Junior problems

J409. Solve the equation

$$\log\left(1 - 2^x + 5^x - 20^x + 50^x\right) = 2x.$$

Proposed by Adrian Andreescu, Dallas, Texas, USA

Solution by Arkady Alt, San Jose, CA, USA

Let $u := 2^x, v := 5^x$. Then $u, v > 0$ and

$$\log(1 - 2^x + 5^x - 20^x + 50^x) = 2x \iff 1 - 2^x + 5^x - 20^x + 50^x = 10^{2x} \iff$$

$$1 - u + v - u^2v + uv^2 = u^2v^2 \iff 1 - u + v - u^2v + uv^2 - u^2v^2 = 0 \iff$$

$$(1 - uv)(1 + uv) - (u - v)(1 + uv) = 0 \iff (1 + uv)(1 - uv - u + v) = 0 \iff$$

$$(1 + uv)(1 + v)(1 - u) = 0 \iff u = 1$$

because $1 + uv, 1 + v > 0$.
Hence $2^x = 1 \iff x = 0$. Thus, $x = 0$ is only solution.

Also solved by Daniel Lasaosa, Pamplona, Spain; Aditya Ghosh, Kolkata, India; Alessandro Ventullo, Milan, Italy; Konstantinos Metaxas, 1st Ag. Dimitrios High School, Athens, Greece; Joel Schlosberg, Bayside, NY, USA; John Bellos, Senior High School, Larissa, Greece; Julio Cesar Mohnsam, IF Sul, Campus Pelotas, RS, Brazil; Nandansai Dasireddy, Hyderabad, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nestor Rodríguez, Angel Mejía, Universidad Autónoma de Santo Domingo, Dominican Republic; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Rajdeep Majumder, Drgapur, India; Roberto Ruiz, Universidad Nacional de Ingeniería, Managua, Nicaragua; Shuborno Das, Ryan International School, Bangalore, India; Upamanyu Mukharji, The Heritage School, Kolkata, India; Titu Zvonaru, Comănești, România; Polyahedra, Polk State College, FL, USA; Konstantinos Metaxas, 1st Ag. Dimitrios High School, Athens, Greece; Vincelot Ravoson, Lycée Henri IV, Paris, France; Celine Lee, Chinese International School, Hong Kong; Nikos Kalapodis, Patras, Greece; P.V.Swaminathan, Smart Minds Academy, Chennai, India.

J410. Let a, b, c, d be real numbers such that $a^2 \le 2b$ and $c^2 < 2bd$. Prove that

$$x^4 + ax^3 + bx^2 + cx + d > 0$$

for all $x \in \mathbb{R}$.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

Solution by Polyahedra, Polk State College, USA

Notice that $2b \ge a^2 \ge 0$ and $2bd > c^2 \ge 0$. So $b > 0$ and for all $x \in \mathbb{R}$,

$$x^4 + ax^3 + bx^2 + cx + d = x^2 \left[\left(x + \frac{a}{2} \right)^2 + \frac{2b - a^2}{4} \right] + \frac{b}{2} \left(x + \frac{c}{b} \right)^2 + \frac{2bd - c^2}{2b} > 0.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Konstantinos Metaxas, 1st Ag. Dimitrios High School, Athens, Greece; Soo Young Choi, Vestal High School, Vestal, NY, USA; Vincelot Ravoson, Lycée Henri IV, Paris, France; Arkady Alt, San Jose, CA, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Alessandro Ventullo, Milan, Italy; Corneliu Mănescu-Avram, Ploieşti, Romania; Joel Schlosberg, Bayside, NY, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, România; Prajnanaswaroopa S, Bangalore, Karnataka, India.

J411. Find all primes p and q such that

$$\frac{p^3 - 2017}{q^3 - 345} = q^3.$$

Proposed by Titu Andreescu, University of Dallas at Texas, USA

Solution by Alessandro Ventullo, Milan, Italy

The given equation is equivalent to

$$p^3 - 2017 = q^6 - 345q^3.$$

Reducing modulo 7, we get

$$p^3 - 1 \equiv q^6 - 2q^3 \pmod 7 \iff p^3 \equiv (q^3 - 1)^2 \pmod 7.$$

Since $q^3 \equiv 0, 1, 6 \pmod 7$, then $(q^3 - 1)^2 \equiv 0, 1, 4 \pmod 7$. Since $p^3 \equiv 0, 1, 6 \pmod 7$, we get $(q^3 - 1)^2 \equiv 0, 1 \pmod 7$. If $(q^3 - 1)^2 \equiv 0 \pmod 7$, then $p^3 \equiv 0 \pmod 7$, which gives $p = 7$. But the equation $q^6 - 345q^3 + 1674 = 0$ gives no integer solutions. If $(q^3 - 1)^2 \equiv 1 \pmod 7$, then $q^3 \equiv 0 \pmod 7$, which gives $q = 7$. So, $p^3 - 2017 = -686$, i.e. $p^3 = 1331$, which gives $p = 11$. So, $p = 11$ and $q = 7$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Konstantinos Metaxas, 1st Ag. Dimitrios High School, Athens, Greece; Polyahedra, Polk State College, FL, USA; Soo Young Choi, Vestal High School, Vestal, NY, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, România.

J412. Let $a \geq b \geq c$ be positive real numbers. Prove that Let $a \geq b \geq c$ be positive real numbers. Prove that

$$(a-b+c)\left(\frac{1}{a+b}-\frac{1}{b+c}+\frac{1}{c+a}\right) \leq \frac{1}{2}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Albert Stadler, Herrliberg, Switzerland

Let $r := a-b \geq 0, s := b-c \geq 0$. Then $a = c+r+s, b = c+s$ and

$$\frac{1}{2}-(a-b+c)\left(\frac{1}{a+b}-\frac{1}{b+c}+\frac{1}{c+a}\right) = \frac{1}{2}-(c+r+s-c-s+c)\left(\frac{1}{c+r+s+c+s}-\frac{1}{c+s+c}+\frac{1}{c+c+r+s}\right) =$$

$$= \frac{4cr^2+2r^3+8c^2s+4crs+3r^2s+8cs^2+rs^2+2s^3}{2(2c+s)(2c+r+s)(2c+r+2s)} \geq 0.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Konstantinos Metaxas, 1st Ag. Dimitrios High School, Athens, Greece; Polyahedra, Polk State College, FL, USA; Vincelot Ravoson, Lycée Henri IV, Paris, France; Arkady Alt, San Jose, CA, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Alessandro Ventullo, Milan, Italy; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Rajdeep Majumder, Drgapur, India; Shuborno Das, Ryan International School, Bangalore, India; Titu Zvonaru, Comănești, România.

J413. Solve in integers the system of equations

$$\begin{cases} x^2y + y^2z + z^2x - 3xyz = 23, \\ xy^2 + yz^2 + zx^2 - 3xyz = 25. \end{cases}$$

Proposed by Adrian Andreescu, Dallas, USA

Solution by Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua
Subtracting the first from the second equation:

$$(xy^2 + yz^2 + zx^2) - (x^2y + y^2z + z^2x) = 2$$

And using the identity $(xy^2 + yz^2 + zx^2) - (x^2y + y^2z + z^2x) = (x - y)(y - z)(z - x)$, considering that the product and the sum of $x - y$, $y - z$ and $z - x$ are 2 and 0 respectively, we can conclude that the are two possibilities for those three integers: $2, -1, -1$ and $-2, 1, 1$.

