Junior problems

J421. Let a and b be positive real numbers. Prove that

$$\frac{6ab-b^2}{8a^2+b^2}<\sqrt{\frac{a}{b}}.$$

Proposed by Adrian Andreescu, Dallas, USA

Solution by Polyahedra, Polk State College, USA

By the AM-GM inequality,

$$8a^2\sqrt{a}+b^2\sqrt{a}+b^2\sqrt{b}\geq 3\left(8a^2\sqrt{a}\cdot b^2\sqrt{a}\cdot b^2\sqrt{b}\right)^{1/3}=6ab\sqrt{b}.$$

Equality holds if and only if $8a^2\sqrt{a}=b^2\sqrt{a}=b^2\sqrt{b}$, which implies that $a=b=0$. Hence, for $a,b>0$, $\left(8a^2+b^2\right)\sqrt{a}>\left(6ab-b^2\right)\sqrt{b}$, completing the proof.

Also solved by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Alessandro Ventullo, Milan, Italy; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Adnan Ali, NIT Silchar, Assam, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nguyen Ngoc Tu, Ha Giang, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Robert Bosch, USA and Jorge Erick, Brazil; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Elaine Moon, Peddie School, Hightstown, NJ, USA; Jio Jeong, Seoul International School, South Korea; Joehyun Kim, Fort Lee High School, NJ, USA; Jiwon Park, St. Andrew's School, Middletown, DE, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Erica Choi, Blair Academy, Blairstown, NJ, USA; Nikos Kalapodis, Patras, Greece; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy.

J422. Let ABC be an acute triangle and let M be the midpoint of BC. The circle of diameter AM intersects the sides BC, AC, AB in X, Y, Z, respectively. Let U be that point on the side AC such that $MU = MC$. The lines BU and AX intersect in T and the lines CT and AB intersect in R. Prove that $MB = MR$.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution by Daniel Lasaosa, Pamplona, Spain

Since AM is diameter, $\angle AXM = 90°$, and X is the foot of the altitude from A onto BC. Since M is the midpoint of BC, U is on the circle with diameter BC, or $\angle BUC = 90°$, and U is the foot of the altitude from B onto CA. Then, T is the orthocenter where the altitudes BU and AX mean, or $\angle BRC = 90°$, and R is on the circle with diameter BC and center M. The conclusion follows.

Also solved by Elaine Moon, Peddie School, Hightstown, NJ, USA; Jio Jeong, Seoul International School, South Korea; Joehyun Kim, Fort Lee High School, NJ, USA; Jiwon Park, St. Andrew's School, Middletown, DE, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Polyahedra, Polk State College, USA; Erica Choi, Blair Academy, Blairstown, NJ, USA; Nikos Kalapodis, Patras, Greece; Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Nguyen Ngoc Tu, Ha Giang, Vietnam; Robert Bosch, USA; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Titu Zvonaru, Comănești, Romania.

J423. (a) Prove that for any real numbers a, b, c

$$a^2 + \left(2 - \sqrt{2}\right) b^2 + c^2 \geq \sqrt{2}(ab - bc + ca).$$

(b) Find the best constant k such that for all real numbers a, b, c,

$$a^2 + kb^2 + c^2 \geq \sqrt{2}(ab + bc + ca).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Robert Bosch, USA and Jorge Erick, Brazil

(a) The inequality is

$$(2 - \sqrt{2})b^2 - \sqrt{2}(a - c)b + a^2 + c^2 - \sqrt{2}ca \geq 0.$$

The left side can be considered as a quadratic equation in b. Clearly, the leading coefficient is positive, so it is sufficient to prove the discriminant Δ is nonpositive.

$$\begin{aligned} \Delta &= 2(a - c)^2 - 4(2 - \sqrt{2})(a^2 + c^2 - \sqrt{2}ac) \leq 0, \\ &\Leftrightarrow (6 - 4\sqrt{2})a^2 + (6 - 4\sqrt{2})c^2 + (12 - 8\sqrt{2})ac \geq 0, \\ &\Leftrightarrow (a + c)^2 \geq 0. \end{aligned}$$

