Junior Problems

J415. Prove that for all real numbers x, y, z at least one of the numbers

$$2^{3x-y} + 2^{3x-z} - 2^{y+z+1}$$
$$2^{3y-z} + 2^{3y-x} - 2^{z+x+1}$$
$$2^{3z-x} + 2^{3z-y} - 2^{x+y+1}$$

is nonnegative.

Proposed by Adrian Andreescu, Dallas, USA

J416. Find all positive real numbers a and b for which

$$\frac{ab}{ab+1} + \frac{a^2b}{a^2+b} + \frac{ab^2}{a+b^2} = \frac{1}{2}(a+b+ab).$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

J417. Solve in positive real numbers the equation

$$\frac{x^2+y^2}{1+xy} = \sqrt{2-\frac{1}{xy}}$$

Proposed by Adrian Andreescu, Dallas, Texas, USA

J418. Prove that the following inequality holds for all $a, b, c \in [0, 1]$

$$a + b + c + 3abc \geq 2(ab + bc + ca).$$

Proposed by Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J419. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^4+b+c^4} + \frac{1}{b^4+c+a^4} + \frac{1}{c^4+a+b^4} \leq \frac{3}{a+b+c}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J420. Let ABC be a triangle and let A, B, C be the magnitudes of its angles, expressed in radians. Prove that if A, B, C and $\cos A, \cos B, \cos C$ are geometric sequences, then the triangle is equilateral.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Senior Problems

S415. Let

$$f(x) = \frac{(2x-1)6^x}{2^{2x-1} + 3^{2x-1}}.$$

Evaluate

$$f\left(\frac{1}{2018}\right) + f\left(\frac{3}{2018}\right) + \cdots + f\left(\frac{2017}{2018}\right)$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S416. Let $f : \mathbb{N} \longrightarrow \{\pm 1\}$ be a function such that $f(mn) = f(m)f(n)$, for all $m, n \in \mathbb{N}$. Prove that there are infinitely many n such that $f(n) = f(n+1)$.

Proposed by Oleksi Krugman, University College London, UK

S417. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \le \frac{3(a^2+b^2+c^2)}{2(a+b+c)}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S418. Let a, b, c, d be positive real numbers such that $abcd \ge 1$. Prove that

$$\frac{a+b}{a+1} + \frac{b+c}{b+1} + \frac{c+d}{c+1} + \frac{d+a}{d+1} \le a+b+c+d.$$

Proposed by An Zhenping, Xianyang Normal University, China

S419. Solve the system of equations:

$$x(x^4 - 5x^2 + 5) = y$$
$$y(y^4 - 5y^2 + 5) = z$$
$$z(z^4 - 5z^2 + 5) = x.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S420. Let T be the Toricelli point of a triangle ABC. Prove that

$$(AT + BT + CT)^2 \le AB \cdot BC + BC \cdot CA + CA \cdot AB$$

Proposed by Nguyen Viet Chung, Hanoi University of Science, Vietnam

Undergraduate Problems

U415. Prove that the polynomial $P(X) = X^4 + iX^2 - 1$ is irreducible in the ring of polynomials over Gauss integers.

Proposed by Mircea Becheanu, University of Bucharest, Romania

U416. For any root $z \in \mathbb{C}$ of the polynomial $X^4 + iX^2 - 1$ we denote $w_z = z + \frac{2}{z}$. Let $f(x) = x^2 - 3$. Prove that

$$|(f(w_z) - 1)f(w_z - 1)f(w_z + 1)|$$

is an integer that does not depend on z.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U417. Prove that for any $n \geq 14$ and for any real number x, $0 < x < \frac{\pi}{2n}$, the following inequality holds:

$$\frac{\sin 2x}{\sin x} + \frac{\sin 3x}{\sin 2x} + ... + \frac{\sin(n+1)x}{\sin nx} < 2 \cot x.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

U418. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\sqrt{16a^2 + 9} + \sqrt{16b^2 + 9} + \sqrt{16c^2 + 9} \leq 1 + \frac{14}{3}(a + b + c).$$

Proposed by An Zhenping, Xianyang Normal University, China

U419. Let $p > 1$ be a natural number. Prove that

$$\lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{\sqrt[p]{k}} - \frac{p}{p-1}(n^{\frac{p-1}{p}} - 1) \right) \in (0, 1).$$

Proposed by Alessandro Ventullo, Milan, Italy

U420. Find the least length of a segment whose endpoints are on the hyperbola $xy = 5$ and ellipse $\frac{x^2}{4} + 4y^2 = 2$, respectively.

Proposed by Titu Andreescu, USA and Oleg Mushkarov, Bulgaria

Olympiad Problems

O415. Let $n > 2$ be an integer. An $n \times n$ square is divided into n^2 unit squares. Find the maximum number of unit squares that can be painted in such a way that every 1×3 rectangle contains at least one unpainted unit square.

Proposed by Magauin Armanzhan, Kazakhstan and Nairi Sedrakyan, Armenia

O416. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 - abc = 4$. Find the minimum of $(ab - c)(bc - a)(ca - b)$ and all triples (a, b, c) for which the minimum is attained.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O417. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_1^2 + x_2^2 + \cdots + x_n^2 \le 1$. Prove that

$$|x_1| + |x_2| + \cdots + |x_n| \le \sqrt{n}\left(1 + \frac{1}{n}\right) + n^{\frac{n-1}{2}} x_1 x_2 \ldots x_n.$$

When does equality occur?

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O418. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{a^5}{c^3 + 1} + \frac{b^5}{a^3 + 1} + \frac{c^5}{b^3 + 1} \ge \frac{3}{2}$$

Proposed by Konstantinos Metaxas, Athens, Greece

O419. Let $x_1, x_2, \ldots, x_n$ be real numbers in the interval $(0, \pi/2)$. Prove that

$$\frac{1}{n^2}\left(\frac{\tan x_1}{x_1} + \cdots + \frac{\tan x_n}{x_n}\right)^2 \le \frac{\tan^2 x_1 + \cdots + \tan^2 x_n}{x_1^2 + \cdots + x_n^2}$$

Proposed by Mircea Becheanu, University of Bucharest, Romania

O420. Let $n \ge 2$ and let $A = \{1, 4, \ldots, n^2\}$ be the set of the first n nonzero perfect squares. A subset B of A is called *Sidon* if whenever $a + b = c + d$ for $a, b, c, d \in B$, we have $\{a, b\} = \{c, d\}$. Prove that A contains a *Sidon* subset of size at least $Cn^{1/2}$ for some absolute constant $C > 0$. Can the exponent $1/2$ be improved?

Proposed by Cosmin Pohoaţă, Caltech, USA