The LHS of the second equation can be expressed as $z(x - y)^2 + y(x - z)(y - z)$. There are several cases, and is necessary to check:

- If $x - y = 2$, $y - z = -1$ and $z - x = -1$, then the second equation becomes $4z - y = 25$, and we have the solution $(x, y, z) = (9, 7, 8)$
- If $x - y = -1$, $y - z = 2$ and $z - x = -1$, then the second equation becomes $z + 2y = 25$, and we have the solution $(x, y, z) = (8, 9, 7)$
- If $x - y = -1$, $y - z = -1$ and $z - x = 2$, then the second equation becomes $z + 2y = 25$, and we have the solution $(x, y, z) = (7, 8, 9)$
- If $x - y = -2$, $y - z = 1$ and $z - x = 1$, then the second equation becomes $4z - y = 25$, and we have not solution in integers
- If $x - y = 1$, $y - z = -2$ and $z - x = 1$, then the second equation becomes $z + 2y = 25$, and we have not solution in integers
- If $x - y = 1$, $y - z = 1$ and $z - x = -2$, then the second equation becomes $z + 2y = 25$, and we have not solution in integers

The integers solutions of the system are $(x, y, z) = \{(9, 7, 8), (8, 9, 7), (7, 8, 9)\}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Konstantinos Metaxas, 1st Ag. Dimitrios High School, Athens, Greece; Polyahedra, Polk State College, FL, USA; Soo Young Choi, Vestal High School, Vestal, NY, USA; Vincelot Ravoson, Lycée Henri IV, Paris, France; Celine Lee, Chinese International School, Hong Kong; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Alessandro Ventullo, Milan, Italy; Corneliu Mănescu-Avram, Ploieşti, Romania; Joel Schlosberg, Bayside, NY, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, România.

J414. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\geq a^2+b^2+c^2.$$

Proposed by Dragoljub Miloševići, Gornji Milanovac, Serbia

Solution by the author

Using the Cauchy – Schwarz's inequality we have

$$\left(\frac{a\sqrt{a}}{c}\cdot c\sqrt{a}+\frac{b\sqrt{b}}{a}\cdot a\sqrt{b}+\frac{c\sqrt{c}}{b}\cdot b\sqrt{c}\right)^2\leq\left(\frac{b^3}{a^2}+\frac{c^3}{b^2}+\frac{a^3}{c^2}\right)(ba^2+cb^2+ac^2),$$

that is,

$$\frac{b^3}{a^2}+\frac{c^3}{b^2}+\frac{a^3}{c^2}\geq\frac{(a^2+b^2+c^2)^2}{ba^2+cb^2+ac^2}. \tag{1}$$

The inequality $a^2+b^2+c^2\geq ba^2+cb^2+ac^2$ is equivalent to

$$(a+b+c)(a^2+b^2+c^2)\geq 3(ba^2+cb^2+ac^2), \tag{2}$$

because $a+b+c=3$.

Now, the inequality (2) is equivalent to

$$a^3+b^3+c^3+ab^2+bc^2+ca^2\geq 2(ba^2+cb^2+ac^2)\Leftrightarrow$$
$$\Leftrightarrow (a^3-ba^2)+(b^3-cb^2)+(c^3-ac^2)+(ab^2-ba^2)+(bc^2-cb^2)+(ca^2-ac^2)\geq 0$$
$$\Leftrightarrow (a-b)(a^2-ab)+(b-c)(b^2-bc)+(c-a)(c^2-ca)\geq 0$$
$$\Leftrightarrow a(a-b)^2+b(b-c)^2+c(c-a)^2\geq 0.$$

The conclusion follows from (1) and (2).

Also solved by Polyahedra, Polk State College, USA; Daniel Lasaosa, Pamplona, Spain; Konstantinos Metaxas, 1st Ag. Dimitrios High School, Athens, Greece; Nikos Kalapodis, Patras, Greece; Prajnanaswaroopa S, Bangalore, Karnataka, India; Arkady Alt, San Jose, CA, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Adamopoulos Dionysios, 4th Junior High School, Pyrgos, Greece; Aditya Ghosh, Kolkata, India; Alessandro Ventullo, Milan, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Jamal Gadirov, Istanbul, Turkey; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Rajdeep Majumder, Drgapur, India; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, România; Kevin Soto Palacios Huarmey, Perú.

Senior problems

S409. Solve in real numbers the equation

$$2\sqrt{x-x^2}-\sqrt{1-x^2}+2\sqrt{x+x^2}=2x+1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Emily Krales, College at Brockport, SUNY, NY, USA

The domain of the equation is given by

$$\begin{cases} x-x^2\geq 0 \\ 1-x^2\geq 0 \\ x+x^2\geq 0 \end{cases}$$

The first has solution $x\in[0,1]$, the second $x\in[-1,1]$ and the third $x\in(-\infty,-1]\cup[0,\infty)$.
Therefore the domain is $x\in[0,1]$
The equation is equivalent to

$$2\sqrt{x-x^2}+2\sqrt{x+x^2}=2x+1+\sqrt{1-x^2}$$

Squaring both sides we get

$$4x-4x^2+4x+4x^2+8\sqrt{x^2-x^4}=(2x+1)^2+1-x^2+2(2x+1)\sqrt{1-x^2}$$

$$\iff 8x+8\sqrt{x^2(1-x^2)}=4x^2+4x+2-x^2+2(2x+1)\sqrt{1-x^2}$$

$$\iff 8x\sqrt{1-x^2}-2(2x+1)\sqrt{1-x^2}=3x^2-4x+2$$

$$(4x-2)\sqrt{1-x^2}=3x^2-4x+2$$

Squaring again we get

$$(16x^2-16x+4)(1-x^2)=9x^4+16x^2+4-24x^3+12x^2-16x$$

$$\iff -16x^4+16x^3+12x^2-16x+4=9x^4-24x^3+28x^2-16x+4$$

$$\iff 25x^4-40x^3+16x^2=0 \iff x^2(5x-4)^2=0.$$

therefore, the only solution is $x=4/5$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Soo Young Choi, Vestal High School, Vestal, NY, USA; Vincelot Ravoson, Lycée Henri IV, Paris, France; Celine Lee, Chinese International School, Hong Kong; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Arkady Alt, San Jose, CA, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Titu Zvonaru, Comănești, România; Abhay Chandra, IIT Delhi, New Delhi, India; Alessandro Ventullo, Milan, Italy; Jamal Gadirov, Istanbul, Turkey; Dona Ghosh, JU, Jadavpur, India; Duy Quan Tran, University of Medicine and Pharmacy at Ho Chi Minh City, Vietnam; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Roberto Ruiz, Universidad Nacional de Ingeniería, Managua, Nicaragua; Albert Stadler, Herrliberg, Switzerland; Vu Tan Khang, Hungyen Specialized High School, Hungyen, Vietmam.