(b) $k = 2 + \sqrt{2}$. The proof is by the sum of squares. The inequality to be proved is equivalent to

$$\frac{\sqrt{2}}{2}(a - c)^2 + \frac{\sqrt{2} - 1}{\sqrt{2}}\left(a - (1 + \sqrt{2})b\right)^2 + \frac{\sqrt{2} - 1}{\sqrt{2}}\left(c - (1 + \sqrt{2})b\right)^2 + \left(k - (2 + \sqrt{2})\right)b^2 \geq 0,$$

clearly true, the constant $k = 2 + \sqrt{2}$ is optimal considering $b = 1$ and $a = c = 1 + \sqrt{2}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Elaine Moon, Peddie School, Hightstown, NJ, USA; Jio Jeong, Seoul International School, South Korea; Joehyun Kim, Fort Lee High School, NJ, USA; Jiwon Park, St. Andrew's School, Middletown, DE, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Polyahedra, Polk State College, USA; Erica Choi, Blair Academy, Blairstown, NJ, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Adnan Ali, NIT Silchar, Assam, India; Arkady Alt, San Jose, CA, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

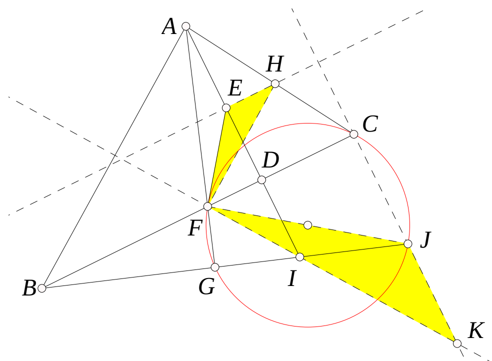
J424. Let ABC be a triangle, D be the foot of the altitude from A and E and F be points on the segments AD BC, respectively, such that

$$\frac{AE}{DE}=\frac{BF}{CF}.$$

Let G be the foot of the perpendicular from B to AF. Prove that EF is tangent to the circumcircle of triangle CFG.

Proposed by Marius Stănean, Zalău, Romania

Solution by Polyahedra, Polk State College, USA



Locate H on AC such that $EHallelBC$. Then $\frac{BF}{FC}=\frac{AE}{ED}=\frac{AH}{HC}$, so $HFallelAB$. Extend AD to intersect BG at I. Since F is the orthocenter of $\triangle ABI$, $IF\perp HF$. Suppose that the line through C and parallel to AD intersects BI and FI at J and K, respectively. Then $\angle JKI=\angle EHF=\angle FBA$. Also, $\angle KIJ=\angle FIB=\angle BAF$, thus $\triangle IKJ\sim\triangle ABF$. Hence,

$$\begin{aligned}\frac{FK}{JK} &= \frac{FK}{IK}\cdot\frac{IK}{JK}=\frac{FC}{DC}\cdot\frac{AB}{FB}=\frac{FC}{DC}\cdot\frac{BA}{FH}\cdot\frac{FH}{BF}\\ &= \frac{FC}{DC}\cdot\frac{BC}{FC}\cdot\frac{FH}{BF}=\frac{FH}{DC}\cdot\frac{AC}{AH}=\frac{FH}{DC}\cdot\frac{DC}{EH}=\frac{FH}{EH}.\end{aligned}$$

Therefore, $\triangle FKJ\sim\triangle FHE$. Consequently, $\angle KFJ=\angle HFE$, so $JF\perp EF$. This completes the proof, since the circumcenter of $\triangle CFG$ is the midpoint of FJ.

Also solved by Daniel Lasaosa, Pamplona, Spain; Elaine Moon, Peddie School, Hightstown, NJ, USA; Jio Jeong, Seoul International School, South Korea; Joehyun Kim, Fort Lee High School, NJ, USA; Jiwon Park, St. Andrew's School, Middletown, DE, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Robert Bosch, USA.

J425. Prove that for any positive real numbers a, b, c

$$\left(\sqrt{3}-1\right)\sqrt{ab+bc+ca}+3\sqrt{\frac{abc}{a+b+c}} \le a+b+c.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia
Using AM-GM inequlaity, we get

$$\begin{aligned}\sqrt{3(ab+bc+ca)} &\le \sqrt{2(ab+bc+ca)+\frac{a^2+b^2}{2}+\frac{b^2+c^2}{2}+\frac{c^2+a^2}{2}}\\ &= \sqrt{(a+b+c)^2} = a+b+c\end{aligned}$$

$$\begin{aligned}(a+b+c)(ab+bc+ca) \ge 3\sqrt[3]{abc}\cdot 3\sqrt[3]{(abc)^2} &= 9abc\\ \Leftrightarrow\ \frac{9abc}{a+b+c} &\le ab+bc+ca\\ \Leftrightarrow\ 3\sqrt{\frac{abc}{a+b+c}} &\le \sqrt{ab+bc+ca}\\ \Leftrightarrow\ 3\sqrt{\frac{abc}{a+b+c}}-\sqrt{ab+bc+ca} &\le 0\end{aligned}$$

and the conclusion follows.

