Junior problems

J415. Prove that for all real numbers x, y, z at least one of the numbers

$$2^{3x-y} + 2^{3x-z} - 2^{y+z+1}$$

$$2^{3y-z} + 2^{3y-x} - 2^{z+x+1}$$

$$2^{3z-x} + 2^{3z-y} - 2^{x+y+1}$$

is nonnegative.

Proposed by Adrian Andreescu, Dallas, USA

Solution by the author

We argue by contradiction assuming that all the three numbers are negative.

Then

$$2^{y+z+1} > 2^{3x-y} + 2^{3x-z} \geq 2 \cdot 2^{\frac{3x-y+3x-z}{2}},$$

by the AM-GM Inequality. It follows that

$$y + z + 1 > \frac{3x - y + 3x - z}{2} + 1,$$

implying $y + z > 2x$. Adding this with the analogous $z + x > 2y$ and $x + y > 2z$ yields a contradiction.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Daniel Lasaosa, Pamplona, Spain; Nikos Kalapodis, Patras, Greece; Alessandro Ventullo, Milan, Italy; Arkady Alt, San Jose, CA, USA; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Duy Quan Tran, University of Medicine and Pharmacy at Ho Chi Minh City, Vietnam; Kousik Sett, West Bengal, Hooghly, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paul Revenant, Lycée du Parc, Lyon, France; Albert Stadler, Herrliberg, Switzerland; Stephanie Li, New York, NY, USA; Titu Zvonaru, Comănești, România; Polyahedra, Polk State College, FL, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Alan Yan, Princeton Junction, NJ, USA; Ervin Ramirez, Universidad Nacional de Ingenieria, Nicaragua; Prajnanaswaroopa S., Bangalore, Karnataka, India.

J416. Find all positive real numbers a and b for which

$$\frac{ab}{ab+1}+\frac{a^2b}{a^2+b}+\frac{ab^2}{a+b^2}=\frac{1}{2}(a+b+ab).$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution by Polyahedra, Polk State College, USA

By the AM-GM inequality, $ab+1\ge 2\sqrt{ab}$, $a^2+b\ge 2a\sqrt{b}$, and $a+b^2\ge 2b\sqrt{a}$. Thus

$$\begin{aligned}&2(a+b+ab)-4\left(\frac{ab}{ab+1}+\frac{a^2b}{a^2+b}+\frac{ab^2}{a+b^2}\right)\ge 2(a+b+ab)-2(\sqrt{ab}+a\sqrt{b}+b\sqrt{a})\\&=(\sqrt{a}-\sqrt{b})^2+(\sqrt{a}-\sqrt{ab})^2+(\sqrt{b}-\sqrt{ab})^2\ge 0,\end{aligned}$$

with equalities if and only if $a=b=1$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Arkady Alt, San Jose, CA, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Duy Quan Tran, University of Medicine and Pharmacy at Ho Chi Minh City, Vietnam; Konstantinos Metaxas, Athens, Greece; Kousik Sett, West Bengal, Hooghly; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nguyen Ngoc Tu, Ha Giang, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, România; Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy; Rajarshi Kanta Ghosh, Kolkata, India.

J417. Solve in positive real numbers the equation

$$\frac{x^2+y^2}{xy+1} = \sqrt{2-\frac{1}{xy}}.$$

Proposed by Adrian Andreescu, Dallas, Texas, USA

Solution by Jamal Gadirov, Istanbul University, Istanbul, Turkey

Let take square of both sides :

$$\frac{(x^2+y^2)^2}{(1+xy)^2} = \frac{2xy-1}{xy}$$

After some simplifications we will get

$$xy(x^4+y^4)+1 = 3x^2y^2$$

Since x, y positive real numbers by $AM-GM$ inequality we have

$$x^5y+y^5x+1 \geq 3\sqrt[3]{x^6y^6} = 3x^2y^2$$

That is equation has solution if and only if whenever equality case holds. Thus, it has solution iff $x^5y = y^5x = 1$. Now if $x^5y = y^5x$ then $xy(x-y)(x+y)(x^2+y^2) = 0$. Clearly $x, y \neq 0$. Therefore there are two cases :

Case 1: If $x = y$ then $x^6 = 1$ (roots of unity) and only real solution is $x = 1 = y$.

Case 2: If $x = -y$ then $x^6 = -1$ which has no real solution.

So, we conlude that there is only $(x, y) = \{(1, 1)\}$ solution.

Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, FL, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Alan Yan, Princeton Junction, NJ, USA; Ervin Ramirez, Universidad Nacional de Ingenieria, Nicaragua; Konstantinos Metaxas, Athens, Greece; Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy; Alessandro Ventullo, Milan, Italy; Arkady Alt, San Jose, CA, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Duy Quan Tran, University of Medicine and Pharmacy at Ho Chi Minh City, Vietnam; Kousik Sett, West Bengal, Hooghly, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paul Revenant, Lycée du Parc, Lyon, France; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, România.

J418. Prove that the following inequality holds for all $a, b, c \in [0, 1]$

$$a + b + c + 3abc \geq 2(ab + bc + ca).$$

Proposed by Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain
Note that the proposed inequality is equivalent to

$$a(1-b)(1-c) + b(1-c)(1-a) + c(1-a)(1-b) \geq 0,$$

clearly true since all factors in all terms are nonnegative. Note further that if $c \neq 0$, then either $a = 1$ or $b = 1$, hence wlog by symmetry in the problem, we may assume that $c \in \{0, 1\}$. If $c = 1$ then equality is equivalent to $(1-a)(1-b)$, ie $a = 1$ or $b = 1$ but not necessarily both. On the other hand, if $c = 0$ equality is equivalent to $a + b = 2ab$. But since $ab \leq a$ with equality iff $a = 1$ or $b = 0$ because $a, b \in [0, 1]$, and similarly $ab \leq b$, we have $a + b \leq a + b$, or since equality must hold, either $a = b = 0$ or $a = b = 1$. We conclude that equality holds in the proposed inequality iff $(a, b, c) = (0, 0, 0)$ or iff (a, b, c) is a permutation of $(1, 1, k)$ where k is any real in $[0, 1]$.

Also solved by Polyahedra, Polk State College, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Alan Yan, Princeton Junction, NJ, USA; Prajnanaswaroopa S., Bangalore, Karnataka, India; Konstantinos Metaxas, Athens, Greece; Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy; Rajarshi Kanta Ghosh, Kolkata, India; Arkady Alt, San Jose, CA, USA; Kevin Soto Palacios, Huarmey-Perú; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Adamopoulos Dionysios, 3rd High School, Pyrgos, Greece; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Duy Quan Tran, University of Medicine and Pharmacy at Ho Chi Minh City, Vietnam; Kousik Sett, West Bengal, Hooghly, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paraskevi-Andrianna Maroutsou, High School of Evangeliki, Athens, Greece; Paul Revenant, Lycée du Parc, Lyon, France; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland.