S410. Let ABC be a triangle with orthocenter H and circumcenter O. We denote $\angle AOH = \alpha$, $\angle BOH = \beta$, $\angle COH = \gamma$. Prove that

$$\left(\sin^2\alpha + \sin^2\beta + \sin^2\gamma\right)^2 = 2\left(\sin^4\alpha + \sin^4\beta + \sin^4\gamma\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain

Note that the proposed inequality rewrites as

$$2\left(\sin^2\alpha\sin^2\beta + \sin^2\beta\sin^2\gamma + \sin^2\gamma\sin^2\alpha\right) - \left(\sin^4\alpha + \sin^4\beta + \sin^4\gamma\right) = 0,$$

ie by Heron's formula, $\sin\alpha, \sin\beta, \sin\gamma$ are the sidelengths of a triangle with area zero, or two out of these three values add up to the third. Now, applying the Sine Law to AOH, we have

$$\frac{\sin\alpha}{AH} = \frac{\sin\angle OAH}{OH},$$

where since $\angle BAH = \angle CAO = 90^\circ - B$, we have $\angle OAH = |A - 180^\circ + 2B| = |B - C|$, and similarly $\angle OBH = |C - A|$ and $\angle OCH = |A - B|$. Moreover, it is well known that $AH = 2R|\cos A|$, and similarly for BH and CH. If ABC is not obtuse, assume wlog by the symmetry in the problem that $90^\circ \geq A \geq B \geq C$, or

$$\frac{OH}{2R}(\sin\alpha + \sin\gamma) = \cos A\sin(B - C) + \cos C\sin(A - B) =$$

$$= \cos C\sin A\cos B - \cos A\cos B\sin C = \cos B\sin(A - C) = \frac{OH}{2R}\sin\beta,$$

or $\sin\beta = \sin\gamma + \sin\alpha$. Now assume that ABC is obtuse, denote by B the obtuse angle, assume wlog that $B > A \geq C$, and note that in the previous relation, $\sin(A - B)$ changes sign, and so does $\cos B$, all other terms remaining unchanged, and yielding $\sin\gamma = \sin\alpha + \sin\beta$. The conclusion follows.

Also solved by Titu Zvonaru, Comănești, România; Albert Stadler, Herrliberg, Switzerland; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece.

S411. Solve in real numbers the system of equations

$$\begin{cases} \sqrt{x} - \sqrt{y} = 45, \\ \sqrt[3]{x-2017} - \sqrt[3]{y} = 2. \end{cases}$$

Proposed by Adrian Andreescu, Dallas, Texas, USA

Solution by Emily Krales, College at Brockport, SUNY, NY, USA

$$\sqrt{x} = 45 + \sqrt{y} \iff x = 2025 + 90\sqrt{y} + y$$

By substituting this into the second equation I get

$$\sqrt[3]{8 + y + 90\sqrt{y}} = 2 + \sqrt[3]{y}$$

$$\iff 8 + y + 90\sqrt{y} = 8 + 12\sqrt[3]{y} + 6\sqrt[3]{y^2} + y \iff 90\sqrt{y} = 12\sqrt[3]{y} + 6\sqrt[3]{y^2}$$

$$\iff 15\sqrt{y} = 2\sqrt[3]{y} + \sqrt[3]{y^2}$$

Let

$$u = \sqrt[3]{y}.$$

The equation becomes

$$15\sqrt{u^3} = 2u + u^2 \iff 225u^3 = (2u + u^2)^2 \iff 225u^3 = 4u^2 + 4u^3 + u^4$$

$$\iff u^4 - 221u^3 + 4u^2 = 0 \iff u^2(u^2 - 221u + 1) = 0$$

with solutions

$$u = 0, \qquad u = \frac{221 \pm 15\sqrt{217}}{2}$$

When

$$u = 0$$

we get

$$y = 0 \qquad \text{and} \qquad x = 2025$$

When

$$u = \frac{221 \pm 15\sqrt{217}}{2}$$

we get

$$y = \frac{10791209 \pm 732555\sqrt{217}}{2} \qquad \text{and} \qquad x = \frac{11090909 \pm 752625\sqrt{217}}{2}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Konstantinos Metaxas, 1st Ag. Dimitrios High School, Athens, Greece; Vincelot Ravoson, Lycée Henri IV, Paris, France; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Arkady Alt, San Jose, CA, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Titu Zvonaru, Comănești, România; Albert Stadler, Herrliberg, Switzerland; Alessandro Ventullo, Milan, Italy; Jamal Gadirov, Istanbul, Turkey; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece.

S412. Let a, b, c be positive real numbers such that

$$\frac{1}{\sqrt{1+a^3}}+\frac{1}{\sqrt{1+b^3}}+\frac{1}{\sqrt{1+c^3}} \leq 1.$$

Prove that $a^2+b^2+c^2 \geq 12$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece

By Titu's Lemma in B.C.S Inequality, it is

$$1 \geq \frac{1}{\sqrt{1+a^3}}+\frac{1}{\sqrt{1+b^3}}+\frac{1}{\sqrt{1+c^3}} \geq \frac{9}{\sqrt{1+a^3}+\sqrt{1+b^3}+\sqrt{1+c^3}},$$

hence $\sqrt{1+a^3}+\sqrt{1+b^3}+\sqrt{1+c^3} \geq 9$.
By B.C.S. Inequality we deduce that

$$[(a+1)+(b+1)+(c+1)][(a^2-a+1)+(b^2-b+1)+(c^2-c+1)] \geq (\sqrt{1+a^3}+\sqrt{1+b^3}+\sqrt{1+c^3})^2.$$

Therefore, we obtain that $(a+b+c+3)(a^2+b^2+c^2-a-b-c+3) \geq 81 (1)$ holds.
Let $s = a+b+c$ and $p = ab+bc+ca$.
Then $a^2+b^2+c^2 = s^2-2p$ and we may write (1) as

$$(s+3)(s^2-2p-s+3) \geq 81 \Leftrightarrow s^2-2p \geq \frac{s^2+72}{s+3}.$$

However, it is obvious that $\frac{s^2+72}{s+3} \geq 12 \Leftrightarrow (s-6)^2 \geq 0$ holds.
Hence, we obtain that $s^2-2p \geq 12 \Leftrightarrow a^2+b^2+c^2 \geq 12$ as wanted.
Equality holds if and only if $a = b = c = 2$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Konstantinos Metaxas, 1st Ag. Dimitrios High School, Athens, Greece; Soo Young Choi, Vestal High School, Vestal, NY, USA; Nikos Kalapodis, Patras, Greece; Kevin Soto Palacios Huarmey, Perú; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Albert Stadler, Herrliberg, Switzerland; Adamopoulos Dionysios, 4th Junior High School, Pyrgos, Greece; Aditya Ghosh, Kolkata, India; Alessandro Ventullo, Milan, Italy; Jamal Gadirov, Istanbul, Turkey; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Vu Tan Khai, Bguyen Tat Thanh Secondary school, Vietnam.

S413. Let n be a composite integer. Given that n divides $\binom{n}{2}, \ldots, \binom{n}{k-1}$ and n does not divide $\binom{n}{k}$, prove that k is prime.

Proposed by Robert Bosch, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Let p be any prime which divides n. Then, $(n-1)!$ and $(n-p)!$ are divisible by exactly the same power of p, since $n-p$ and n are consecutive multiples of p. Or since $p!$ is divisible by p, we have $\frac{1}{n}\binom{n}{p} = \frac{(n-1)!}{p!(n-p)!}$ is not an integer since the power of p in the denominator is larger by 1 than the power of p in the numerator.