Also solved by Daniel Lasaosa, Pamplona, Spain; Arkady Alt, San Jose, CA, USA; Elaine Moon, Peddie School, Hightstown, NJ, USA; Jio Jeong, Seoul International School, South Korea; Joehyun Kim, Fort Lee High School, NJ, USA; Jiwon Park, St. Andrew's School, Middletown, DE, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Erica Choi, Blair Academy, Blairstown, NJ, USA; Polyahedra, Polk State College, USA; Nikos Kalapodis, Patras, Greece; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Adnan Ali, NIT Silchar, Assam, India; Nikolaos Evgenidis, Aristotle University of Thessaloniki, Greece; Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA; Jamal Gadirov, Istanbul University, Istanbul; Kevin Soto Palacios Huarmey, Perú; Nguyen Ngoc Tu, Ha Giang, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Robert Bosch, USA; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Prajnanaswaroopa S, Bangalore, India.

J426. Find all 4-tuples (x, y, z, t) of positive integers which satisfy the equation:

$$xyz + yzt + ztx + txy = xyzt + 3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Polyahedra, Polk State College, USA

Assume that $x \geq y \geq z \geq t \geq 1$. Then $xyzt + 3 \leq 4xyz$, thus $t \in \{1, 2, 3\}$.

If $t = 1$, then $3 = xy + yz + zx$, so $x = y = z = 1$.

Next, consider $t = 2$. Then $xyz + 3 = 2(xy + yz + zx) \leq 6xy$, so $z \in \{2, 3, 4, 5\}$. If $z = 2$, then $3 = 4(x + y)$, an impossibility. If $z = 3$, then $xy + 3 = 6(x + y) \leq 12x$, so $x = \frac{3(2y-1)}{y-6}$ and $y \in \{7, 8, 9, 10, 11\}$, which lead to the solutions $(y, x) = (7, 39)$ and $(9, 17)$. If $z = 4$, then $2xy + 3 = 8(x + y)$, an impossibility since 3 is odd. If $z = 5$, then $3xy + 3 = 10(x + y) \leq 20x$, so $x = \frac{10y-3}{3y-10}$ and $y \in \{5, 6\}$, which lead to no solution.

Finally, consider $t = 3$. Then $2xyz + 3 = 3(xy + yz + zx) \leq 9xy$, so $z \in \{3, 4\}$. If $z = 3$, then $xy + 1 = 3(x + y) \leq 6x$, so $x = \frac{3y-1}{y-3}$ and $y \in \{4, 5\}$, which lead to the solutions $(y, x) = (4, 11)$ and $(5, 7)$. If $z = 4$, then $5xy + 3 = 12(x + y) \leq 24x$, so $y = 4$, which leads to no solution.

In conclusion, the solutions for (x, y, z, t) are all the permutations of $(1, 1, 1, 1)$, $(2, 3, 7, 39)$, $(2, 3, 9, 17)$, $(3, 3, 4, 11)$, and $(3, 3, 5, 7)$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Joehyun Kim, Fort Lee High School, NJ, USA; Jiwon Park, St. Andrew's School, Middletown, DE, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Erica Choi, Blair Academy, Blairstown, NJ, USA; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Adnan Ali, NIT Silchar, Assam, India; Joel Schlosberg, Bayside, NY, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

Senior problems

S421. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{a^2}{\sqrt{1+a}} + \frac{b^2}{\sqrt{1+b}} + \frac{c^2}{\sqrt{1+c}} \geq 2.$$

Proposed by Constantinos Metaxas, Athens, Greece

Solution by Alessandro Ventullo, Milan, Italy

We prove the stronger inequality

$$\frac{a^2}{\sqrt{1+a}} + \frac{b^2}{\sqrt{1+b}} + \frac{c^2}{\sqrt{1+c}} \geq \frac{3}{\sqrt{2}}.$$

