PIECEWISE TELESCOPING AND APPLICATIONS TO FOURIER SERIES

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ABSTRACT. This article highlights a technique for evaluating sums that we call *piecewise telescoping*, which involves re-labeling indices in the sum to make a telescoping scheme apparent. Then we use piecewise telescoping to rewrite Fourier Series to verify their convergence. Finally, theorems for special classes of Fourier Cosine Series are introduced to apply these principles, with accompanying examples.

1. INTRODUCTION TO PIECEWISE TELESCOPING

As you may recall from an algebra class, a *telescoping series* is one for which expanding the sum causes the vast majority of the terms to cancel. One of the most well-known examples of a telescoping series, found in Richard Rusczyk's book "Intermediate Algebra," [Rs] is the sum

$$\sum_{k=1}^{\infty} \frac{1}{k(k+2)} \tag{1.1}$$

The method for evaluating this is to write the partial fraction decomposition of the sum, and then the cancellation becomes obvious:

$$\sum_{k=1}^{\infty} \frac{1}{2} \cdot \frac{1}{k} - \frac{1}{2} \cdot \frac{1}{k+2} = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \ldots\right). \tag{1.2}$$

This sum ends up being equivalent to $\frac{3}{4}$ [Rs]. While we will not need to use partial fraction decomposition again in this paper, the resulting representation of the sum is what motivates the concept of piecewise telescoping.

More formally, we define *piecewise telescoping* as the act of rewriting an infinite sum over a piecewise-defined sequence in such a way that the sum's telescoping property becomes apparent. Piecewise telescoping is done by rewriting each term of the infinite sum as a finite sum of elements in the sequence. That said, explaining this technique is most effective with an example.

Example 1.1: Let $\{b_n\}_{n=1}^{\infty}$ be the sequence defined such that $b_n = \frac{2}{n+1}$ when n is odd and $b_n = -\frac{2}{n+2}$ when n is even. Compute $\sum_{n=1}^{\infty} b_n$.

Solution: We partition the infinite sum based on whether the value of index n is odd or even (this is valid since every natural number has a unique parity):

$$\sum_{n=1}^{\infty} b_n = \sum_{n \geq 1,\ n\ odd}^{\infty} b_n + \sum_{n \geq 2,\ n\ even}^{\infty} b_n. \tag{1.3}$$

The above identity is a staple of every piecewise telescoping computation. Note that if n is odd, there is a positive integer k where $n = 2k-1$. Likewise, if n is even, there is a positive integer k where $n = 2k$. Then we can rewrite the above expression as

$$\sum_{n \geq 1,\ n\ odd}^{\infty} b_n + \sum_{n \geq 2,\ n\ even}^{\infty} b_n = \sum_{k=1}^{\infty} b_{2k-1} + b_{2k}. \tag{1.4}$$

The first term always has an odd index, and the second term always has an even index, so

$$\sum_{k=1}^{\infty} b_{2k-1} + b_{2k} = \sum_{k=1}^{\infty} \frac{2}{(2k-1)+1} - \frac{2}{2k+2} = \sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+1}. \tag{1.5}$$

Clearly the sum above evaluates to $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + ... = 1$. This method is very efficient in abusing the telescoping nature of this sum. Note you could not directly compute the two sums in (1.3) separately and add them together because they both diverge [in opposite directions].

2. Special Representations of Fourier Series

The technique of piecewise telescoping is especially helpful for studying a specific subgroup of Fourier Series, particularly of Fourier Cosine Series. To navigate towards the class of functions most relevant to this technique, we first state a theorem on Fourier Series and one of its corollaries. They are discussed in more detail in Zachmanoglou's book [Za].