Let now p be precisely the smallest prime which divides n, and let j be any integer larger than 1 and less than p, or j is coprime with n. For such a j, write $\frac{1}{n}\binom{n}{j}$ as an irreducible fraction $\frac{u}{v}$, and note that since $A = \binom{n}{j}$ and $B = \binom{n-1}{j-1}$ are both integers, we have

$$\frac{u}{v} = \frac{(n-1)!}{j!(n-j)!} = \frac{A}{n} = \frac{B}{j},$$

or v divides n and j. But since n and j are coprime, $v = 1$, or $\binom{n}{j}$ is divisible by n for all $j = 2, 3, \ldots, p-1$. It follows that k is precisely the smallest prime p which divides n.

S414. Prove that for any positive integers a and b

$$\left(a^6-1\right)\left(b^6-1\right)+\left(3a^2b^2+1\right)(2ab-1)(ab+1)^2$$

is the product of at least four primes, not necessarily distinct.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

Let

$$N=(a^6-1)(b^6-1)+(3a^2b^2+1)(2ab-1)(ab+1)^2.$$

If $a=b=1$, then $N=16=2\cdot 2\cdot 2\cdot 2$. If $b=1$ and $a>1$, then $N=(3a^2+1)(2a-1)(a+1)^2$ is the product of four factors greater than 1 and so is the product of at least four primes. Assume that $a\geq b>1$. Let $s=a+b$, $p=ab$ and $d=a-b$. Observe that $d^2=(a-b)^2=(a+b)^2-4ab=s^2-4p$. Clearly, $s\geq 4$, $p\geq 4$ and $d\geq 0$. We have

$$\begin{aligned}
a^6+b^6 &= (a^2+b^2)(a^4+b^4)-(ab)^2(a^2+b^2)\\
&= (a^2+b^2)(a^4+b^4-a^2b^2)\\
&= (s^2-2p)(s^4+p^2-4s^2p).
\end{aligned}$$

Hence,

$$\begin{aligned}
N &= p^6-(s^2-2p)(s^4+p^2-4s^2p)+1+(3p^2+1)(2p-1)(p+1)^2\\
&= (p^2+4p-s^2)(p^4+2p^3+p^2s^2+p^2-2ps^2+s^4)\\
&= (p^2-d^2)[(p^2+s^2-p)^2-d^2p^2]\\
&= (p-d)(p+d)(p^2+s^2-p-dp)(p^2+s^2-p+dp).
\end{aligned}$$

Since $a\geq b>1$, then $p-d>1$ and $p+d>1$. Moreover, $p^2+s^2-p-dp=p(p-d)+s^2-p>p+s^2-p=s^2>1$ and $p^2+s^2-p+dp\geq p(p-1)+s^2>1$, so N is the product of four factors greater than 1 and so is the product of at least four primes.

Also solved by Daniel Lasaosa, Pamplona, Spain; Vincelot Ravoson, Lycée Henri IV, Paris, France; Celine Lee, Chinese International School, Hong Kong; Joel Schlosberg, Bayside, NY, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Albert Stadler, Herrliberg, Switzerland.

U409. Evaluate

$$\lim_{x\to 0} \frac{2\sqrt{1+x}+2\sqrt{2^2+x}+\cdots+2\sqrt{n^2+x}-n(n+1)}{x}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Joel Schlosberg, Bayside, NY, USA

Since

$$\lim_{x\to 0} 2\sqrt{1+x}+2\sqrt{2^2+x}+\cdots+2\sqrt{n^2+x}-n(n+1) =$$

$$= 2\left[1+2+\cdots+n-\frac{n(n+1)}{2}\right] = 0,$$

L'Hospital's rule may be applied, so

$$\lim_{x\to 0} \frac{2\sqrt{1+x}+2\sqrt{2^2+x}+\cdots+2\sqrt{n^2+x}-n(n+1)}{x}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{2^2+x}}+\cdots+\frac{1}{\sqrt{n^2+x}}}{1} = 1+\frac{1}{2}+\cdots+\frac{1}{n}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Konstantinos Metaxas, 1st Ag. Dimitrios High School, Athens, Greece; Vincelot Ravoson, Lycée Henri IV, Paris, France; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Prajnanaswaroopa S, Bangalore, Karnataka, India; Arkady Alt, San Jose, CA, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Shuborno Das, Ryan International School, Bangalore, India; Aditya Ghosh, Kolkata, India; Alessandro Ventullo, Milan, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Corneliu Mănescu-Avram, Ploieşti, Romania; Henry Ricardo, Westchester Area Math Circle; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Roberto Ruiz, Universidad Nacional de Ingeniería, Managua, Nicaragua; Albert Stadler, Herrliberg, Switzerland.

U410. Let a, b, c be real numbers such that $a+b+c=5$. Prove that

$$\left(a^2+3\right)\left(b^2+3\right)\left(c^2+3\right) \geq 192.$$

Proposed by Marius Stănean, Zalău, România

Solution by Arkady Alt, San Jose, CA, USA

Due to symmetry we can assume that $c := \min\{a, b, c\}$ .Then $c \leq \dfrac{5}{3}$ and $a+b \geq \dfrac{10}{3}$.
Let $p := a+b$. Since $(a^2+3)(b^2+3) \geq 3(a+b)^2 \iff (ab-3)^2 \geq 0$ suffice to prove the inequality

$$3(a+b)^2\left(c^2+3\right) \geq 192 \iff (a+b)^2\left(c^2+3\right) \geq 64$$

or, in more compact notation, prove

$$p^2\left(\left((5-p)^2+3\right)\right) \geq 64 \iff p^2\left(p^2-10p+28\right) \geq 64.$$

Since $p \geq \dfrac{10}{3}$ we have

$$p^2\left(p^2-10p+28\right)-64 = p^4-10p^3+28p^2-64 =$$

$$\left(p^2-2p-4\right)(p-4)^2 \geq 0 \text{ (because } p^2-2p-4=(p-1)^2-5 \geq \left(\frac{10}{3}-1\right)^2-5=\frac{4}{9}>0.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Konstantinos Metaxas, 1st Ag. Dimitrios High School, Athens, Greece; Prajnanaswaroopa S, Bangalore, Karnataka, India; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Alessandro Ventullo, Milan, Italy; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Albert Stadler, Herrliberg, Switzerland.

U411. Let e be a positive integer. For any positive integer m denote by $\omega(m)$ the number of distinct prime divisors of m. We say that m is awesome if it has $\omega(m)^e$ digits in base ten. Prove that there are only finitely many awesome numbers.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Daniel Lasaosa, Pamplona, Spain

We claim that the proposed result is true for $e = 1$, but not true for any other positive integer e.

Case 1: When $e = 1$, consider the product of the first k primes, also known as primorial, which is known to be asymptotically of the form $\exp(k\ln(k))$, ie a positive integer K and a real constant C exist such that, for all $k \geq K$, the product of the first k primes is at least $C \cdot \exp(k\ln(k))$. We claim that a positive integer K' exists such that $K' \geq K$, and for all $k \geq K'$, we have $C \cdot \exp(k\ln(k)) > 10^k$, ie the product of the first k primes has more than k digits. Indeed, taking natural logarithm on both sides, this is equivalent to $\ln(C) + k\ln(k) > k\ln(10)$, or to $\ln(k) > \ln(10) - \frac{\ln(C)}{k}$. As k grows, the LHS grows without bounds, and the RHS is bounded by $\ln(10)$, or indeed such a K' exists. Note therefore that, if m has $\omega(m) = k$ distinct prime divisors for $k \geq K'$, then m is at least the product of k distinct primes, ie at least the product of the first k primes, hence it has more than k digits. Awesome numbers cannot have K' digits or more, hence they are all less than $10^{K'}$, or there is a finite number of them.

