(2xw+1)^2$ must be an integer. Since $(2xw+1)^2$ is odd, then $2 \mid yz$. Since y and z are primes, then $y = 2$ or $z = 2$. By symmetry, we can assume $y = 2$. We have

$$x^3 - z^3 + w^3 + z(2xw+1)^2 = 2025.$$

Since 2025 is odd, then the LHS must be odd and it's easy to see that x, z, w cannot be all odd. Moreover, the given equation can be written as

$$x^3 + w^3 + z(2xw - z + 1)(2xw + z + 1) = 2025.$$

Since $z(2xw - z + 1)(2xw + z + 1)$ is always even, then $x^3 + w^3$ must be odd, which implies that at least one of x and w is even, i.e. $x = 2$ or $w = 2$. By symmetry, we can assume $x = 2$. We have

$$w^3 + z(4w - z + 1)(4w + z + 1) = 2017.$$

If $4w - z + 1 \geq 0$, then $w^3 \leq 2017$, so $w \leq 11$. An easy check gives $w = 7$ and $z = 2$. So, $(x, y, z, w) \in \{(2, 2, 2, 7), (7, 2, 2, 2)\}$.

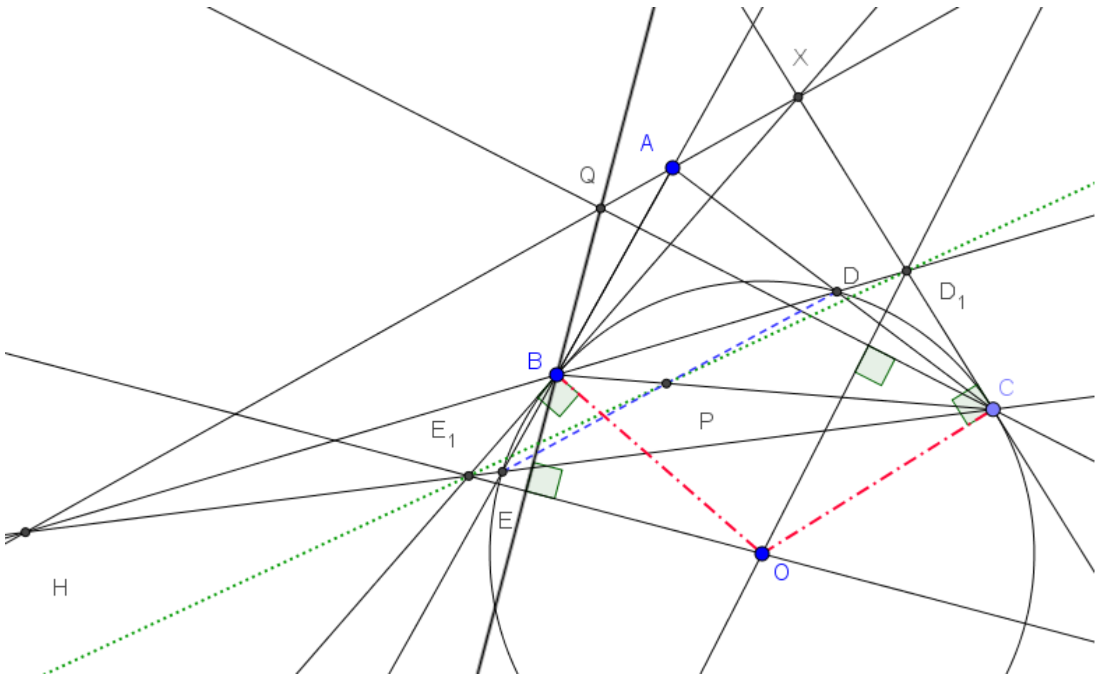
Remark: The case $4w - z + 1 \leq 0$ remains open.

O407. Let ABC be a triangle, O a point in the plane and ω a circle of center O passing through B and C such that it intersects AC in D and AB in E. Let H be the intersection of BD and CE and D_1 and E_1 be the intersection points of the tangents lines to ω at C and B with BD and CE respectively. Prove that AH and the perpendiculars from B and C to OE_1 and OD_1 respectively, are concurrent.