Theorem 2.1: Let f be a differentiable function where f and f' are both sectionally continuous on $[-\pi, \pi]$ and f is also 2π-periodic. Then f has a Fourier Series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)). \tag{2.1}$$

Moreover, we are guaranteed the following integrals exist for all $n \in \mathbb{N}^+$:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx \tag{2.2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx \tag{2.3}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx. \tag{2.4}$$

Theorem 2.2: Let f be a function that satisfies the conditions of Theorem 2.1 and is an even function on $[-\pi, \pi]$. Then f's Fourier Series reduces to a *Fourier Cosine Series*. That is, its Fourier Coefficients are

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x)\cos(nx)dx \ \forall n \in \mathbb{N}^0 \tag{2.5}$$

$$b_n \;=\; 0\;\forall n \in \mathbb{N}^0. \tag{2.6}$$

Theorem 2.2 narrows down the collection of functions for which Theorem 2.1 holds and lets us impose additional conditions. That is, the functions we are most interested in are characterized by Theorem 2.2. We state and prove a corollary that will identify which such functions are most readily analyzed by piecewise telescoping.

Theorem 2.3: Let f be a function that satisfies the conditions of Theorem 2.2 that also has $a_n \;=\; 0$ for all odd natural numbers n. Then there exists a sequence of coefficients $\{c_k\}_{k\,=\,1}^{\infty} \subset \mathbb{R}$ such that the Fourier [Cosine] Expansion of f can be written as

$$f(x) \;=\; \sum_{k\,=\,0}^{\infty} c_k \cos(2k\pi x). \tag{2.7}$$

over the interval $x \in [-\pi, \pi]$.

Proof: By the result of Theorem 2.2, we know that f possesses the Fourier Cosine Expansion with coefficients from (2.5):

$$f(x) \;=\; \sum_{n\,=\,0}^{\infty} a_n \cos(n\pi x). \tag{2.8}$$

We claim that the sequence $\{c_k\}_{k\,=\,1}^{\infty}$ exists so that $c_k \;=\; a_{2k}\;\forall k \in \mathbb{N}^0$. This is verified by utilizing a variant of (1.3) and (1.4) where the summand is $a_n \cos(n\pi x)$ instead of b_n, and the indices are slightly different to account for the $n \;=\; 0$ term:

$$f(x) \;=\; \sum_{n\,=\,0}^{\infty} a_n \cos(n\pi x) \;=\; \sum_{k\,=\,0}^{\infty} a_{2k} \cos(2k\pi x) + a_{2k+1} \cos((2k+1)\pi x). \tag{2.9}$$

Since all of the odd-indexed coefficients are zero, the second term in the summand equals zero, and it follows that

$$f(x) \;=\; \sum_{k\,=\,0}^{\infty} a_{2k} \cos(2k\pi x) \;:=\; \sum_{k\,=\,0}^{\infty} c_k \cos(2k\pi x), \tag{2.10}$$

as desired. □

Finally we can prove that a few other sums absolutely converge on $\mathbb{R}$. The conditions we impose on the next theorem are based on the class of functions studied in Theorem 2.3.

Theorem 2.4: Let $\{b_n\}_{n\,=\,1}^{\infty} \subset \mathbb{R}$ be such that $\sum_{n\,=\,1}^{\infty} b_{2n}$ converges absolutely. Then the following functions also converge absolutely for all $x \in \mathbb{R}$:

$$\sum_{n\,=\,1}^{\infty} 2b_{2n} \cos^2(n\pi x) - b_{2n} \tag{2.11}$$

$$\sum_{n\,=\,1}^{\infty} b_{2n} - 2b_{2n} \sin^2(n\pi x) \tag{2.12}$$

$$\sum_{n=1}^{\infty} b_{2n}\cos(2n\pi x). \tag{2.13}$$

Proof:

i) The following bound holds by properties of absolute value and $\sin^2$ and $\cos^2$ being bounded above by 1:

$$\sum_{n=1}^{\infty} 2b_{2n}\cos^2(n\pi x) - b_{2n} \le \sum_{n=1}^{\infty} |2b_{2n}\cos^2(n\pi x) - b_{2n}| \le \tag{2.14}$$