Let $f(x) = \dfrac{x^2}{\sqrt{1+x}}$. Since $f''(x) = \dfrac{3x^2+8x+8}{4(x+1)^2\sqrt{x+1}}$, then $f''(x) > 0$ for all $x > 0$, so f is convex on $(0, +\infty)$. By Jensen's Inequality, we have

$$f\left(\frac{a+b+c}{3}\right) \leq \frac{f(a)+f(b)+f(c)}{3},$$

i.e.

$$\frac{a^2}{\sqrt{1+a}} + \frac{b^2}{\sqrt{1+b}} + \frac{c^2}{\sqrt{1+c}} \geq 3 \cdot \frac{\left(\frac{a+b+c}{3}\right)^2}{\sqrt{1+\frac{a+b+c}{3}}} = \frac{(a+b+c)^2}{\sqrt{9+3(a+b+c)}}.$$

By the AM-GM Inequality, we have $a+b+c \geq 3\sqrt[3]{abc} = 3$. Set $x = a+b+c$. Observe that the function $g(x) = \dfrac{x^2}{\sqrt{9+3x}}$ is increasing on $[3, +\infty)$, so $g(x) \geq g(3) = \dfrac{3}{\sqrt{2}}$ and the conclusion follows.

Also solved by Daniel Lasaosa, Pamplona, Spain; Joehyun Kim, Fort Lee High School, NJ, USA; Jiwon Park, St. Andrew's School, Middletown, DE, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Erica Choi, Blair Academy, Blairstown, NJ, USA; Nikos Kalapodis, Patras, Greece; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Adnan Ali, NIT Silchar, Assam, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Paraskevi-Andrianna Maroutsou, Charters Sixth Form, Sunningdale, England, UK; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Robert Bosch, USA; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

S422. Solve in positive integers the equation

$$u^2 + v^2 + x^2 + y^2 + z^2 = uv + vx - xy + yz + zu + 3.$$

Proposed by Adrian Andreescu, Dallas, USA

Solution by Alessandro Ventullo, Milan, Italy

The given equation can be written as

$$(u-v)^2 + (v-x)^2 + (x+y)^2 + (y-z)^2 + (z-u)^2 = 6.$$

Since u, v, x, y, z are positive integers, then $x + y \geq 2$, which gives $(x+y)^2 \geq 4$. Since $(x+y)^2 \leq 6$, we conclude that $x + y = 2$, so $x = y = 1$ and

$$(u-v)^2 + (v-1)^2 + (1-z)^2 + (z-u)^2 = 2.$$

So, exactly two of the summands on the LHS are equal to 1 and the other are equal to 0. We have six cases.

(i) $(u-v)^2 = (v-1)^2 = 1$ and $(1-z)^2 = (z-u)^2 = 0$. From the last equations we get $u = z = 1$ and from the first equations we get $v = 2$.

(ii) $(u-v)^2 = (1-z)^2 = 1$ and $(v-1)^2 = (z-u)^2 = 0$. From the last equations we get $v = 1$ and $u = z$ and from the first equations we get $u = z = 2$.

(iii) $(u-v)^2 = (z-u)^2 = 1$ and $(v-1)^2 = (1-z)^2 = 0$. From the last equations we get $v = z = 1$ and from the first equations we get $u = 2$.

(iv) $(v-1)^2 = (1-z)^2 = 1$ and $(u-v)^2 = (z-u)^2 = 0$. From the last equations we get $u = v = z$ and from the first equations we get $u = v = z = 2$.

(v) $(v-1)^2 = (z-u)^2 = 1$ and $(u-v)^2 = (1-z)^2 = 0$. From the last equations we get $u = v$ and $z = 1$ and from the first equations we get $u = v = 2$.

(vi) $(1-z)^2 = (z-u)^2 = 1$ and $(u-v)^2 = (v-1)^2 = 0$. From the last equations we get $u = v = 1$ and from the first equations we get $z = 2$.