Case 2: When $e \geq 2$, consider again the product of the first k primes, which we now prove that is less than $10^{k^2} \leq 10^{k^e}$ for all $k \geq K'$. Note that there exists another constant $D > C$ such that for all $k \geq K$, the product of the first k primes is less than $D \cdot \exp(k\ln(k))$, and $D \cdot \exp(k\ln(k)) < 10^{(k^2)}$ is equivalent to $\ln(D) + k\ln(k) < \ln(10)k^2$, or $\ln(10) > \frac{\ln(k)}{k} + \ln(D)k^2$. The LHS is a positive real constant, whereas the RHS decreases to values arbitrarily close to 0, or indeed for any $e \geq 2$, there exists a K'' such that for all $k \geq K''$, the product of the first k primes is less than $10^{(k^2)}$, ie less than $10^{(k^e)}$ for each $e \geq 2$. Now, if the product is larger than $10^{(k^e)-1}$, this product is awesome. Otherwise, multiply by 2 as may times as needed until obtaining a number which is larger than or equal to $10^{(k^e)-1}$. The first such number will be less than $2 \cdot 10^{(k^e)-1} < 10^{(k^e)}$, or it will have k^e digits, and will thus be awesome. Since this happens for every positive integer $k > K'$, given e there are infinitely many awesome numbers, at least one for each $k \geq K''$.

Also solved by Albert Stadler, Herrliberg, Switzerland.

U412. Let $P(x)$ be a monic polynomial with real coefficients, of degree n, which has n real roots. Prove that if

$$P(c) \le \left(\frac{b^2}{a}\right)^n$$

then $P\left(ax^2+2bx+c\right)$ has at least one real root.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Alessandro Ventullo, Milan, Italy

If $P(x)$ is an n-th degree monic polynomial with real coefficients and with n real roots $\alpha_1, \dots, \alpha_n$, then

$$P(x) = (x-\alpha_1)(x-\alpha_2)\cdot\ldots\cdot(x-\alpha_n).$$

Hence,

$$P(c) = (c-\alpha_1)(c-\alpha_2)\cdot\ldots\cdot(c-\alpha_n) \le \left(\frac{b^2}{a}\right)^n$$

and

$$P(ax^2+2bx+c) = (ax^2+2bx+c-\alpha_1)(ax^2+2bx+c-\alpha_2)\cdot\ldots\cdot(ax^2+2bx+c-\alpha_n).$$

Assume by contradiction that $P(ax^2+2bx+c)$ has no real roots. Then, each factor has negative discriminant, i.e.

$$\begin{array}{rcl} b^2-a(c-\alpha_1) & < & 0 \\ b^2-a(c-\alpha_2) & < & 0 \\ \vdots & \vdots & \vdots \\ b^2-a(c-\alpha_n) & < & 0. \end{array} \iff \begin{array}{rcl} b^2 & < & a(c-\alpha_1) \\ b^2 & < & a(c-\alpha_2) \\ \vdots & \vdots & \vdots \\ b^2 & < & a(c-\alpha_n). \end{array}$$

Multiplying side by side all these inequalities, we get $(b^2)^n < a^n(c-\alpha_1)(c-\alpha_2)\cdot\ldots\cdot(c-\alpha_n)$, i.e.

$$\left(\frac{b^2}{a}\right)^n < (c-\alpha_1)(c-\alpha_2)\cdot\ldots\cdot(c-\alpha_n),$$

contradiction.

Also solved by Daniel Lasaosa, Pamplona, Spain; Vincelot Ravoson, Lycée Henri IV, Paris, France; Prajnanaswaroopa S, Bangalore, Karnataka, India; Arkady Alt, San Jose, CA, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece.

U413. Let ABC be a triangle and let a, b, c be the lengths of sides BC, CA, AB, respectively. The tangency points of the incircle with sides BC, CA, AB are denoted by A', B', C'.

(a) Prove that the segments of lengths $AA' \sin A, BB' \sin B, CC' \sin C$ are the sides of a triangle.

(b) If $A_1B_1C_1$ is such a triangle, compute in terms of a, b, c the ratio

$$\frac{K[A_1B_1C_1]}{K[ABC]}.$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

Solution by Daniel Lasaosa, Pamplona, Spain

(a) It is well known that $BA' = \frac{c+a-b}{2}$ and $CA' = \frac{a+b-c}{2}$, or by Stewart's theorem,

$$AA'^2 = \frac{(c+a-b)b^2 + (a+b-c)c^2}{2a} - \frac{(c+a-b)(a+b-c)}{4} =$$

$$= \frac{3ab^2 + 3ac^2 - 2abc + 2b^2c + 2bc^2 - a^3 - 2b^3 - 2c^3}{4a},$$

and similarly for BB' and CC'. Denote now $a_1 = AA' \sin A$, $b_1 = BB' \sin B$, $c_1 = CC' \sin C$, and denoting by R the circumradius of ABC, we have

$$16R^2{a_1}^2 = 4a^2{a_1}^2 = 3a^2b^2 + 3a^2c^2 - 2a^2bc + 2ab^2c + 2abc^2 - a^4 - 2ab^3 - 2ac^3 =$$

$$= 16S^2 + (b+c-a)^2(b-c)^2 = 16S^2 + 16R^2r^2(\cos C - \cos B)^2,$$

where we have used Heron's formula for the area $S = K[ABC]$ of triangle ABC, and we have also used that

$$b + c - a = 8R \cos\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}, \qquad b - c = 4R \sin\frac{A}{2} \sin\frac{B-C}{2},$$

$$2\cos\frac{A}{2} \sin\frac{B-C}{2} = \cos C - \cos B, \qquad r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}.$$

It follows after some algebra that

$$16R^4K[A_1B_1C_1]^2 = R^4\left(2{a_1}^2{a_2}^2 + 2{a_2}^2{a_3}^2 + 2{a_3}^2{a_1}^2 - {a_1}^4 - {a_2}^4 - {a_3}^4\right) =$$

$$= 3S^4 + 4R^2S^2r^2\left(\cos^2 A + \cos^2 B + \cos^2 C - \cos A \cos B - \cos B \cos C - \cos C \cos A\right).$$

This expression is clearly positive, since S is positive and $u^2 + v^2 + w^2 - uv - vw - wu \geq 0$ for any u, v, w because of the scalar product inequality. It follows that a_1, b_1, c_1 are indeed the sides of a non-degenerate triangle. Note that the expression would be zero if two out of a_1, b_1, c_1 would add up to the third, and negative if one of a_1, b_1, c_1 would be larger than the sum of the other two.

(b) Now, using that $2Rr(a + b + c) = 4RS = abc$ and using the Cosine Law, we obtain after some algebra that

$$\frac{K[A_1B_1C_1]}{K[ABC]} = \frac{2ab + 2bc + 2ca - a^2 - b^2 - c^2}{16R^2} =$$

$$= \frac{\left(2ab + 2bc + 2ca - a^2 - b^2 - c^2\right)\left(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\right)}{16a^2b^2c^2}.$$

Also solved by Albert Stadler, Herrliberg, Switzerland.

