Junior Problems

J421. Let a and b be positive real numbers. Prove that

$$\frac{6ab-b^2}{8a^2+b^2}<\sqrt{\frac{a}{b}}.$$

Proposed by Adrian Andreescu, Dallas, USA

J422. Let ABC be an acute triangle and let M be the midpoint of BC. The circle of diameter AM intersects the sides BC, AC, AB in X, Y, Z, respectively. Let U be that point on the side AC such that $MU = MC$. The lines BU and AX intersect in T and the lines CT and AB intersect in R. Prove that $MB = MR$.

Proposed by Mihaela Berindeanu, Bucharest, Romania

J423. (a) Prove that for any real numbers a, b, c

$$a^2+(2-\sqrt{2})b^2+c^2\geq\sqrt{2}(ab-bc+ca).$$

(b) Find the best constant k such that for all real numbers a, b, c,

$$a^2+kb^2+c^2\geq\sqrt{2}(ab+bc+ca).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J424. Let ABC be a triangle, D be the foot of the altitude from A and E and F be points on the segments AD BC, respectively, such that

$$\frac{AE}{DE}=\frac{BF}{CF}.$$

Let G be the foot of the perpendicular from B to AF. Prove that EF is tangent to the circumcircle of triangle CFG.

Proposed by Marius Stănean, Zalău, Romania

J425. Prove that for any positive real numbers a, b, c

$$(\sqrt{3}-1)\sqrt{ab+bc+ca}+3\sqrt{\frac{abc}{a+b+c}}\leq a+b+c.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J426. Find all 4-tuples (x, y, z, t) of positive integers which satisfy the equation:

$$xyz+yzt+ztx+txy=xyzt+3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Senior Problems

S421. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{a^2}{\sqrt{1+a}} + \frac{b^2}{\sqrt{1+b}} + \frac{c^2}{\sqrt{1+c}} \geq 2.$$

Proposed by Constantinos Metaxas, Athens, Greece

S422. Solve in positive integers the equation

$$u^2 + v^2 + x^2 + y^2 + z^2 = uv + vx - xy + yz + zu + 3.$$

Proposed by Adrian Andreescu, Dallas, USA

S423. Let $0 \leq a, b, c \leq 1$. Prove that

$$(a+b+c+2)\left(\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca}\right) \leq 10.$$

Proposed by An Zenping, Xianyang Normal University, China

S424. Let p and q be prime numbers such that p^2+pq+q^2 is a perfect square. Prove that p^2-pq+q^2 is prime.

Proposed by Alessandro Ventullo, Milan, Italy

S425. Let a, b, c be positive real numbers. Prove that

$$\sqrt{a^2-ab+b^2} + \sqrt{b^2-bc+c^2} + \sqrt{c^2-ca+a^2} \leq \sqrt{(a+b+c)\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right)}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S426. Prove that in any triangle ABC the following inequality holds:

$$\frac{r_a}{\sin\frac{A}{2}} + \frac{r_b}{\sin\frac{B}{2}} + \frac{r_c}{\sin\frac{C}{2}} \geq 2\sqrt{3}s.$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

U421. Find all pairs a and b of distinct positive integers for which there is a polynomial P with integer coefficients such that

$$P(a^3) + 7(a + b^2) = P(b^3) + 7(b + a^2).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U422. Let a and b be complex numbers and let $(a_n)_{n\geq 0}$ be the sequence defined by $a_0 = 2$, $a_1 = a$ and

$$a_n = aa_{n-1} + ba_{n-2},$$

for $n \geq 2$. Write a_n as a polynomial in a and b.

Proposed by Dorin Andrica and Grigore Călugăreanu, Romania

U423. Find the maximum and minimum of

$$f(x) = \sqrt{\sin^4 x + \cos^2 x + 1} + \sqrt{\cos^4 x + \sin^2 x + 1}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U424. Let a be a real number such that $|a| > 2$. Prove that if $a^4 - 4a^2 + 2$ and $a^5 - 5a^3 + 5a$ are rational numbers, then a is a rational number as well.

Proposed by Mircea Becheanu, University of Bucharest, Romania

U425. Let p be a prime number and let G be a group of order p^3. Define $\Gamma(G)$ the graph whose vertices are the noncentral conjugacy class sizes of G and two vertices are joined if and only if the two associated conjugacy class sizes are not coprime. Determine the structure of $\Gamma(G)$.

Proposed by Alessandro Ventullo, Milan, Italy

U426. Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a continuous function and let $(x_n)_{n\geq 1}$ be the sequence defined by

$$x_n = \sum_{k=0}^{n} \cos\left(\frac{1}{\sqrt{n}} f\left(\frac{k}{n}\right)\right) - \alpha n^{\beta},$$

where α and β are real numbers. Evaluate $\lim_{n\to\infty} x_n$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

O421. Prove that for any real numbers a, b, c, d,

$$a^2 + b^2 + c^2 + d^2 + \sqrt{5}\,\min\{a^2, b^2, c^2, d^2\} \geq (\sqrt{5} - 1)(ab + bc + cd + da).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O422. Let $P(x)$ be a polynomial with integer coefficients which has an integer root. Prove that if p and q are distinct odd primes such that $P(p) = p < 2q - 1$ and $P(q) = q < 2p - 1$, then p and q are twin primes.

Proposed by Alessandro Ventullo, Milan, Italy

O423. Prove that in any triangle ABC,

$$\sqrt{\frac{1}{r_b^2} + \frac{1}{r_c} + 1} + \sqrt{\frac{1}{r_c^2} + \frac{1}{r_b} + 1} \geq 2\sqrt{\frac{1}{h_a^2} + \frac{1}{h_a} + 1}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O424. For a positive integer n, we define $f(n)$ to be the number of $2's$ that appear (as digits) after writing the numbers $1, 2, \ldots, n$ in their decimal expansion. For example, $f(22) = 6$ because 2 appears once in the numbers $2, 12, 20, 21$ and it appears twice in the number 22. Prove that there are finitely many numbers n such that $f(n) = n$.

Proposed by Enrique Trevinio, Lake Forest College, USA

O425. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + abc = 4$ and let k be a nonnegative real number. Prove that

$$a + b + c + \sqrt{k\left(k - 1 + \frac{a^2 + b^2 + c^2}{3}\right)} \leq k + 3.$$

Proposed by Marius Stănean, Zalău, Romania

O426. Let a, b, c be positive numbers such that

$$\frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} = 1.$$

Prove that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{a+b+c}{2}\ .$$

Proposed by An Zhenping, Xianyang Normal University, China