In conclusion,

$$(u, v, x, y, z) \in \{(1,2,1,1,1), (2,1,1,1,2), (2,1,1,1,1), (2,2,1,1,2), (2,2,1,1,1), (1,1,1,1,2)\}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Jio Jeong, Seoul International School, South Korea; Joehyun Kim, Fort Lee High School, NJ, USA; Jiwon Park, St. Andrew's School, Middletown, DE, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Erica Choi, Blair Academy, Blairstown, NJ, USA; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Prajnanaswaroopa S, Bangalore, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Adnan Ali, NIT Silchar, Assam, India; Nikolaos Evgenidis, Aristotle University of Thessaloniki, Greece; Jenny Johannes, College at Brockport SUNY, NY, USA; Paraskevi-Andrianna Maroutsou, Charters Sixth Form, Sunningdale, England, UK; Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

S423. Let $0 \le a, b, c \le 1$. Prove that

$$(a+b+c+2)\left(\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca}\right) \le 10.$$

Proposed by An Zenping, Xianyang Normal University, China

Solution by Robert Bosch, USA

Making the substitution

$$\begin{aligned} a+b+c &= x, \\ ab+bc+ca &= y, \\ abc &= z, \end{aligned}$$

yields

$$(10x-(x+2)x)z+10(1+y)-(x+2)(3+2y) \ge 0.$$

This expression is an increasing linear function of the variable z, so let us try to prove the inequality assuming $z=0$, equivalent to $c=0$ (without loss of generality). Say,

$$(a+b+2)(3+2ab) \le 10(1+ab),$$

or expanding

$$3(a+b)+2ab(a+b) \le 4+6ab.$$

Clearly $(1-a)(1-b) \ge 0$ or $1+ab \ge a+b$. So, now the inequality to be proved becomes

$$2(ab)^2-ab-1 \le 0.$$

Let $ab=\lambda$ and $f(\lambda)=2\lambda^2-\lambda-1$, for $\lambda \in [0,1]$. This parabola is convex, looking at the endpoints we get $f(0)=-1<0$ and $f(1)=0$. Done. We have equality in the original inequality for $a=b=1$ and $c=0$ for example.

Also solved by Joehyun Kim, Fort Lee High School, NJ, USA; Jiwon Park, St. Andrew's School, Middletown, DE, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

S424. Let p and q be prime numbers such that p^2+pq+q^2 is a perfect square. Prove that p^2-pq+q^2 is prime.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Daniel Lasaosa, Pamplona, Spain

If $q=2$ then $(p+1)^2 < p^2+pq+q^2 = p^2+2p+4 < (p+2)^2$, and p^2+pq+q^2 is not a perfect square. If $p=q$ then $p^2+pq+q^2=3p^2$, clearly not a perfect square. It follows that p,q must be distinct odd primes, and wlog by symmetry we may assume $p>q$. Now, denoting $u=p+q$ and denoting by v the positive integer such that $v^2=p^2+pq+q^2$, we have

$$(u-v)(u+v)=u^2-v^2=pq,$$

for either $u+v=pq$ and $u-v=1$, or $u+v=p$ and $u-v=q$. Now, $p+q<u+v=p<p+q$ cannot occur, or $v=pq-p-q=p+q-1$, or

$$3=pq-2p-2q+4=(p-2)(q-2).$$

But since p,q are distinct odd primes, $q\geq 3$ and $p\geq 5$, for $(p-2)(q-2)\geq 3$, and equality must hold. It follows that (p,q) is a permutation of $(3,5)$, and $p^2-pq+q^2=19$ is indeed a prime. The conclusion follows.

Also solved by Constantinos Metaxas, Athens, Greece; Jio Jeong, Seoul International School, South Korea; Joehyun Kim, Fort Lee High School, NJ, USA; Jiwon Park, St. Andrew's School, Middletown, DE, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Erica Choi, Blair Academy, Blairstown, NJ, USA; Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Arkady Alt, San Jose, CA, USA; Albert Stadler, Herrliberg, Switzerland; Adnan Ali, NIT Silchar, Assam, India; Jamal Gadirov, Istanbul University, Istanbul, Turkey; Joel Schlosberg, Bayside, NY, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua; Robert Bosch, USA; Titu Zvonaru, Comănești, Romania.

S425. Let a, b, c be positive real numbers. Prove that

$$\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}\le\sqrt{(a+b+c)\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina

Since $x^2-xy+y^2\ge 0$ for all real x and y, using Cauchy-Schwarz inequality we have

$$\sqrt{(a+b+c)\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)}=\sqrt{(a+b+c)\cdot\left(\frac{a^2}{b}-a+b+\frac{b^2}{c}-b+c+\frac{c^2}{a}-c+a\right)}=$$

$$=\sqrt{(b+c+a)\left(\frac{a^2-ab+b^2}{b}+\frac{b^2-bc+c^2}{c}+\frac{c^2-ca+a^2}{a}\right)}\ge\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}$$

Equality holds only if $a=b=c$.

