Junior Problems

J409. Solve the equation

$$\log\left(1-2^x+5^x-20^x+50^x\right)=2x.$$

Proposed by Adrian Andreescu, Dallas, Texas, USA

J410. Let a,b,c,d be real numbers such that $a^2\le 2b$ and $c^2<2bd$. Prove that

$$x^4+ax^3+bx^2+cx+d>0$$

for all $x\in\mathbb{R}$.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

J411. Find all primes p and q such that

$$\frac{p^3-2017}{q^3-345}=q^3.$$

Proposed by Titu Andreescu, University of Dallas at Texas, USA

J412. Let $a\ge b\ge c$ be positive real numbers. Prove that

$$(a-b+c)\left(\frac{1}{a+b}-\frac{1}{b+c}+\frac{1}{c+a}\right)\le\frac{1}{2}$$

Proposed by An Zhenping, Xianyang Normal University, China

J413. Solve in integers the system of equations

$$x^2y+y^2z+z^2x-3xyz=23$$
$$xy^2+yz^2+zx^2-3xyz=25.$$

Proposed by Adrian Andreescu, Dallas, USA

J414. Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\ge a^2+b^2+c^2.$$

Proposed by Dragoljub Miloševići, Gornji Milanovac, Serbia

Senior Problems

S409. Solve in real numbers the equation

$$2\sqrt{x-x^2}-\sqrt{1-x^2}+2\sqrt{x+x^2}=2x+1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S410. Let ABC be a triangle with orthocenter H and circumcenter O. We denote $\angle AOH = \alpha, \angle BOH = \beta, \angle COH = \gamma$. Prove that

$$\left(\sin^2\alpha+\sin^2\beta+\sin^2\gamma\right)^2=2\left(\sin^4\alpha+\sin^4\beta+\sin^4\gamma\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S411. Solve in real numbers the system of equations

$$\begin{cases}\sqrt{x}-\sqrt{y}=45\\ \sqrt[3]{x-2017}-\sqrt[3]{y}=2.\end{cases}$$

Proposed by Adrian Andreescu, Dallas, Texas, USA

S412. Let a, b, c be positive real numbers such that

$$\frac{1}{\sqrt{1+a^3}}+\frac{1}{\sqrt{1+b^3}}+\frac{1}{\sqrt{1+c^3}}\le 1.$$

Prove that $a^2+b^2+c^2\ge 12$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S413. Let n be a composite integer. Given that n divides $\binom{n}{2}, \ldots, \binom{n}{k-1}$ and n does not divides $\binom{n}{k}$, prove that k is prime.

Proposed by Robert Boch, USA

S414. Prove that for any positive integers a and b

$$(a^6-1)(b^6-1)+(3a^2b^2+1)(2ab-1)(ab+1)^2$$

is the product of at least four primes, not necessarily distinct.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Undergraduate Problems

U409. Evaluate

$$\lim_{x\to 0} \frac{2\sqrt{1+x}+2\sqrt{2^2+x}+\cdots+2\sqrt{n^2+x}-n(n+1)}{x}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U410. Let a, b, c be real numbers such that $a+b+c=5$. Prove that

$$(a^2+3)(b^2+3)(c^2+3) \geq 192.$$

Proposed by Marius Stănean, Zalău, România

U411. Let e be a positive integer. For any positive integer m denote by $\omega(m)$ the number of distinct prime divisors of m. We say that m is *awesome* if it has $\omega(m)^e$ digits in base ten. Prove that there are only finitely many awesome numbers.

Proposed by Alessandro Ventullo, Milan, Italy

U412. Let $P(x)$ be a monic polynomial with real coefficients, of degree n, which has n real roots. Prove that if

$$P(c) \leq \left(\frac{b^2}{a}\right)^n$$

then $P(ax^2+2bx+c)$ has at least one real root.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U413. Let ABC be a triangle and let a, b, c be the lengths of sides BC, CA, AB, respectively. The tangency points of the incircle with sides BC, CA, AB are denoted by A', B', C'.

(a) Prove that the segments of lengths $AA'\sin A$, $BB'\sin B$, $CC'\sin C$ are the sides of a triangle.

(b) If $A_1B_1C_1$ is such a triangle, compute in terms of a, b, c the ratio

$$\frac{K[A_1B_1C_1]}{K[ABC]}.$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

U414. Let $p < q < 1$ be positive real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the conditions:

(i) $f(px+f(x)) = qf(x)$ for all real numbers x,

(ii) $\displaystyle\lim_{x\to 0} \frac{f(x)}{x}$ exists and is finite.

Proposed by Florin Stănescu, Găeşti, România

Olympiad Problems

O409. Find all positive integers n for which there are $n+1$ digits in base 10, not necessarily distinct, such that at least $2n$ permutations of those digits produce $(n+1)$-digit perfect squares, with leading zeros not allowed. Note that two different permutations are considered distinct even if they lead to the same digit string due to repetition among the digits.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O410. On each cell of a chess board it is written a number equal to the amount of the rectangles that contain this cell. Find the sum of all the numbers.

Proposed by Robert Bosch, USA

O411. For a positive integer n denote by $S(n)$ the sum of all prime divisors of n. (For example, S(1)=0; S(2)=2; S(45)=8.) Find all positive integers n such that $S(n) = S(2^n + 1)$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

O412. Let $ABCDE$ be a convex pentagon such that $AC = AD = AB + AE$ and $BC + CD + DE = BD + CE$. Lines AB and AE intersect CD at F and G, respectively. Prove that

$$\frac{1}{AF} + \frac{1}{AG} = \frac{1}{AC}$$

Proposed by Anton Vasilyev, Astana, Kazakhstan

O413. Let ABC be an acute triangle. Prove that:

$$\text{a)} \frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \geq \frac{a+b+c}{R+r}$$

$$\text{b)} \frac{b+c}{m_a} + \frac{c+a}{m_b} + \frac{a+b}{m_c} \geq \frac{4(a+b+c)}{3R}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O414. Characterize all positive integers n with the following property: for any two coprime divisors $a < b$ of n, $b - a + 1$ is also a divisor of n.

Proposed by Vlad Matei, University of Wisconsin, Madison, USA