J419. Let a,b,c be positive real numbers such that $abc=1$. Prove that

$$\frac{1}{a^4+b+c^4}+\frac{1}{b^4+c+a^4}+\frac{1}{c^4+a+b^4}\le\frac{3}{a+b+c}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, CA, USA

In homogeneous form the original inequality becomes

$$\sum_{cyc}\frac{abc}{a^4+ab^2c+c^4}\le\frac{3}{a+b+c}.$$

Since

$$\frac{abc}{a^4+ab^2c+c^4}=\frac{b}{\frac{a^3}{c}+b^2+\frac{c^3}{a}}\text{ and }\frac{a^3}{c}+\frac{c^3}{a}\ge a^2+b^2\iff a^4+c^4\ge a^3c+ac^3\iff$$

$$\left(a^2+ac+c^2\right)(a-c)^2\ge 0$$

then

$$\frac{b}{\frac{a^3}{c}+b^2+\frac{c^3}{a}}\le\frac{b}{a^2+b^2+c^2}$$

and, therefore,

$$\sum_{cyc}\frac{abc}{a^4+ab^2c+c^4}\le\sum_{cyc}\frac{b}{a^2+b^2+c^2}=\frac{a+b+c}{a^2+b^2+c^2}.$$

And also we have

$$\frac{a+b+c}{a^2+b^2+c^2}\le\frac{3}{a+b+c}\iff(a+b+c)^2\le 3\left(a^2+b^2+c^2\right)\iff$$

$$(a-b)^2+(b-c)^2+(c-a)^2\ge 0.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Nikos Kalapodis, Patras, Greece; Polyahedra, Polk State College, FL, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Konstantinos Metaxas, Athens, Greece; Kevin Soto Palacios, Huarmey-Perú; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Abhay Chandra, Indian Institute of Technology, New Delhi, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nguyen Ngoc Tu, Ha Giang, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, România.

J420. Let ABC be a triangle and let A, B, C be the magnitudes of its angles, expressed in radians. Prove that if A, B, C and $\cos A, \cos B, \cos C$ are geometric sequences, then the triangle is equilateral.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

First solution by Joel Schlosberg, Bayside, NY, USA

First, we show that $\triangle ABC$ is acute. If $\triangle ABC$ has a nonacute angle, then being larger than the other angles it must be the first or last in the geometric sequence A, B, C. Then $\cos A \cdot \cos C$, as a product with one nonpositive factor, is nonpositive; while $\cos^2 B$, the square of a nonzero quantity, is positive. Thus $\cos A \cdot \cos C \neq \cos^2 B$, contradicting the assumption that $\cos A, \cos B, \cos C$ is a geometric sequence.

Define $f : (0, \pi/2) \to \mathbb{R}$ by $x \mapsto \ln \cos x$. Since the cosine function is strictly decreasing on $(0, \pi/2)$ and the logarithm function is strictly increasing on $(0, \pi/2)$, f is strictly decreasing. Since by the AM-GM inequality $\sqrt{AC} \leq \frac{A+C}{2}$ with equality iff $A = C$,

$$\ln \cos \sqrt{AC} = f\left(\sqrt{AC}\right) \geq f\left(\frac{A+C}{2}\right)$$

with equality iff $A = C$. Since $f''(x) = -\sec^2 x < 0$, f is strictly concave, so by Jensen's inequality

$$f\left(\frac{A+C}{2}\right) \geq \frac{f(A) + f(C)}{2} = \ln \sqrt{\cos A \cos C}$$

with equality iff $A = C$. If A, B, C and $\cos A, \cos B, \cos C$ are geometric sequences, then $\ln \cos \sqrt{AC} = \ln \cos B = \ln \sqrt{\cos A \cos C}$, so equality in the above inequalities implies that $A = C$. Since $B = \sqrt{AC}$ has the common value as well, $\triangle ABC$ is equilateral.

Second solution by Polyahedra, Polk State College, USA

Suppose that $B^2 = AC$ and $\cos^2 B = \cos A \cos C$. Then A and C must be acute. By the AM-GM inequality and the fact that $\cos x$ is concave and decreasing for $x \in (0, \frac{\pi}{2})$,

$$\cos A \cos C \leq \left(\frac{\cos A + \cos C}{2}\right)^2 \leq \cos^2 \frac{A+C}{2} \leq \cos^2 \sqrt{AC} = \cos^2 B,$$

with equalities if and only if $A = C$. Hence $B = A = C$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Kousik Sett, West Bengal, Hooghly, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

S415. Let

$$f(x) = \frac{(2x-1)6^x}{2^{2x-1}+3^{2x-1}}.$$

Evaluate

$$f\left(\frac{1}{2018}\right) + f\left(\frac{3}{2018}\right) + \cdots + f\left(\frac{2017}{2018}\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

Observe that

$$f(1-x) = \frac{(1-2x)6^{1-x}}{2^{1-2x}+3^{1-2x}} = \frac{(1-2x)6^x}{2^{2x-1}+3^{2x-1}} = -f(x),$$

i.e. $f(x) + f(1-x) = 0$.

Therefore,

$$\begin{aligned}
\sum_{k=1}^{1009} f\left(\frac{2k-1}{2018}\right) &= \sum_{k=1}^{504}\left(f\left(\frac{2k-1}{2018}\right) + f\left(\frac{2018-(2k-1)}{2018}\right)\right) + f\left(\frac{1009}{2018}\right)\\
&= f\left(\frac{1009}{2018}\right)\\
&= f\left(\frac{1}{2}\right)\\
&= 0.
\end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Nikos Kalapodis, Patras, Greece; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Konstantinos Metaxas, Athens, Greece; Luca Ferrigno, Universitá degli studi di Tor Vergata, Roma, Italy; Rajarshi Kanta Ghosh, Kolkata, India; Arkady Alt, San Jose, CA, USA; Kevin Soto Palacios, Huarmey-Perú; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Kousik Sett, West Bengal, Hooghly, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Paul Revenant, Lycée du Parc, Lyon, France; Albert Stadler, Herrliberg, Switzerland; Tamoghno Kandar, Mumbai, India; Xingze Xu, Hangzhou Foreign Languages School A-level Centre, China; Titu Zvonaru, Comănești, România.

S416. Let $f : \mathbb{N} \to \{\pm 1\}$ be a function such that $f(mn) = f(m)f(n)$, for all $m, n \in \mathbb{N}$. Prove that there are infinitely many n such that $f(n) = f(n+1)$.

Proposed by Oleksi Krugman, University College London, UK

Solution by Paul Revenant, Lycée du Parc, Lyon, France

Suppose that there are only a finite number of n such that $f(n) = f(n+1)$. Then there exists an even integer N such that:

$$\forall n \geq N, \quad f(n) = -f(n+1).$$

Denote $\epsilon = f(N)$. By induction, $f(N+2k) = \epsilon$ and $f(N+2k+1) = -\epsilon$ for $k \in \mathbb{N}$. Since N is even, the values of even integers greater than N are ϵ, and those of odd integers greater than N are $-\epsilon$.

However, for all integer n, the given equality shows that $f(n^2) = f(n)^2$. Hence, $f(n^2)$ is nonnegative, and then $f(n^2) = 1$. Since n^2 can take even and odd values greater than n, we face with a contradiction, because even and odd integers greater than N shouldn't have the same value by f.