U414. Let $p < q < 1$ be positive real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the conditions:

(i) $f(px + f(x)) = qf(x)$ for all real numbers x,

(ii) $\lim_{x\to 0} \dfrac{f(x)}{x}$ exists and its finite.

Proposed by Florin Stănescu, Găeşti, România

Solution by Alessandro Ventullo, Milan, Italy

Clearly, $f(x) = 0$ for all $x \in \mathbb{R}$ is a solution to the problem. Let $f \neq 0$. Observe that by condition (ii) it must be $f(0) = 0$. We have two cases.

(i) $\displaystyle\lim_{x\to 0} \frac{f(x)}{x} = \ell \in \mathbb{R} \setminus \{0\}$. Then, from condition (i), we get

$$\frac{f(px + f(x))}{px + f(x)} = \frac{qf(x)}{x(p + \frac{f(x)}{x})}$$

and if $x \to 0$ we get

$$\ell = \frac{q\ell}{p + \ell} \iff \ell = q - p.$$

So, $f(x) = (q - p)x$ is a solution to the problem.

(ii) $\displaystyle\lim_{x\to 0} \frac{f(x)}{x} = 0$. Then, $f(x) \sim Cx^{1+\delta}$ if $x \to 0$, where $C \neq 0$ and $\delta > 0$. Then, from condition (i), we get

$$\frac{f(px + f(x))}{(px + f(x))^{1+\delta}} = \frac{qf(x)}{x^{1+\delta}(p + \frac{f(x)}{x})^{1+\delta}}$$

and if $x \to 0$ we get

$$C = \frac{qC}{p^{1+\delta}} \iff p^{1+\delta} = q,$$

contradiction.

We conclude that the only solutions are $f(x) = 0$ and $f(x) = (q - p)x$.

Also solved by Mohammed Kharbach, Gasco, Abu Dhabi, UAE.

O409. Find all positive integers n for which there are $n+1$ digits in base 10, not necessarily distinct, such that at least $2n$ permutations of those digits produce $(n+1)$-digit perfect squares, with leading zeros not allowed. Note that two different permutations are considered distinct even if they lead to the same digit string due to repetition among the digits.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

For $n=1$, we need a two digit number ab such that the two digit number ba is also a perfect square. Since two-digit perfect squares are $16, 25, 36, 49, 64, 81$, none of them have two identical digits, and no two of them share the same two distinct digits, $n=1$ does not satisfy the proposed condition.

For $n=2$, consider $144=12^4$ and $441=21^2$, or at least $4=2n$ permutations of $(1,4,4)$ result in 3-digit perfect squares, hence $n=2$ satisfies the proposed condition.

For $n=3$, consider $1444=38^2$, or at least $3!=6=2n$ permutations of $(1,4,4,4)$ result in 4-digit perfect squares, hence $n=3$ satisfies the proposed condition.

For any positive integer $m\geq 2$, let $n=2m$ be any even positive integer larger than 2. Chose $2m$ digits equal to 0, and one digit equal to 1, and consider the $n!\geq 3!n=6n$ permutations that leave the 1 in first place. All these permutations produce number $10^n=10^{2m}$, clearly a perfect square, or every even integer $n\geq 4$ satisfies the proposed condition.

For any positive integer $m\geq 2$, let $n=2m+1$ be any odd positive integer larger than 3. Choose m digits equal to 1, $m+1$ digits equal to 2 and one digit equal to 5, and consider the $m!(m+1)!$ permutations that leave all 1's first, all 2's next, and finally the 5. Notice that such a $2m+2=n+1$-digit number can be expressed as

$$\frac{10^{2k}-1}{9}+\frac{10^{k+1}-1}{9}+3=\frac{10^{2k}+10^{k+1}+25}{9}=\left(\frac{10^k+5}{3}\right)^2,$$

which is indeed a perfect square since $10^k+5\equiv 1+5\equiv 0 \pmod 3$. Since $m!\geq 2$ and $(m+1)!\geq 2(m+1)$, the number of permutations of the given set of digits which form a perfect square is at least $4(m+1)=2(n+1)$, and every odd integer $n\geq 5$ satisfies the proposed condition.

It follows that all positive integers except for $n=1$ satisfy the proposed condition.

Also solved by Alessandro Ventullo, Milan, Italy.

O410. On each cell of a chess board it is written a number equal to the amount of the rectangles that contain this cell. Find the sum of all the numbers.

Proposed by Robert Bosch, USA

Solution by Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua
Consider a row (or column) of the board and label the cells from 1 to n. The number of rectangles in that row (or column) containing the cell labeled by i is: $i(n+1-i)$.

Since for a cell the labeling by rows does not influence the labeling in columns, then by multiplication principle, the amount of rectangles in the board that contain a cell labeled in row by i and in column by j is: $ij(n+1-i)(n+1-j)$.

Now, if you consider the chessboard the sum requested is:

$$S = \sum_{i=1}^{8}\sum_{j=1}^{8} ij(9-i)(9-j) = \left(\sum_{i=1}^{8} 9i - i^2\right)^2$$
$$= \left(9\frac{8\cdot 9}{2} - \frac{8\cdot 9\cdot 17}{6}\right)^2 = 120^2 = 14400.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Lee Jae Woo, Hamyang-gun, South Korea; Albert Stadler, Herrliberg, Switzerland; Alessandro Ventullo, Milan, Italy; Joel Schlosberg, Bayside, NY, USA; Shuborno Das, Ryan International School, Bangalore, India.

O411. For a positive integer n denote by $S(n)$ the sum of all prime divisors of n. (For example, $S(1) = 0$, $S(2) = 2$, $S(45) = 8$.) Find all positive integers n such that $S(n) = S(2^n + 1)$.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Daniel Lasaosa, Pamplona, Spain

For $n = 1$, we have $S(2^n + 1) = S(3) = 3 > 0 = S(n)$, while for $n = 2$ we have $S(2^n + 1) = S(5) = 5 > 2 = S(n)$, and for $n = 3$ we obtain $S(2^n + 1) = S(9) = S(3^2) = 3 = S(3) = S(n)$, or the only solution with $n \leq 3$ is $n = 3$.

Let now $n \geq 4$, in which case we can easily prove that $S(n) \leq n$, with equality iff n is prime. If n is prime, equality trivially holds, whereas if $n = p^\alpha$ is a power of a prime with $\alpha > 1$, then $S(n) = p < p^\alpha = n$. By induction on the number u of distinct prime divisors of n, note now that if $n = {p_1}^{\alpha_1} \cdot {p_2}^{\alpha_2} \cdots {p_u}^{\alpha_u}$ with $p_1 < p_2 < \cdots < p_u$ and $\alpha_1, \alpha_2, \ldots, \alpha_u$ positive integers, then we have

$$S(n) = p_1 + p_2 + \cdots + p_n < 2(p_2 + \cdots + p_n) < p_1(p_2 + \cdots + p_n) < {p_1}^{\alpha_1} \left({p_2}^{\alpha_2} \cdots {p_u}^{\alpha_u}\right) = n,$$