$$\sum_{n=1}^{\infty} |b_{2n}(\cos^2(n\pi x) - 1)| + \sum_{n=1}^{\infty} |b_{2n}\cos^2(n\pi x)| \le \tag{2.15}$$

$$\sum_{n=1}^{\infty} |b_{2n}|\sin^2(n\pi x) + \sum_{n=1}^{\infty} |b_{2n}|\cos^2(n\pi x) \le 2\sum_{n=1}^{\infty} |b_{2n}|. \tag{2.16}$$

Thus $\sum_{n=1}^{\infty} 2b_{2n}\cos^2(n\pi x) - b_{2n}$ converges absolutely by the Comparison Test.

ii) This proof is virtually identical to the first:

$$\sum_{n=1}^{\infty} b_{2n} - 2b_{2n}\sin^2(n\pi x) \le \sum_{n=1}^{\infty} |b_{2n} - 2b_{2n}\sin^2(n\pi x)| \le \tag{2.17}$$

$$\sum_{n=1}^{\infty} |b_{2n}(1 - \sin^2(n\pi x))| + \sum_{i=1}^{\infty} |b_{2n}\sin^2(n\pi x)| \le \tag{2.18}$$

$$\sum_{n=1}^{\infty} |b_{2n}|\cos^2(n\pi x) + \sum_{i=1}^{\infty} |b_{2n}|\sin^2(n\pi x) \le 2\sum_{n=1}^{\infty} |b_{2n}|, \tag{2.19}$$

so the result follows by the Comparison Test.

iii) We can reduce this to the case proven in part 2 by means of the cosine double-angle identity:

$$\sum_{n=1}^{\infty} b_{2n}\cos(2n\pi x) \;=\; \sum_{n=1}^{\infty} b_{2n}(1 - 2\sin^2(n\pi x)) \;=\; \sum_{n=1}^{\infty} b_{2n} - 2b_{2n}\sin^2(n\pi x). \tag{2.20}$$

Since $\sum_{n=1}^{\infty} b_{2n}$ converges absolutely, we are done by the result of part 2. $\square$

Remark: One could also prove part 3 first, using the fact that cosine is bounded above by 1 and below by -1, and then using the double-angle identity for cosine to immediately prove parts 1 and 2, but this proof is more elegant.

3. Further Discussion and Examples

One may note that Theorem 2.4 requires $\sum_{n=1}^{\infty} b_{2n}$ must converge absolutely in order for it to be valid. We first present an example to show that it does not suffice for $\sum_{n=1}^{\infty} b_n$ to converge absolutely, and this demonstration again uses piecewise telescoping.

Example 3.1: Let $\{b_n\}_{n=1}^{\infty}$ be the sequence where $b_n = \frac{2}{n+1}$ if n is odd and $b_n = \frac{4}{n^2} - \frac{2}{n}$ if n is even. Note we can rewrite the infinite sum of these terms as

$$\sum_{n=1}^{\infty} b_n = \sum_{k=1}^{\infty} b_{2k-1} + b_{2k} = \tag{3.1}$$

$$\sum_{k=1}^{\infty} \frac{2}{2k-1+1} + \frac{4}{4k^2} - \frac{2}{2k} = \sum_{k=1}^{\infty} \frac{1}{k^2}, \tag{3.2}$$

which [absolutely] converges by the p-series test. However the sum of $\{b_n\}$'s terms of even index equals

$$\sum_{k=1}^{\infty} \frac{4}{k^2} - \frac{2}{k} = 2\sum_{k=1}^{\infty} \frac{1-k}{k^2}. \tag{3.3}$$

This subsequence is monotone, so by the Integral Test we see that the integral

$$\int_{1}^{\infty} \frac{1-k}{k^2} dk \tag{3.4}$$

diverges, so the sum diverges.