Finally, there are infinitely many n such that $f(n) = f(n+1)$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Konstantinos Metaxas, Athens, Greece; Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy; Rajarshi Kanta Ghosh, Kolkata, India; Alessandro Ventullo, Milan, Italy; Joel Schlosberg, Bayside, NY, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Shuborno Das, Ryan International School, Bangalore, India; Albert Stadler, Herrliberg, Switzerland.

S417. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\leq\frac{3\left(a^2+b^2+c^2\right)}{2(a+b+c)}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

First solution by Albert Stadler, Herrliberg, Switzerland

We note that

$$\frac{3(a^2+b^2+c^2)}{2(a+b+c)}-\frac{a^2}{a+b}-\frac{b^2}{b+c}-\frac{c^2}{c+a}=\frac{\sum_{\text{symm}}a^4b-\sum_{\text{symm}}a^3b^2}{2(a+b)(b+c)(c+a)(a+b+c)}\geq 0$$

by Muirhead's inequality.

Second solution by Daniel Lasaosa, Pamplona, Spain

After multiplying by the (clearly positive) product of denominators and rearranging terms, the proposed inequality is equivalent to

$$ab(s-c)(a-b)^2+bc(s-a)(b-c)^2+ca(s-b)(c-a)^2\geq 0,$$

which is clearly true since all three terms in the LHS are nonnegative. Now, since no two of a, b, c are zero, $s-a=b+c, s-b=c+a, s-c=a+b$ are all positive. If $abc=0$, then wlog by symmetry we may consider $c=0$, and the inequality becomes $ab(a+b)(a-b)^2\geq 0$, where $ab(a+b)>0$, and equality holds iff $a=b$. If $abc\neq 0$, then $ab(s-c), bc(s-a), ca(s-b)$ are positive, or equality holds iff $a=b=c$. Thus equality holds in the proposed inequality iff either $a=b=c$ takes a positive real value, or (a,b,c) is a permutation of $(k,k,0)$, where k is any positive real value.

Also solved by Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Ervin Ramirez, Universidad Nacional de Ingenieria, Nicaragua; Prajnanaswaroopa S., Bangalore, Karnataka, India; Konstantinos Metaxas, Athens, Greece; Nguyen Ngoc Tu, Ha Giang, Vietnam; Arkady Alt, San Jose, CA, USA; Kevin Soto Palacios, Huarmey-Perú; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Kousik Sett, West Bengal, Hooghly, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nguyen Ngoc Tu, Ha Giang, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paul Revenant, Lycée du Parc, Lyon, France; Titu Zvonaru, Comănești, România; P.V.Swaminathan, Smart Minds Academy, Chennai, India.

S418. Let a, b, c, d be positive real numbers such that $abcd \geq 1$. Prove that

$$\frac{a+b}{a+1} + \frac{b+c}{b+1} + \frac{c+d}{c+1} + \frac{d+a}{d+1} \leq a+b+c+d.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by the author

From the condition $abcd \geq 1$, we can find $\lambda \geq 1$ and $x, y, z, t > 0$ such that $a = \lambda x$, $b = \lambda y$, $c = \lambda z$, $d = \lambda t$ and $xyzt = 1$. Then the given inequality becomes equivalent to

$$\sum_{\text{cyc}} \frac{x+y}{1+\lambda x} \leq \sum_{\text{cyc}} x$$

Now, it is sufficient to prove that

$$\sum_{\text{cyc}} \frac{x+y}{1+x} \leq \sum_{\text{cyc}} x.$$

This inequality is equivalent to

$$\sum_{\text{cyc}} \frac{1+xy}{1+x} \geq 4,$$

Let $x = \frac{n}{m}$, $y = \frac{p}{n}$, $z = \frac{q}{p}$, $t = \frac{m}{q}$, where $m, n, p, q > 0$. Then, we have

$$\sum_{\text{cyc}} \frac{1+xy}{1+x} = (m+p)\left(\frac{1}{m+n} + \frac{1}{p+q}\right) + (n+q)\left(\frac{1}{n+p} + \frac{1}{q+m}\right) \geq$$

$$\geq \frac{4(m+p)}{m+n+p+q} + \frac{4(n+q)}{m+n+p+q} = 4.$$

S419. Solve the system of equations

$$x(x^4 - 5x^2 + 5) = y$$
$$y(y^4 - 5y^2 + 5) = z$$
$$z(z^4 - 5z^2 + 5) = x$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

First, we seek real solutions in the interval $[-2, 2]$. Let $x = 2\cos t$, $t \in [0, \pi]$. Then $y = 2(16\cos^5 t - 20\cos^3 t + 5) = 2\cos 5t$, $z = \cos 5(5t) = \cos 25t$, and so $x = 2\cos 5(25t) = 2\cos 125t$. Hence $\cos 125t = \cos t$, implying $125t - t = 2k(\pi), k = 0, 1, \ldots, 124$ or $125t + t = 2k'(\pi)$, $k' = 1, 2, \ldots, 126$.

We obtain $63 + 62$ distinct solutions: $2\cos k\left(\frac{\pi}{62}\right)$, $k = 0, 1, ..., 62$ and $2\cos k\left(\frac{\pi}{63}\right), k' = 1, ..., 62$, as $2\cos 0$ and $2\cos(\pi)$ have already been found.

We do not need to look for more solutions since all 125 solutions we found, $\left(2\cos k\frac{\pi}{62}, 2\cos 5k\frac{\pi}{62}, 2\cos 25k\frac{\pi}{62}\right)$, $k = 0, 1, ..., 62$ and $\left(2\cos k\frac{\pi}{63}, 2\cos 5k\frac{\pi}{63}, 2\cos 25k\frac{\pi}{63}\right)$, $k = 0, 1, ..., 62$ are distinct and the given system has $5 \times 5 \times 5 = 125$ solutions.

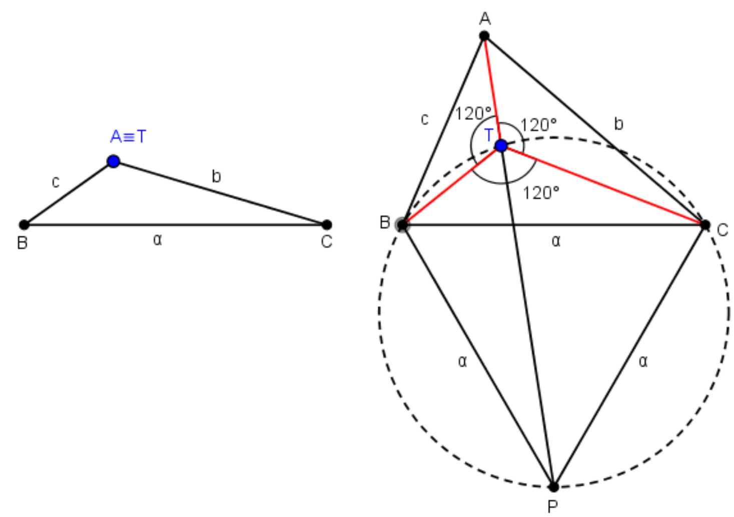
Also solved by Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ervin Ramirez, Universidad Nacional de Ingenieria, Nicaragua; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Angel Mejía, Universidad Autónoma de Santo Domingo, Dominican Republic; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina.