where in the first inequality we have used that $p_1 < p_2 \leq p_2 + \cdots + p_n$, and in the last inequality we have used the hypothesis of induction applied to $n = {p_2}^{\alpha_2} \cdots {p_u}^{\alpha_u}$, and that $p_1 \leq {p_1}^{\alpha_1}$. Now, as a particular case of Zsigmondy's theorem, for all integer $n \geq 4$ there exists a prime p_n which divides $2^{2n} - 1$ but does not divide $2^m - 1$ for all $2n - 1 \geq m \geq 1$. Note that we avoid here the exceptions $n = 6$, and $n = 2$ with $a + b = 2$, in the usual formulation $a^n - b^n$ with $a > b$ positive coprime integers of Zsigmondy's theorem. Now, since p_n does not divide $2^n - 1$ but divides $2^{2n} - 1 = (2^n + 1)(2^n - 1)$, then p_n divides $2^n + 1$, or $S(2^n + 1) \geq p_n$ with equality if $2^n + 1$ is prime. Moreover, if $p_n \leq 2n$, then $2^{p_n - 1} - 1$ is divisible by p_n by Fermat's "little" theorem, since p_n is clearly odd because $2^{2n} - 1$ is odd, in contradiction with the choice of p_n. Or $p_n \geq 2n + 1 > n$, and $S(2^n + 1) \geq p_n > n \geq S(n)$. There can therefore be no solutions with $n \geq 3$.

The only solution is therefore $n = 3$, for which $S(2^3 + 1) = S(3) = 3$.

Also solved by Albert Stadler, Herrliberg, Switzerland; Rajdeep Majumder, Drgapur, India.

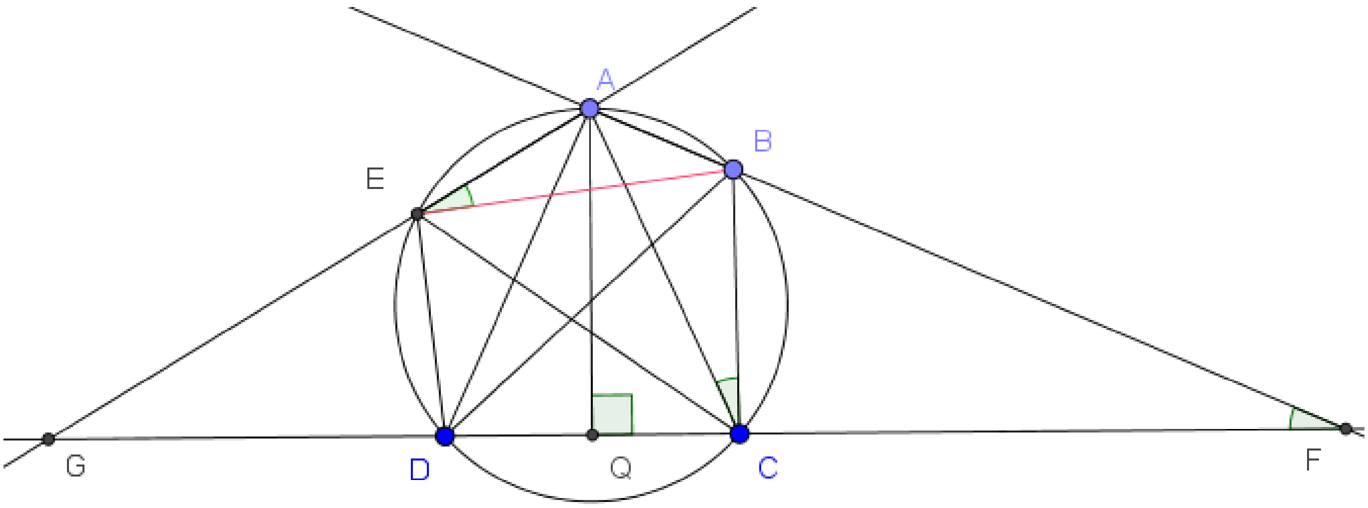
O412. Let $ABCDE$ be a convex pentagon such that $AC = AD = AB + AE$ and $BC + CD + DE = BD + CE$. Lines AB and AE intersect CD at F and G, respectively. Prove that

$$\frac{1}{AF} + \frac{1}{AG} = \frac{1}{AC}.$$

Proposed by Anton Vasilyev, Astana, Kazakhstan

Solution by Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece

First, denote as Q the midpoint of CD. Since ACD is an isosceles triangle, it is $AQ \perp CD$.



Ptolemy's Inequality in convex quadrilaterals $ABCD$ and $ACDE$ yields

$$AB \cdot CD + BC \cdot AD \geq AC \cdot BD (1)$$

and

$$AC \cdot DE + AE \cdot CD \geq AD \cdot CE (2).$$

Adding up the above inequalities and using $AC = AD$, we get

$$AC(BC + DE) + CD(AB + AE) \geq AC(BD + CE)$$

or equivalently by the given conditions

$$AC(BC + DE) + AC \cdot CD \geq AC(BD + CE).$$

Since in the last one the equality holds, it must also hold on (1) and (2) which means that $ABCD$ and $ACDE$ are cyclic and A, B, C, D, E all lie on the same circle.

Then, it follows that $AG \cdot GE = GC \cdot GD = GQ^2 - GC^2$ and

$$AF \cdot FB = FC \cdot FD = FQ^2 - QC^2.$$

Furthermore, using that $AQ \perp CD$ and the above relations, it is

$$AG^2 - GQ^2 = AF^2 - FQ^2 \Leftrightarrow AG^2 - AG \cdot GE = AF^2 - AF \cdot FB \Leftrightarrow AE \cdot AG = AB \cdot AF$$

and as a result $EBFG$ is also cyclic.
Then, we have that $\angle BFC = \angle AEB = \angle ACB$ because $ABCE$ is also cyclic. But this means that AC is the tangent of the circumcircle of triangle BFC and as a result $AC^2 = AF \cdot AB$.
From all the above results, we have

$$\frac{1}{AF} + \frac{1}{AG} = \frac{AB}{AF \cdot AB} + \frac{AE}{AG \cdot AE} = \frac{AC}{AF \cdot AB} = \frac{AC}{AC^2} = \frac{1}{AC}$$

as wanted.

Also solved by Daniel Lasaosa, Pamplona, Spain.

O413. Let ABC be an acute triangle. Prove that:

$$\text{a) } \frac{a}{m_a}+\frac{b}{m_b}+\frac{c}{m_c}\geq\frac{a+b+c}{R+r}$$
$$\text{b) } \frac{b+c}{m_a}+\frac{c+a}{m_b}+\frac{a+b}{m_c}\geq\frac{4\,(a+b+c)}{3R}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Evgenidis Nikolaos, M.N.Raptou High School, Larissa, Greece

Denote as d_a, d_b, d_c the distances from the circumcenter O of the triangle to the corresponding sides. By Triangle's Inequality we have

$$R+d_a\geq m_a$$
$$R+d_b\geq m_b$$
$$R+d_c\geq m_c.$$

a) Using the above, it is

$$\frac{a}{m_a}+\frac{b}{m_b}+\frac{c}{m_c}\geq\frac{a^2}{aR+ad_a}+\frac{b^2}{bR+bd_b}+\frac{c^2}{cR+cd_c}$$

and by B.C.S Inequality we have

$$\frac{a^2}{aR+ad_a}+\frac{b^2}{bR+bd_b}+\frac{c^2}{cR+cd_c}\geq\frac{(a+b+c)^2}{(a+b+c)R+ad_a+bd_b+cd_c}.$$

