# A Generalization of the Rearrangement Inequality

Jan Holstermann

**Abstract**

In this article we present a generalization of the rearrangement inequality and show its applications by solving a USAJMO problem. The classical rearrangement inequality deals with two sequences $a_i, b_i, i = 1, 2, ..., n$, which are both sorted in ascending order. It states that $\sum_{i=1}^{n} b_i a_{n-i+1} \leqslant \sum_{i=1}^{n} b_i a_{\sigma(i)} \leqslant \sum_{i=1}^{n} b_i a_i$ for any permutation $\sigma$ of the numbers $1, 2, ..., n$. In the generalization the $b_i a_j$ terms are replaced by $f_i(a_j)$.

## 1 Main Result

To start off we will prove the version with only two summands on each side.

**Lemma 1.** *Let $a_1 \leqslant a_2$ be real numbers and $f_i : [a_1, a_2] \to \mathbb{R}, i = 1, 2$ be functions, such that $f_1'(x) \leqslant f_2'(x)$ for all $x \in [a_1, a_2]$. We have*

$$f_1(a_2) + f_2(a_1) \leqslant f_1(a_1) + f_2(a_2) \tag{1}$$

*Proof.* We have

$$\begin{aligned} & f_1'(x) \leqslant f_2'(x) \\ \Rightarrow & \int_{a_1}^{a_2} f_1'(x)dx \leqslant \int_{a_1}^{a_2} f_2'(x)dx \\ \Leftrightarrow & f_1(a_2) - f_1(a_1) \leqslant f_2(a_2) - f_2(a_1) \\ \Leftrightarrow & f_1(a_2) + f_2(a_1) \leqslant f_1(a_1) + f_2(a_2). \end{aligned}$$

□

Let's see our lemma in action with the first examples.

**Example 1.** Let $a, b$ be real numbers, such that $a \leqslant b$. Prove that

$$(a+1)^2 + 2e^b \geqslant (b+1)^2 + 2e^a.$$

*Solution.* Define $f_1(x) = (x+1)^2$ and $f_2(x) = 2e^x$. Note that

$$f_1'(x) = 2(x+1) \leqslant 2e^x = f_2'(x).$$

Thus, we can conclude that

$$f_1(a) + f_2(b) \geqslant f_1(b) + f_2(a),$$

which is exactly the inequality we wanted to prove.

**Example 2.** (AM-GM) Let $a, b$ be nonnegative real numbers. Prove that

$$\frac{a+b}{2} \geqslant \sqrt{ab}.$$

*Solution.* Let $a \leqslant b, f_1(x) = \frac{x+b}{2}$ and $f_2(x) = \sqrt{xb}$. For $x \in [a, b]$ we have

$$f_1'(x) = \frac{1}{2} \leqslant \frac{\sqrt{\frac{b}{x}}}{2} = f_2'(x).$$

Therefore, our lemma asserts that

$$\frac{a+b}{2} - \sqrt{ab} = f_1(a) - f_2(a) \geqslant f_1(b) - f_2(b) = b - b = 0,$$

which finishes the proof.

Now, we are prepared to attack the main result.

**Definition 1.** *Let $\sigma$ and $\pi$ be permutations of the numbers $1, 2, ..., n$. We say that $\sigma$ strongly minorates $\pi$ if for any $k \in \{1, 2, ..., n\}$ the following condition holds: For any $i \in \{1, 2, ..., n-k+1\}$ the $i$th largest term of $\sigma(k), \sigma(k+1), ..., \sigma(n)$ is less than or equal to the $i$th largest term of $\pi(k), \pi(k+1), ..., \pi(n)$.*

**Theorem 1.** *Let $a_1 \leqslant a_2 \leqslant ... \leqslant a_n$ be real numbers and $f_i : [a_1, a_n] \to \mathbb{R}, i = 1, 2, ..., n$ be functions, such that $f_1'(x) \leqslant f_2'(x) \leqslant ... \leqslant f_n'(x)$ for all $x \in [a_1, a_n]$. For the permutations $\sigma, \pi$ of the numbers $1, 2, ..., n$, such that $\sigma$ strongly minorates $\pi$, we have*

$$\sum_{i=1}^{n} f_i(a_{\sigma(i)}) \leqslant \sum_{i=1}^{n} f_i(a_{\pi(i)}). \tag{2}$$

*Proof.* Let $d_i(j) = f_i(a_j) - f_{i-1}(a_j)$. Our lemma translates to $d_i(j) \leqslant d_i(k)$ for any $j \leqslant k$. Thus, we have

$$\begin{aligned}
\sum_{i=1}^{n} f_i(a_{\sigma(i)}) &= \sum_{i=1}^{n} f_1(a_{\sigma(i)}) + d_2(\sigma(i)) + d_3(\sigma(i)) + ... + d_i(\sigma(i)) \\
&= \sum_{i=1}^{n} f_1(a_{\sigma(i)}) + \sum_{j=2}^{n} d_j(\sigma(j)) + d_j(\sigma(j+1)) + ... + d_j(\sigma(n)) \\
&\leqslant \sum_{i=1}^{n} f_1(a_{\pi(i)}) + \sum_{j=2}^{n} d_j(\pi(j)) + d_j(\pi(j+1)) + ... + d_j(\pi(n)) \\
&= \sum_{i=1}^{n} f_i(a_{\pi(i)}).
\end{aligned}$$