S420. Let T be the Toricelli point of a triangle ABC. Prove that

$$(AT+BT+CT)^2 \le AB\cdot BC+BC\cdot CA+CA\cdot AB.$$

Proposed by Nguyen Viet Chung, Hanoi University of Science, Vietnam

Solution by Nikos Kalapodis, Patras, Greece



If triangle ABC contains an angle of $120°$ or more then it is well-known that the Torricelli point is the vertex of this obtuse angle. So, (without loss of generality) if $A \ge 120°$, then $T \equiv A$ and the given inequality becomes $(AB+CA)^2 \le AB\cdot BC+BC\cdot CA+CA\cdot AB$ or $(b+c)^2 \le ab+bc+ca$. Thus, in this case, it is enough to prove that $b^2+c^2+bc \le a(b+c)$ (1).
Since $A \ge 120°$, we have that $\cos A \le -\frac{1}{2}$ or $-\cos A \ge \frac{1}{2}$ and by the law of cosines we obtain $a^2=b^2+c^2-2bc\cos A \ge b^2+c^2+bc$. Taking into account the triangle inequality we have $b^2+c^2+bc \le a^2 < a(b+c)$, so (1) is true.

If triangle ABC has no angle greater than $120°$ then it is well-known that the Torricelli point is that point inside triangle ABC at which $\angle TAB=\angle TBC=\angle TCA=120°$. Furthermore T lies on AP where P is the vertex of equilateral triangle PBC drawn outwardly of triangle ABC.
Then $AT+BT+CT=AT+TP=AP$ (We have that $TP=BT+CT$ by the Ptolemy's theorem on quadrilateral $TBPC$).
By the law of cosines in triangle BAP we obtain that

$$AP^2=a^2+c^2-2ac\cos\angle ABP=a^2+c^2-2ac\cos(B+60°)=a^2+c^2-2ac\left(\frac{1}{2}\cos B-\frac{\sqrt{3}}{2}\sin B\right)=$$

$$a^2+c^2-ac\cos B+\sqrt{3}ac\sin B=a^2+c^2-\frac{a^2+c^2-b^2}{2}+2\sqrt{3}S=\frac{a^2+b^2+c^2+4\sqrt{3}S}{2}.$$

Therefore we have to prove that $\frac{a^2+b^2+c^2+4\sqrt{3}S}{2} \le ab+bc+ca$ or
$4\sqrt{3}S \le 2(ab+bc+ca)-a^2-b^2-c^2$. But this is the well-known Hadwiger-Finshler's inequality and we are done.

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Kousik Sett, West Bengal, Hooghly, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

U415. Prove that the polynomial $P(X) = X^4 + iX^2 - 1$ is irreducible in the ring of polynomials over Gauss integers.

Proposed by Mircea Becheanu, University of Bucharest, Romania

Solution by Daniel Lasaosa, Pamplona, Spain

Let z be a root of the polynomial. Note that

$$z^2 = \frac{-i \pm \sqrt{-1+4}}{2} = \frac{-i + \sqrt{3}}{2}, \qquad |z|^4 = \left|z^2\right|^2 = \frac{1+3}{4} = 1,$$

or $|z| = 1$ for any root of the polynomial. Now, the Gauss integers with modulus 1 are $1, i, -1, -i$, whose squares are $1, -1, 1, -1$, clearly different from $\frac{-i+\sqrt{3}}{2}$. Or the roots of $P(X)$ are not Gauss integers, hence $P(X)$ does not admit any factor of the form $X - z$ in the ring of polynomials over the Gauss integers, and if $P(X)$ is not irreducible in this ring, then polynomials $Q(X) = X^2 + aX + b$ and $R(x) = X^2 + cX + d$ exist such that $P(X) = Q(X)R(X)$ and where a, b, c, d are Gauss integers. But

$$Q(X)R(X) = X^4 + (a+c)X^3 + (ac+b+d)X^2 + (bc+ad)X + bd,$$

or $c = -a$. If $a = c = 0$ then $Q(X)R(X) = X^4 + (b+d)X^2 + bd$, or $b + d = i$, $bd = -1$, and b, d are the roots of $r^2 - ir - 1 = 0$, for

$$r = \frac{i \pm \sqrt{-1+4}}{2} = \frac{i \pm \sqrt{3}}{2},$$

which are clearly not Gauss integers. If $c = -a \neq 0$, then $b = d$, and $2b - a^2 = i$, $b^2 = -1$, for $b = \pm i$, and $a^2 = 2b - i \in \{i, -3i\}$. But if $a^2 = i$ then $a = \pm\frac{1+i}{\sqrt{2}}$ is not a Gauss integer, and if $a^2 = -3i$ then $a = \pm\sqrt{3}\frac{1-i}{2}$ is not a Gauss integer. The conclusion follows.

Also solved by Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Prajnanaswaroopa S., Bangalore, Karnataka, India; Brian Zilli, Hofstra University, Hempstead, NY, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Albert Stadler, Herrliberg, Switzerland.

U416. For any root $z \in \mathbb{C}$ of the polynomial $X^4 + iX^2 - 1$ we denote $w_z = z + \frac{2}{z}$. Let $f(x) = x^2 - 3$. Prove that

$$|(f(w_z) - 1)\, f(w_z - 1) f(w_z + 1)|$$

is an integer that does not depend on z.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Paul Revenant, Lycée du Parc, Lyon, France
Let z be a root of the polynomial $X^4 + iX^2 - 1$. After computation, we get:

$$|(f(w_z) - 1) f(w_z - 1) f(w_z + 1)| = \left| z^6 + \frac{64}{z^6} \right|.$$

Now, we remark that $z^6 = (z^2 - i)(z^4 + iz^2 - 1) - i = -i$, and then:

$$\left| z^6 + \frac{64}{z^6} \right| = \left| -i - \frac{64}{i} \right| = 63.$$

Finally, $|(f(w_z - 1) f(w_z - 1) f(w_z + 1)|$ is an integer that does not depend on z.

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Kousik Sett, West Bengal, Hooghly, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina.

U417. *Prove that for any $n \geq 14$ and for any real number x, $0 < x < \frac{\pi}{2n}$, the following inequality holds:*

$$\frac{\sin 2x}{\sin x} + \frac{\sin 3x}{\sin 2x} + \ldots + \frac{\sin(n+1)x}{\sin nx} < 2\cot x$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

Lemma:

For $0 < x < \frac{\pi}{2n}$, $\frac{\sin(k+1)x}{\sin kx} \leq 1 + \frac{1}{k}$, $0 \leq k \leq n$

Proof of the Lemma:

We need to prove

$$\int_0^{kx} \cos y dy + \int_{kx}^{(k+1)x} \cos y dy \leq (1 + \frac{1}{k}) \int_0^{kx} \cos dy$$

that is

$$\int_{kx}^{(k+1)x} \cos y dy \leq \frac{1}{k} \int_0^{kx} \cos y dy$$

This is implied by

$$x\cos(kx) \leq \frac{\sin kx}{k} \quad \Longleftrightarrow \quad f(x) \doteq \sin(kx) - kx\cos(kx) \geq 0$$

$$f'(x) = k\cos(kx) - x\cos(kx) + k^2 x\sin(kx) = k^2 x \sin(kx), \;\; kx \leq \pi/2$$