Observe that $ad_a+bd_b+cd_c=2S=(a+b+c)r$. Substituting this in the above inequality and making some simplifications we finally obtain that

$$\frac{a}{m_a}+\frac{b}{m_b}+\frac{c}{m_c}\geq\frac{a+b+c}{R+r},$$

with equality holding if and only if the triangle is equilateral. b) Same as in a) it is

$$\frac{b+c}{m_a}+\frac{c+a}{m_b}+\frac{a+b}{m_c}\geq\frac{(b+c)^2}{(b+c)(R+d_a)}+\frac{(c+a)^2}{(c+a)(R+d_b)}+\frac{(a+b)^2}{(a+b)(R+d_c)}$$

and by B.C.S Inequality in the last one, we obtain

$$\frac{b+c}{m_a}+\frac{c+a}{m_b}+\frac{a+b}{m_c}\geq\frac{4(a+b+c)^2}{(b+c)d_a+(c+a)d_b+(a+b)d_c+2R(a+b+c)}.$$

Observe that $ad_a+bd_b+cd_c=2S=(a+b+c)r$. As a result we can write $(b+c)d_a+(c+a)d_b+(a+b)d_c=(a+b+c)(d_a+d_b+d_c)-(a+b+c)r$.
By Carnot's theorem it is $d_a+d_b+d_c=R+r$, so we have

$$(b+c)d_a+(c+a)d_b+(a+b)d_c=(a+b+c)R$$

and after some simplifications we finally obtain

$$\frac{b+c}{m_a}+\frac{c+a}{m_b}+\frac{a+b}{m_c}\geq\frac{4(a+b+c)}{3R},$$

with equality holding if and only if the triangle is equilateral.

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O414. Characterize all positive integers n with the following property: for any two coprime divisors $a < b$ of n, $b - a + 1$ is also a divisor of n.

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Solution by the author

It is obvious that n equal to a prime power will work. Besides these trivial ones we will also the solutions $n = 12, 24$ and the general solutions $n = p(2p-1)$ where p and $2p-1$ are primes, $n = p^2(p^2+p-1)$ where p and p^2+p-1 are primes.

To find the rest of the solutions we first show that n cannot have more than two prime divisors. Indeed assuming the contrary, say $p < q < r$ are the least three of it's prime divisor. Note that $q-p+1$ is a divisor of n so it must be a power of p, thus $q \equiv -1(\mod p) \star$.

Then by the condition we have that $r-p+1$ and $r-q+1$ are also prime divisor of n. Since they are smaller than n it follows that there are nonnegative integers a, b, c, d such that $r = p+p^aq^b-1 = q+p^cq^d-1$. Thus $p+p^aq^b = q+p^cq^d$. If $d > 0$ it would follow that b must be equal to 0. Similarly if $a > 0$ then c must be zero.

If follows thus that either by redenoting $r = p+p^a-1 = q+q^b-1$ or $r = p+q^a-1 = q+p^b-1$.

In the first case it is easy to see that $b \geq 2$. It follows that $r > q^2 > pq$ thus $r-pq+1$ is a divisor also of n. Thus we can find s and t such that $r = pq+q^sp^t-1 = q+q^b-1$. This further leads to $p+q^{s-1}p^t = 1+q^{b-1}$ and if $s > 1$ then $q|p-1$ which is a contradiction with $p < q$. Thus $s = 1$. Now let us leverage the second writing of r. We have that $pq+qp^t = p+p^a$ so $q+qp^{t-1} = 1+p^{a-1}$. Since $t > 1$ and $a > 1$ we have by $q \equiv 1(\mod p)$ and thus by $\star$ we get that $p = 2$.

This means $q^{b-1} = 1+2^t$ and this well known to have only the solution $q = 3$, $b = 3$ and this gives $r = 29$ which cannot be written as 2^a+1.

Our second case is when $r = p+q^a-1 = q+p^b-1$. If $a \geq 2$ then applying the same trick we can find s, t such that $r = pq+q^sp^t-1 = p+q^a-1$ and this is an immediate contradiction since both s and t must be equal to 0. Thus $r = p+q-1$. But in this situation since $pq > p+q$ it follows that $pq-r+1$ is also a divisor of n. Replacing we have $pq-p-q+2$ is a divisor of n. Since $q > p$ we have q and $pq-p-q+2$ are coprime so applying again the assumption $pq-p-2q+3$ is a divisor of n. By induction $pq-p-kq+(k+1)$ is a divisor of n for any $k \leq p-2$.

Setting $k = p-2$ we obtain that $2q-1$ is also a divisor of n. If $p = 3$ then using $\star$ we have $3|2q-1$ so $2q-1$ has prime divisor less than $\dfrac{2q-1}{3} < q$ and thus $2q-1$ has to be a power of 3. Now we can use the fact that $q-3+1 = q-2$ is also a power of 3 and the only solution is $q = 3$ which is a contradiction.

If $p \neq 3$ this is coprime to p by using $\star$ again. Thus $2q-p$ is a divisor of n. Since $q \geq 2p-1$ by $\star$ we obviously have $q+p-1 = r \leq 2q-p < 2q+2p-2 = 2r$ and if we don't have equality in the LHS then $2q-p-r+1 = q-2p+2$ is a divisor of n. This is smaller than q so it must be a power of p but using $\star$ it is 1 mod p which is a contradiction.

It follows that r must be equal to $2q-p$ and thus $q = 2p-1$. Now going back to $pq-p-kq+(k+1) = 2p(p-k)+1$ is a divisor of n for any $k \leq p-2$, I claim that we can find a k such that $p+1|2p(p-k)+1$ which is equivalent to $4k+5 \equiv 0(\mod p+1)$. The only thing left is to check is that indeed $k \leq p-2$ which is an easy one since we need to deal only with $k = p-1, p, p+1$ and none will work for $p > 3$ prime.

Thus $p+1$ must divide n and since $p > 3$ was the smallest prime divisor this leads to a contradiction.

So suppose $n = p^a q^b$ is a solution and satisfies the conditions of the hypothesis where $p < q$. If $b \geq 2$, then $q^2 - p + 1$ has to be a power of p, since q and $p - 1$ are coprime and this leads to a contradiction using $\star$. Thus $n = p^a q$. We know $q = p^x + p - 1$ for $1 \leq x \leq a$.

I claim that if $x \geq 2$ then $x = a = 2$. Indeed we know that $a \geq x \geq 2$ so $p^2 | n$ and is coprime to q and thus $q - p^2 + 1$ has to be a divisor of n and so it has to be a power of p. Thus $q = p^x + p - 1 = p^y + p^2 - 1$ for some $y \geq 1$ and the claim follows easily. Thus we obtain $n = p^2(p^2 + p - 1)$ as a solution to our problem where p is a prime and also $p^2 + p - 1$ is a prime.

If $x = 1$ we have $q = 2p - 1$. If $a \geq 2$ then $p^2 - q + 1 = p^2 - 2p + 2$ is also a divisor of n and it can only be p. Thus $p = 2$ and $q = 3$. In this case we obtain that $a = 2, 3$ since if $a \geq 4$ then $2^4 - 3 + 1 = 14$ must divide n which is a contradiction. We obtain the solutions $n = 6, 12, 24$.

Finally we are left with $a = 1$ and any $n = p(2p - 1)$ with p prime works.

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