The other restriction change we want to discuss for Theorem 2.4 is if $\sum_{n=1}^{\infty} b_{2n}$ converges conditionally rather than absolutely. Clearly our proof of Theorem 2.4 is no longer valid because it revolves around bounding our sums above by $2\sum_{n=1}^{\infty} |b_{2n}|$. However, the result of Theorem 2.4 is still true. We will use the sum (2.11) as an example. One may initially think it suffices to prove that

$$\sum_{n=1}^{\infty} 2b_{2n}\cos^2(n\pi x) - b_{2n} \leq \sum_{n=1}^{\infty} b_{2n}. \tag{3.5}$$

Since convergence need not be absolute, this assertion alone does not eliminate the possibility that (2.11) diverges to $-\infty$. Thus we must also prove the lower bound

$$\sum_{n=1}^{\infty} 2b_{2n}\cos^2(n\pi x) - b_{2n} \geq -\sum_{n=1}^{\infty} b_{2n}. \tag{3.6}$$

The proofs of these two bounds are virtually identical and relatively simple, and we leave the details as an exercise.

To conclude this paper, we provide one more example that unifies the techniques discussed in this article with some results from real analysis (as one may expect when dealing with convergence of sums).

Example 3.2: Prove that the following limit converges $\forall x \in \mathbb{R}$:

$$\lim_{n\to\infty}\sum_{m=1}^{n}\sum_{k=0}^{m}\frac{-4\cos^2(k\pi x)\sin^2((m-k)\pi x)+2\sin^2((m-k)\pi x)+2\cos^2(k\pi x)-1}{(16k^4+1)(4m^2+4k^2-4mk+1)} \tag{3.7}$$

Solution: We begin by factoring the numerator:

$$\lim_{n\to\infty}\sum_{m=1}^{n}\sum_{k=0}^{m}\frac{2\cos^2(k\pi x)-1}{16k^4+1}\cdot\frac{1-2\sin^2((m-k)\pi x)}{4m^2+4k^2-4mk+1} \tag{3.8}$$

Let us define sequences $\{a_k\}_{k=0}^{\infty}, \{b_k\}_{k=0}^{\infty} \subset \mathbb{R}$ such that $a_k = \frac{1}{k^2+1}$ and $b_k = \frac{1}{k^4+1}$. Note that $\sum_{k=1}^{\infty}\frac{1}{(2k)^2} = \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges [absolutely] by the p-series test. Moreover, $a_k, b_k < \frac{1}{k^2}$ $\forall k \in \mathbb{N}^+$, so by the Comparison Test, both $\sum_{k=0}^{\infty} a_{2k}$ and $\sum_{k=0}^{\infty} b_{2k}$ converge absolutely. Using this notation we rewrite our limit as

$$\lim_{n\to\infty}\sum_{m=1}^{n}\sum_{k=0}^{m}(2b_{2k}\cos^2(k\pi x)-b_{2k})(a_{2(m-k)}-2a_{2(m-k)}\sin^2((m-k)\pi x)) \tag{3.9}$$

By Theorem 2.4, both of these series converge absolutely:

$$\sum_{k=0}^{\infty}(2b_{2k}\cos^2(k\pi x)-b_{2k}) \tag{3.10}$$

$$\sum_{k=0}^{\infty}(a_{2k}-2a_{2k}\sin^2(k\pi x)) \tag{3.11}$$

We were asked to study the limit of the Cauchy Product of (3.10) and (3.11) (the Cauchy Product is discussed in Rudin's book [Ru]). A theorem in [Ru] states that this limit converges as long as at least one of (3.10) and (3.11) converges absolutely. Since both of these sums converge absolutely, the result follows. □

Note that we needed to use the version of Theorem 2.4 where the sums converged absolutely, as this was necessary to use the theorem on Cauchy products. If (3.10) and (3.11) only converged conditionally, the argument presented above would have been invalid for that reason.

References

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