Thus $f(x) \geq f(0) = 0$ ■

Based on the Lemma, it suffices to show

$$\sum_{k=1}^{n} 1 + \frac{1}{k} < 2\frac{\cos x}{\sin x}, \quad 0 < x \leq \frac{\pi}{2n}, \quad n \geq 14 \tag{1}$$

The concavity of $\cos x$ for $0 \leq x \leq \pi/(2n)$, yields

$$\cos x \geq 1 + \frac{2nx}{\pi}\left(\cos\frac{\pi}{2n} - 1\right)$$

The inequality in (1) is implied by

$$\sum_{k=1}^{n} 1 + \frac{1}{k} < 2\frac{1 + \frac{2nx}{\pi}\left(\cos\frac{\pi}{2n} - 1\right)}{x} = \frac{2}{x} - \frac{4n}{\pi}\sin^2\frac{\pi}{4n}$$

which in turn is implied by (use $(\sin x \leq x)$

$$\sum_{k=1}^{n}\left(1 + \frac{1}{k}\right) < \frac{4n}{\pi} - \frac{\pi}{4n}$$

This will be proven by induction.

Let

$$f(n) \doteq \sum_{k=1}^{n}\left(1 + \frac{1}{k}\right) - \frac{4n}{\pi} + \frac{\pi}{4n}, \quad f(12) \sim -0.1$$

Suppose $f(n) < 0$ for any $12 \le n \le r$. For $n = r + 1$ we need to show

$$\sum_{k=1}^{r} \left(1 + \frac{1}{k}\right) + 1 + \frac{1}{r+1} - \frac{4r+4}{\pi} + \frac{\pi}{4r+4} < 0$$

By using the inductive step we come to

$$\frac{4r}{\pi} - \frac{\pi}{4r} + 1 + \frac{1}{r+1} - \frac{4r+4}{\pi} + \frac{\pi}{4r+4} < 0$$

which is equivalent to

$$g(n) \doteq \frac{\pi - 4}{\pi} + \frac{1}{n+1} - \frac{\pi}{4n} + \frac{\pi}{4n+4} < 0 \qquad \forall\ n \ge 12$$

We have $g(12) \sim -0.2$ and

$$g'(n) = \frac{\pi}{4n^2} - \frac{1}{(n+1)^2} - \frac{4\pi}{16(n+1)^2}$$

and it easy to observe that $g'(n) < 0$ for $n \ge 12$ so concluding the proof.

Also solved by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina.

U418. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\sqrt{16a^2+9}+\sqrt{16a^2+9}+\sqrt{16a^2+9} \le 1+\frac{14}{3}(a+b+c)$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

Let $a = e^x$,
$b = e^y$, $c = e^z$.

The condition $abc = 1$ becomes $x + y + z = 0$.

$$\left(\sqrt{16e^{2x}+9}\right)'' = \frac{32e^{2x}(8e^{2x}+9)}{(16e^{2x}+9)^{\frac{3}{2}}} \ge 0$$

The convexity implies the following standard result for a convex function $f(x)$ defined on $\mathbf{R}$

$$f(x_1)+\ldots+f(x_n) \le f(x_1+\ldots+x_n-(n-1)x_0)+(n-1)f(x_0)$$

for any x_0.

By choosing $x_0 = 0$ and clearly $f(x) = \sqrt{16e^{2x}+9}$ we get

$$\sqrt{16e^{2x}+9}+\sqrt{16e^{2y}+9}+\sqrt{16e^{2z}+9} \le f(0)+2f(0) = 5+10 = 15$$

Moreover

$$1+\frac{14}{3}(a+b+c) \ge 1+\frac{14}{3}3(abc)^{\frac{1}{3}} = 15$$

and we are done.

Also solved by Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Prajnanaswaroopa S., Bangalore, Karnataka, India; Konstantinos Metaxas, Athens, Greece; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland.

U419. Let $p>1$ be a natural number. Prove that

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{\sqrt[p]{k}}-\frac{p}{p-1}\left(n^{\frac{p-1}{p}}-1\right)\right)\in(0,1)$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Arkady Alt, San Jose, CA, USA

Let $a_n:=\sum_{k=1}^{n}\frac{1}{\sqrt[p]{k}}-\frac{p}{p-1}\left(\left(n+1^{\frac{p-1}{p}}\right)-1\right)$ and $b_n:=\sum_{k=1}^{n}\frac{1}{\sqrt[p]{k}}-\frac{p}{p-1}\left(n^{\frac{p-1}{p}}-1\right)$.

First we will prove that $a_n\uparrow\mathbb{N}$ and $b_n\downarrow\mathbb{N}$..

Indeed, by Mean Value Theorem there is $c_n\in(n+1,n+2)$ such that

$$a_{n+1}-a_n=\frac{1}{\sqrt[p]{n+1}}-\frac{p}{p-1}\left((n+2)^{\frac{p-1}{p}}-(n+1)^{\frac{p-1}{p}}\right)=$$

$$\frac{1}{\sqrt[p]{n+1}}-\frac{1}{\sqrt[p]{c_n}}>\frac{1}{\sqrt[p]{n+1}}-\frac{1}{\sqrt[p]{n+1}}=0$$

and similarly, there is $c_n\in(n,n+1)$

$$b_n-b_{n+1}=-\frac{1}{\sqrt[p]{n+1}}+\frac{p}{p-1}\left((n+1)^{\frac{p-1}{p}}-n^{\frac{p-1}{p}}\right)=$$

$$\frac{1}{\sqrt[p]{c_n}}-\frac{1}{\sqrt[p]{n+1}}>\frac{1}{\sqrt[p]{n+1}}-\frac{1}{\sqrt[p]{n+1}}=0$$

Since, $a_n\uparrow\mathbb{N},b_n\downarrow\mathbb{N}$ and $a_n<b_n$, $n\in\mathbb{N}$ then $a_n<b_m$ for any $n,m\in\mathbb{N}$.

Indeed,$a_n<a_{n+m}<b_{n+m}<b_m$.In particular $a_n<b_1$ and $a_1<b_n$, $n\in\mathbb{N}$.

Therefore, both sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are convergent.

Let $l:=\lim_{n\to\infty}a_n$ and $u:=\lim_{n\to\infty}b_n$. Since $\lim_{n\to\infty}(b_n-a_n)=0$ then $l=u$.

Let $c:=\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n$ and also $a_n<c<b_n$ because $a_n\uparrow\mathbb{N},b_n\downarrow\mathbb{N}$ and $a_n<b_n$, $n\in\mathbb{N}$.