Proposed by Marius Stanean, Zalau, Romania

Solution by Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece

Denote by P the intersection point of BC and DE and by X the intersection of the tangent lines to ω at B and C. Let Q be the intersection point of the perpendiculars from B and C to OE_1 and OD_1.



It is well known that HA is the polar of P with respect to ω. It is obvious that BC is the polar of X with respect to ω. Hence, by La Hire's theorem X is a point on the polar of P since it is a point on BC. Therefore, points H, A, X are collinear and it suffices to prove that Q belongs to the same line.

Because of this, triangles EE_1B and DD_1C are perspective and as a result, by Desargues' theorem lines ED, E_1D_1 and BC are concurrent. Their common point is obviously P.

Observe that by definition the perpendicular to OE_1 from B is the polar of E_1 with respect to ω. Similarly, the perpendicular to OD_1 from C is the polar of D_1 with respect to ω. Since Q is the intersection of the perpendiculars, by La Hire's theorem, we deduce that E_1D_1 is the polar of Q with respect to ω.

However, P is a point on E_1D_1 so by La Hire's theorem, Q is a point on the polar of P with respect to ω. But we already proved that this polar is the line passing through H, A, X. Hence, points A, H, Q are collinear as we wanted to show.

Also solved by Carlos Yeddiel, Mexico; Jafet Alejandro Baca Obando, IDEAS High School, Sheboygan, WI, USA; Mihail Sarantis, National and Kapodistrian University of Athens, Athens, Greece.

O408. Prove that in any triangle ABC

$$\frac{a}{m_a}+\frac{b}{m_b}+\frac{c}{m_c}\geq 2\sqrt{3}.$$

Proposed by Dragoliub Milosević, Gornji Milanovac, Serbia

Solution by Henry Ricardo, Westchester Area Math Circle

The AM-GM inequality yields $4m_a^2+3a^2\geq 4\sqrt{3}\,am_a$, $4m_b^2+3b^2\geq 4\sqrt{3}\,bm_b$, and $4m_c^2+3c^2\geq 4\sqrt{3}\,cm_c$. Using the known formulas $m_a=\sqrt{(2b^2+2c^2-a^2)/4}$, $m_b=\sqrt{(2c^2+2a^2-b^2)/4}$, and $m_c=\sqrt{(2a^2+2b^2-c^2)/4}$, we have

$$4m_a^2+3a^2\geq 4\sqrt{3}\,am_a \Leftrightarrow a^2+b^2+c^2\geq 2\sqrt{3}\,am_a \tag{1}$$
$$4m_b^2+3b^2\geq 4\sqrt{3}\,bm_b \Leftrightarrow a^2+b^2+c^2\geq 2\sqrt{3}\,bm_b \tag{2}$$
$$4m_c^2+3c^2\geq 4\sqrt{3}\,cm_c \Leftrightarrow a^2+b^2+c^2\geq 2\sqrt{3}\,cm_c. \tag{3}$$

Applying inequalities $(1)-(3)$, we see that

$$\begin{aligned}\frac{a}{m_a}+\frac{b}{m_b}+\frac{c}{m_c} &= \frac{a^2}{am_a}+\frac{b^2}{bm_b}+\frac{c^2}{cm_c}\\ &\geq \frac{2\sqrt{3}\,a^2}{a^2+b^2+c^2}+\frac{2\sqrt{3}\,b^2}{a^2+b^2+c^2}+\frac{2\sqrt{3}\,c^2}{a^2+b^2+c^2}\\ &= 2\sqrt{3}.\end{aligned}$$

Also solved by Nikos Kalapodis, Patras, Greece; Sushanth Sathish Kumar, Jeffery Trail Middle School, CA, USA; Pedro Acosta, West Morris Mendham High School, Mendham, NJ, USA; Arkady Alt, San Jose, CA, USA; Jamal Gadirov, Istanbul, Turkey; Nandansai Dasireddy, Hyderabad, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata, Roma, Italy; Albert Stadler, Herrliberg, Switzerland; Telemachus Baltsavias, Keramies Junior High School, Kefalonia ,Greece; Tran Tien Manh, High School for gifted students of Vinh University, Nghe An, Vietnam; Mihail Sarantis, National and Kapodistrian University of Athens, Athens, Greece.