$\square$

**Corollary 1.** *Let $a_1 \leqslant a_2 \leqslant ... \leqslant a_n$ be real numbers and $f_i : [a_1, a_n] \to \mathbb{R}, i = 1, 2, ..., n$ be functions, such that $f_1'(x) \leqslant f_2'(x) \leqslant ... \leqslant f_n'(x)$ for all $x \in [a_1, a_n]$. For any permutation $\sigma$ of the numbers $1, 2, ..., n$ we have*

$$\sum_{i=1}^{n} f_i(a_{n-i+1}) \leqslant \sum_{i=1}^{n} f_i(a_{\sigma(i)}) \leqslant \sum_{i=1}^{n} f_i(a_i). \tag{3}$$

*Proof.* For the upper bound let $\pi(x) = x$. For the lower bound let $\sigma(x) = n - x + 1$. □

**Remark.** Setting $f_i(x) = b_i x$ we get the classical Rearrangement inequality.

Let's do a real life problem.

**Example 3.** (USAJMO 2012) Let $a, b, c$ be positive real numbers. Prove that

$$\frac{a^3 + 3b^3}{5a + b} + \frac{b^3 + 3c^3}{5b + c} + \frac{c^3 + 3a^3}{5c + a} \geqslant \frac{2}{3}(a^2 + b^2 + c^2).$$

*Solution.* WLOG let $a = min\{a, b, c\}$. Consider the case $b \leqslant c$. Define $f_t(x) = \frac{t^3+3x^3}{5t+x}$. Note that

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}\frac{t^3 + 3x^3}{5t + x} = -\frac{5t^3 + 3t^2x + 225tx^2 + 15x^3}{(5t + x)^3} \leqslant 0.$$

We conclude that $f_t'(x)$ is monotonically decreasing in $t$, so $f_a'(x) \geqslant f_b'(x) \geqslant f_c'(x)$. Thus, we have

$$\begin{aligned} & f_a'(b) + f_b'(c) + f_c'(a) \geqslant f_a'(a) + f_b'(b) + f_c'(c) \\ \Leftrightarrow \; & \frac{a^3 + 3b^3}{5a + b} + \frac{b^3 + 3c^3}{5b + c} + \frac{c^3 + 3a^3}{5c + a} \geqslant \frac{a^3 + 3a^3}{5a + a} + \frac{b^3 + 3b^3}{5b + b} + \frac{c^3 + 3c^3}{5c + c} \\ & = \frac{2}{3}(a^2 + b^2 + c^2). \end{aligned}$$

The other case is analogous.

**Example 4.** (Special Case of Karamata) Let $f$ be a convex, differentiable function and $a, b, d$ be real numbers, such that $0 \leqslant d \leqslant \frac{b-a}{2}$. We have

$$f(a) + f(b) \geqslant f(a + d) + f(b - d). \tag{4}$$

*Solution.* Define $f_1(x) = f\left(a + \frac{d}{2} + x\right)$ and $f_2(x) = f\left(b - \frac{d}{2} + x\right)$. Since $f$ is convex its derivative is monotonically increasing. As $a + \frac{d}{2} \leqslant b - \frac{d}{2}$, we can conclude that

$$f'\left(a + \frac{d}{2} + x\right) \leqslant f'\left(b - \frac{d}{2} + x\right) \Leftrightarrow f_1'(x) \leqslant f_2'(x).$$

Thus, we have

$$f_1\left(-\frac{d}{2}\right) + f_2\left(\frac{d}{2}\right) \geqslant f_1\left(\frac{d}{2}\right) + f_2\left(-\frac{d}{2}\right) \Leftrightarrow f(a) + f(b) \geqslant f(a + d) + f(b - d).$$

## 2 Exercises

*Disclaimer:* Although all problems can be solved by rearranging, there are also other ways to tackle these problems.

**Problem 1.** Let $a, b, c$ be positive and $m, n$ be nonnegative real numbers. Prove that

$$\frac{a^{m+2}}{na+b}+\frac{b^{m+2}}{nb+c}+\frac{c^{m+2}}{nc+a} \geqslant \frac{a^{m+1}+b^{m+1}+c^{m+1}}{n+1}.$$

**Problem 2.** (Turkey JBMO TST 2013) Let $a, b, c$ be positive real numbers, such that $a+b+c=1$. Prove that

$$\frac{a^4+5b^4}{a(a+2b)}+\frac{b^4+5c^4}{b(b+2c)}+\frac{c^4+5a^4}{c(c+2a)} \geqslant 1-ab-bc-ca.$$

**Problem 3.** (Korea National Olympiad 2013) Let $a, b, c$ be positive real numbers, such that $ab+bc+ca=3$. Prove that

$$\frac{(a+b)^3}{\sqrt[3]{2(a+b)(a^2+b^2)}}+\frac{(b+c)^3}{\sqrt[3]{2(b+c)(b^2+c^2)}}+\frac{(c+a)^3}{\sqrt[3]{2(c+a)(c^2+a^2)}} \geqslant 12.$$

**Problem 4.** Let $f : \mathbb{R} \to \mathbb{R}$ be a convex, differentiable function, such that $f(t) \geqslant t\ \forall t \in \mathbb{R}$. Prove that for all real numbers $x$ we have

$$f(x)+f(f(f(x))) \geqslant 2f(f(x)).$$