Thus, we have $a_n<c<b_n \iff$

$$\sum_{k=1}^{n}\frac{1}{\sqrt[p]{k}}-\frac{p}{p-1}\left(\left(n+1^{\frac{p-1}{p}}\right)-1\right)<c<\sum_{k=1}^{n}\frac{1}{\sqrt[p]{k}}-\frac{p}{p-1}\left(n^{\frac{p-1}{p}}-1\right)\iff$$

$$\frac{p}{p-1}\left(n^{\frac{p-1}{p}}-1\right)+c<\sum_{k=1}^{n}\frac{1}{\sqrt[p]{k}}<\frac{p}{p-1}\left(\left(n+1^{\frac{p-1}{p}}\right)-1\right)+c.$$

Since $\frac{p}{p-1}\left(2^{\frac{p-1}{p}}-1\right)\uparrow p\in\mathbb{N}\setminus\{1\}$ then

$$a_1=\sum_{k=1}^{1}\frac{1}{\sqrt[p]{k}}-\frac{p}{p-1}\left(\left((1+1)^{\frac{p-1}{p}}\right)-1\right)=$$

$$1-\frac{p}{p-1}\left(2^{\frac{p-1}{p}}-1\right)\geq 1-\frac{2}{2-1}\left(2^{\frac{2-1}{2}}-1\right)=1-2\left(\sqrt{2}-1\right)=1-\frac{2}{\sqrt{2}+1}=\frac{\sqrt{2}-1}{\sqrt{2}+1}>0$$

and, therefore, $a_1\leq a_n<c\implies c>0$.

Also, since $b_1=\sum_{k=1}^{1}\frac{1}{\sqrt[p]{k}}-\frac{p}{p-1}\left(1^{\frac{p-1}{p}}-1\right)=1$ and $c<b_n\leq b_1$ then $c<1$.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy.

U420. Find the least length of a segment whose endpoints are on the hyperbola $xy = 5$ and ellipse $\frac{x^2}{4} + 4y^2 = 2$, respectively.

Proposed by Titu Andreescu, USA and Oleg Mushkarov, Bulgaria

Solution by the authors

The problem is: for real numbers a, b, c, d find the minimum of $(a-c)^2 + (b-d)^2$ given the constraints $ab = 5$ and $\dfrac{c^2}{4} + 4d^2 = 2$.

We have

$$(a-c)^2 + (b-d)^2 = \left(2\frac{a}{\sqrt{5}} - c\frac{\sqrt{5}}{2}\right)^2 + \left(\frac{b}{\sqrt{5}} - d\sqrt{5}\right)^2 + \left(\frac{1}{5}\right)(a-2b)^2 + 4\frac{ab}{5} - \left(\frac{c^2}{4} + 4d^2\right) \geq 4 - 2 = 2.$$

Hence the minimum is 2, achieved if and only if

$$a = \sqrt{10}, b = \frac{\sqrt{10}}{2}, c = 4\frac{\sqrt{10}}{5}, d = \frac{\sqrt{10}}{10}$$

or

$$a = -\sqrt{10}, b = -\frac{\sqrt{10}}{2}, c = -4\frac{\sqrt{10}}{5}, d = -\frac{\sqrt{10}}{10}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina.

O415. Let $n > 2$ be an integer. An $n \times n$ square is divided into n^2 unit squares. Find the maximum number of unit squares that can be painted in such a way that every 1×3 rectangle contains at least one unpainted unit square.

Proposed by Magauin Armanzhan, Kazakhstan and Nairi Sedrakyan, Armenia

Solution by Daniel Lasaosa, Pamplona, Spain

By "every 1×3 rectangle", we may understand either only those rectangles in a given orientation (ie those containing the unit squares in the intersection of one row and the union of three consecutive columns), or any rectangle of that size regardless of its orientation (ie also those containing the unit squares in the intersection of one column and the union of three consecutive rows). Since the former is a relatively obvious proposition (see the note at the end of this solution), we will assume that the problem actually refers to the second case.

Number the rows and the columns, and denote by (i, j) the unit square in the intersection of the i-th row and the j-th column. Note that it suffices to leave $\left\lfloor \frac{n^2}{3} \right\rfloor$ unit squares unpainted, more specifically those such that $i + j \equiv 1 \pmod 3$. Clearly, any 1×3 rectangle contains three unit squares with the same value of i (or j) and three consecutive values of j (or of i), yielding three consecutive integers as values of $i + j$, one of which will clearly be a multiple of three, and thus unpainted.

If $n = 3k$ is a multiple of 3, this includes the diagonal defined by $i + j = n + 1$ which contains n unit squares, and minor diagonals symmetric two by two with respect to this one, one of such diagonals containing $3, 6, \ldots, n-3$ unit squares on each side of diagonal $i + j = n + 1$, for a total of unpainted unit squares equal to

$$\frac{n(n-3)}{3} + n = \frac{n^2}{3}.$$

If $n = 3k + 1$ for some positive integer k, this includes minor diagonals in the same direction as diagonal $i + j = n + 1$, with respectively $3, 6, \ldots, n-1$ unit squares, and on the other side of diagonal $i + j = n + 1$ with $n-2, n-5, \ldots, 2$ unit squares, for a total of unpainted unit squares equal to

$$\frac{(n+2)(n-1)}{6} + \frac{n(n-1)}{6} = \frac{n^2-1}{3} = \frac{n^2}{3} - \frac{1}{3}.$$

If $n = 3k + 2$, the same argument leads to diagonals containing $3, 6, \ldots, n-2$ unit squares on one side of diagonal $i + j = n + 1$, and diagonals containing $1, 4, \ldots, n-1$ unit squares on the other side of said diagonal, for a total of unpainted unit squares equal to

$$\frac{(n+1)(n-2)}{6} + \frac{n(n+1)}{6} = \frac{n^2-1}{3} = \frac{n^2}{3} - \frac{1}{3}.$$

In all three cases, the total number of unpainted squares can be clearly described by the aforementioned expression $\left\lfloor \frac{n^2}{3} \right\rfloor$. Note now that for $n = 3, 4, 5$, this sufficient number of unpainted squares is also minimum. Indeed, in the 3×3 square, there must be at least one unpainted square in each line, the 4×4 square can be tiled with four 1×3 rectangles and one 3×1 rectangle, leaving one unit square uncovered, and each one of the five tiles must contain at least one unpainted square. For the 5×5 square, note that each row must contain at least one unpainted square, and for each row that contains exactly one unpainted square, it must be its central square. If this happens in more than two rows, there are either two adjacent rows next to one of the edges of the square, or two rows separated by exactly one row, with exactly one unpainted square each, the central square in both cases. Considering the four 3×1 rectangles which contain the other four squares of each one of these rows, their remaining four unit squares, which are on the same line, must be all left unpainted. There are therefore, either at most two rows with exactly one unpainted square and at least three rows with at least two unpainted squares each, or at least one row with at least four unpainted squares and at least one unpainted square in each one of the other for rows, yielding at least 8 unpainted squares.

If the result holds for an $n \times n$ square, note that an $(n+3) \times (n+3)$ square may be partitioned into one $n \times n$ square, which by hypothesis of induction contains at least $\left\lfloor \frac{n^2}{3} \right\rfloor$ unpainted squares, $n+3$ rectangles 3×1 and n rectangles 1×3, for a minimum total number of unpained squares equal to

$$\left\lfloor \frac{n^2}{3} \right\rfloor + (n+3) + n = \left\lfloor \frac{n^2}{3} + 2n + 3 \right\rfloor = \left\lfloor \frac{n^2+6n+9}{3} \right\rfloor = \left\lfloor \frac{(n+3)^2}{3} \right\rfloor,$$

and the result holds for all $n \geq 3$. Thus the maximum number of painted squares is

$$n^2 - \left\lfloor \frac{n^2}{3} \right\rfloor = \left\lceil \frac{2n^2}{3} \right\rceil.$$

Note: For the case where rectangles 1×3 of only one orientation are considered (wlog those oriented along a row), it suffices to see that at least $\left\lfloor \frac{n}{3} \right\rfloor$ unit squares must be left unpainted in each row, otherwise partitioning the row in $\left\lfloor \frac{n}{3} \right\rfloor$ rectangles 1×3 and a remainder of 0, 1 or 2 unit squares in the end, by Dirichlet's pigeonhole principle, at least one of these rectangles would not have an unpainted unit square. On the other hand, numbering the columns from 1 to n and leaving unpainted all squares in the columns which are multiples of 3 clearly solves the problem. It follows that the maximum number of painted squares in this (significantly easier at least at first sight) case is $n^2 - n\left\lfloor \frac{n}{3} \right\rfloor$.

Also solved by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Joel Schlosberg, Bayside, NY, USA.

O416. Let a, b, c real numbers such that $a^2 + b^2 + c^2 - abc = 4$. Find the minimum of $(ab - c)(bc - a)(ca - b)$ and all triples (a, b, c) for which the minimum is attained.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy

$$(ab - c)(bc - a)(ca - b) = (abc)^2 - abc(a^2 + b^2 + c^2) + (a^2b^2 + b^2c^2 + c^2a^2) - abc$$

We change variables
$a + b + c = 3u$, $ab + bc + ca = 3v^2$, $abc = w^3$ so we get $w^3 = 9u^2 - 6v^2 - 4$ and

$$(ab - c)(bc - a)(ca - b) = -9(v^2)^2 + v^2(30 - 36u) - 45u^2 + 54u^3 - 24u + 20$$

This is a concave parabola whose minimum is attained at one of the extreme points of the variable v^2. The theory states that once fixed the values of u and w the extreme values of v occur when $a = b$ (or cyclic) so we set $a = b$ and $a^2 = 2 + c$ from $a^2 + b^2 + c^2 - abc = 4$. Clearly $c \geq -2$.

$$(ab - c)(bc - a)(ca - b)\Big|_{a=b=\pm\sqrt{2+c}} = 2(2 + c)(c - 1)^2 \geq 0$$

Thus the minimum is zero and is attained when

$$(a, b, c) = (0, 0, -2), (\sqrt{3}, \sqrt{3}, 1), (-\sqrt{3}, -\sqrt{3}, 1)$$

and symmetric.

Also solved by Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

O417. Let $x_1, x_2, \ldots, x_n$ be real numbers such that ${x_1}^2 + {x_2}^2 + \cdots + {x_n}^2 \le 1$. Prove that

$$|x_1| + |x_2| + \cdots + |x_n| \le \sqrt{n}\left(1 + \frac{1}{n}\right) + n^{\frac{n-1}{2}} x_1 x_2 \cdots x_n.$$

When does equality occur?

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Joel Schlosberg, Bayside, NY, USA

By the arithmetic mean-quadratic mean inequality,

$$|x_1| + \cdots + |x_n| \le n\sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}} \le \sqrt{n}.$$

By the AM-GM inequality,

$$n^{\frac{n-1}{2}} |x_1 \cdots x_n| = \frac{1}{\sqrt{n}} \left(n \sqrt[n]{x_1^2 \cdots x_n^2}\right)^{n/2} \le \frac{1}{\sqrt{n}} \left(x_1^2 + \cdots + x_n^2\right)^{n/2} = \frac{1}{\sqrt{n}}.$$

Therefore,

$$|x_1| + \cdots + |x_n| \le \sqrt{n} + \frac{1}{\sqrt{n}} - n^{\frac{n-1}{2}} |x_1 \cdots x_n| \le \sqrt{n} + \frac{1}{\sqrt{n}} + n^{\frac{n-1}{2}} x_1 \cdots x_n = \sqrt{n}\left(1 + \frac{1}{n}\right) + n^{\frac{n-1}{2}} x_1 \cdots x_n.$$

Unless $|x_1| = |x_2| = \cdots = |x_n|$ and equivalently $x_1^2 = \cdots = x_n^2$, inequality is strict in the AM-QM and AM-GM inequalities, and thus in the desired inequality.

Let the common absolute value be x. Since $nx^2 = x_1^2 + \cdots + x_n^2 \le 1$,

$$|x_1| + \cdots + |x_n| + n^{\frac{n-1}{2}} x_1 \cdots x_n = nx \pm n^{\frac{n-1}{2}} x^n \le \sqrt{n} + \frac{1}{\sqrt{n}},$$

with strict inequality unless $x = \frac{1}{\sqrt{n}}$. For $x = \frac{1}{\sqrt{n}}$, equality implies that

$$\frac{n}{\sqrt{n}} = \sqrt{n} + \frac{1}{\sqrt{n}} + n^{\frac{n-1}{2}} x_1 \cdots x_n$$

$$x_1 \cdots x_n = -\left(\frac{1}{\sqrt{n}}\right)^n$$

and therefore that an odd number of $x_1, \ldots, x_n$ are negative.

Therefore, an odd number of $x_1, \ldots, x_n$ are $-\frac{1}{\sqrt{n}}$ and all the rest are $\frac{1}{\sqrt{n}}$. Conversely, any such values for $x_1, \ldots, x_n$ yield equality, so those are exactly the conditions when equality occurs.

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Paul Revenant, Lycée du Parc, Lyon, France; Albert Stadler, Herrliberg, Switzerland.

O418. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{a^5}{c^3+1} + \frac{b^5}{a^3+1} + \frac{c^5}{b^3+1} \geq \frac{3}{2}.$$

Proposed by Konstantinos Metaxas, Athens, Greece

Solution by Daniel Lasaosa, Pamplona, Spain
Note first that since $1 = \frac{a^2+b^2+c^2}{3} \leq \frac{a^3+b^3+c^3}{3}$ because of the inequality between quadratic and cubic means, the proposed inequality is weaker than

$$\frac{6a^5}{4c^3+a^3+b^3} + \frac{6b^5}{4a^3+b^3+c^3} + \frac{6c^5}{4b^3+c^3+a^3} \geq a^2 + b^2 + c^2,$$

both being equivalent iff $a = b = c = 1$. Now, this new inequality may rewrite, after multiplying both sides by the product of denominators and rearranging terms, as follows:

$$\begin{gathered}
\sum_{\text{cyc}} a^3 \left(3a^8 + b^8 - 4a^6b^2\right) + \sum_{\text{cyc}} b^3 \left(3b^8 + a^8 - 4a^2b^6\right) + \\
+3\sum_{\text{cyc}} a^3b^3 \left(3a^5 + 2b^5 - 5a^3b^2\right) + \frac{3}{7}\sum_{\text{cyc}} \left(33a^8b^3 + 6b^8c^3 + 10c^8a^3 - 49a^6b^3c^2\right) + \\
+\frac{3}{7}\sum_{\text{cyc}} \left(30a^8b^3 + b^8c^3 + 18c^8a^3 - 49a^6b^2c^3\right) + \\
+\sum_{\text{cyc}} \left(5a^{11} + 5b^{11} + 29a^8b^3 + 2a^3b^8 + 3a^5b^6 - 44a^7b^4\right) \\
+\frac{44}{37}\sum_{\text{cyc}} \left(19a^7b^4 + 5b^7c^4 + 13c^7a^4 - 37a^5b^3c^3\right) + \\
+4\left(a^{11} + b^{11} + c^{11} - a^3b^3c^3\left(a^2 + b^2 + c^2\right)\right) \geq 0.
\end{gathered}$$

Now, all terms in the LHS except for the last one are nonnegative by use of the weighted AM-GM inequality, whereas the last term is nonnegative by use of the power-mean inequality and the AM-GM inequality. The last term is zero iff $a = b = c = 1$, and this condition is sufficient for equality in all other conditions, hence equality holds in the proposed inequality iff $a = b = c = 1$.

Also solved by Luke Robitaille, Euless, Texas, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Arkady Alt, San Jose, CA, USA; Kevin Soto Palacios, Huarmey-Perú; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Roma, Italy; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nguyen Ngoc Tu, Ha Giang, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland.

O419. Let $x_1, x_2, \ldots, x_n$ be real numbers in the interval $(0, \frac{\pi}{2})$. Prove that

$$\frac{1}{n^2}\left(\frac{\tan x_1}{x_1}+\cdots+\frac{\tan x_n}{x_n}\right)^2 \leq \frac{\tan^2 x_1+\cdots+\tan^2 x_n}{{x_1}^2+\cdots+{x_n}^2}.$$

Proposed by Mircea Becheanu, University of Bucharest, Romania

Solution by Arkady Alt, San Jose, CA, USA

First note that $\dfrac{\tan x}{x}$ increase in $(0, \pi/2)$.

Indeed

$$\left(\frac{\tan x}{x}\right)' = \frac{x-\sin x\cdot\cos x}{x^2\cos^2 x} = \frac{(x-\sin x)+\sin x\,(1-\cos x)}{x^2\cos^2 x} > 0$$

Then, because n-tiples $\left(x_1^2, \ldots, x_n^2\right)$ and $\left(\dfrac{\tan^2 x_1}{x_1^2}, \ldots, \dfrac{\tan^2 x_n}{x_n^2}\right)$ agreed in order, that is $sign\left(x_i^2-x_j^2\right) = sign\left(\dfrac{\tan^2 x_i}{x_i^2}-\dfrac{\tan^2 x_j}{x_j^2}\right)$ for any $i, j \in \{1, 2, \ldots, n\}$, by Chebishev's Inequality

$$\sum_{k=1}^{n}\tan^2 x_k = \sum_{k=1}^{n} x_k^2\cdot\frac{\tan^2 x_k}{x_k^2} \geq \sum_{k=1}^{n} x_k^2\cdot\left(\frac{1}{n}\sum_{k=1}^{n}\frac{\tan^2 x_k}{x_k^2}\right).$$

Also, by Quadratic Mean-Arithmetic Mean Inequality

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\tan^2 x_k}{x_k^2} \geq \left(\frac{1}{n}\sum_{k=1}^{n}\frac{\tan x_k}{x_k}\right)^2.$$

Thus,

$$\frac{\sum_{k=1}^{n}\tan^2 x_k}{\sum\limits_{k=1}^{n} x_k^2} \geq \frac{\sum_{k=1}^{n} x_k^2\cdot\left(\frac{1}{n}\sum_{k=1}^{n}\frac{\tan^2 x_k}{x_k^2}\right)}{\sum_{k=1}^{n} x_k^2} \geq \frac{1}{n^2}\left(\sum_{k=1}^{n}\frac{\tan x_k}{x_k}\right)^2.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Albert Stadler, Herrliberg, Switzerland.

O420. Let $n \geq 2$ and let $A = \{1, 4, \ldots, n^2\}$ be the set of the first n nonzero perfect squares. A subset B of A is called *Sidon* if whenever $a + b = c + d$ for $a, b, c, d \in B$, we have $\{a, b\} = \{c, d\}$. Prove that A contains a *Sidon* subset of size at least $Cn^{1/2}$ for some absolute constant $C > 0$.

Can the exponent $1/2$ be improved?

Proposed by Cosmin Pohoata, Caltech, USA

Solution by the author

To show that $A = \left\{1, 4, \ldots, n^2\right\}$ contains a Sidon subset of size at least $Cn^{1/2}$ for some absolute constant $C > 0$, we invoke a theorem from

V. S. Konyagin, An estimate of L1−norm of an exponential sum, The Theory of Approximations of Functions and Operators. Abstracts of Papers of the International Conference Dedicated to Stechkin's 80th Anniversay [in Russian]. Ekaterinburg (2000), 88–89.

If E be the number of solutions to $a^2 + b^2 = c^2 + d^2$ with a, b, c, d all in A and such that $\{a, b\} \neq \{c, d\}$, Konyagin's theorem says that $E \leq cn^{5/2}$ for an absolute constant $c > 0$. Using this result, we can argue as follows. Let $p \in [0, 1]$ and let A' be a random subset of A obtained by considering each element of A with probability p (independently). Such a random set A' may have solutions to $a^2 + b^2 = c^2 + d^2$ with a, b, c, d all in A and such that $\{a, b\} \neq \{c, d\}$, but it can't have too many. In particular, from Konyagin's theorem A' has at most $cn^{5/2}$ such 4-tuples. A 4-tuple may be so that all a, b, c, d are pairwise distinct or $a = c$, in which case $b = -d$ (or vice versa), or $a = b$, in which case c, d satisfy $c^2 + d^2 = 2a^2$. In the latter case, c and d may be distinct or $c = d$, in which case $c = -\sqrt{2}a$. The number of distinct triplets c, d, a so that $c^2 + d^2 = 2a^2$ is upper bounded by $n^2/\sqrt{\log |A|}$ from a classical theorem of Landau, and the number of the other solutions solutions is at most dn for some constant d, which is negligible for the argument to follow.

With this information, we define set A'' by taking the random A' and removing one element from each 4-tuple for which $a^2 + b^2 = c^2 + d^2$. By construction, this set A'' won't have any solutions to $a^2 + b^2 = c^2 + d^2$. The expected size of A'' is equal to

$$E[|A''|] \gg np - n^{5/2}p^4 - \frac{n^2}{\sqrt{\log n}}p^3.$$

Taking $p \, \frac{1}{100}n^{-1/2}$, we get that $E[|A''|] \gg Cn^{1/2}$ for some absolute constant $C > 0$. By pidgeonhole, this means that there must be some Sidon subset A'' of size at least $Cn^{1/2}$. One can improve on this result, if we use a better bound on E. Such bounds are available in the recent literature.

Note that if we avoid Konyagin's theorem, then one can show the existence of a Sidon subset of size $Cn^{1/3}$, but using Landau's theorem to bound E by $n^3/\sqrt{\log n}$. It is also worth mentioning that any set of n real numbers contains a Sidon subset of size $Cn^{1/2}$ for some absolute constant $C > 0$ (not just the set of the first n perfect squares), but the proof is a bit more complicated.