Also solved by Constantinos Metaxas, Athens, Greece; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Nikos Kalapodis, Patras, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Kevin Soto Palacios Huarmey, Perú; Nguyen Ngoc Tu, Ha Giang, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Robert Bosch, USA.

S426. Prove that in any triangle ABC the following inequality holds:

$$\frac{r_a}{\sin\frac{A}{2}}+\frac{r_b}{\sin\frac{B}{2}}+\frac{r_c}{\sin\frac{C}{2}}\geq 2\sqrt{3}s.$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain
We will prove the desired inequality in the equivalent form

$$\frac{1}{\cos\frac{A}{2}}+\frac{1}{\cos\frac{B}{2}}+\frac{1}{\cos\frac{C}{2}}\geq 2\sqrt{3}$$

using the fact that $\tan\frac{A}{2}=\frac{r_a}{s}$, that is, $\frac{r_a}{\sin\frac{A}{2}}=\frac{s}{\cos\frac{A}{2}}$ and observing that $f(x)=\frac{1}{\cos\frac{x}{2}}$ is a convex function in the interval $(0,\pi)$. The analytic criterion for convexity of a function is that its second derivative is positive. Indeed, $f'(x)=2\sin\frac{x}{2}\sec^2\frac{x}{2}$ and $f''(x)=\frac{1}{4}\left(1+\sin^2\frac{x}{2}\right)\sec^3\frac{x}{2}>0$ for $0<x<\pi$.

Thus, by Jensen's inequality,

$$\frac{1}{\cos\frac{A}{2}}+\frac{1}{\cos\frac{B}{2}}+\frac{1}{\cos\frac{C}{2}}\geq 3\cdot\frac{1}{\cos\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}}=3\cdot\frac{1}{\cos\frac{\pi}{6}}=2\sqrt{3},$$

with equality if and only if $A=B=C$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Joehyun Kim, Fort Lee High School, NJ, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Erica Choi, Blair Academy, Blairstown, NJ, USA; Nikos Kalapodis, Patras, Greece; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Adnan Ali, NIT Silchar, Assam, India; Arkady Alt, San Jose, CA, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nikolaos Evgenidis, Aristotle University of Thessaloniki, Greece; Kevin Soto Palacios Huarmey, Perú; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Robert Bosch, USA; Titu Zvonaru, Comănești, Romania.

Undergraduate problems

U421. Find all pairs a and b of distinct positive integers for which there is a polynomial P with integer coefficients such that

$$P(a^3) + 7(a + b^2) = P(b^3) + 7(b + a^2).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

$$\begin{aligned} & P(a^3) + 7(a + b^2) = P(b^3) + 7(b + a^2) \\ & \quad \Rightarrow \begin{cases} P(a^3) - P(b^3) = 7(b + a^2 - a - b^2) = 7(a - b)(a + b - 1) \\ a^3 - b^3 | P(a^3) - P(b^3) \end{cases} \\ & \quad \Rightarrow a^3 - b^3 | 7(a - b)(a + b - 1), \ a \neq b \\ & \quad \Rightarrow a^2 + ab + b^2 | 7(a + b - 1). \end{aligned}$$

We can assume that $a > b$.

$$7(a + b - 1) \geq a^2 + ab + b^2 = (a + b - 1)(a + 1) + b^2 - b + 1 > (a + b - 1)(a + 1)$$
$$\Rightarrow \ 7 > a + 1 \ \Rightarrow \ 6 > a.$$

Hence $b < a < 6$. Furthermore simple calculation shows that $b = 1,\ a = 2$ and $b = 3,\ a = 5$.

If $a = 2, b = 1$ and $a = 1, b = 2$, we get $P(x) = 2x$.

If $a = 5, b = 3$ and $a = 3, b = 5$, we get $P(x) = x$.

Hence we have $a = 2, b = 1$; $a = 1, b = 2$; $a = 5, b = 3$; $a = 3, b = 5$.

Also solved by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Joehyun Kim, Fort Lee High School, NJ, USA; Jiwon Park, St. Andrew's School, Middletown, DE, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Erica Choi, Blair Academy, Blairstown, NJ, USA; Adnan Ali, NIT Silchar, Assam, India; Nikolaos Evgenidis, Aristotle University of Thessaloniki, Greece; Leonard Arkhanhelskyi, Hofstra University, NY, USA; Narayanan P, Vivekananda College, Chennai, India; Robert Bosch, USA.

U422. Let a and b be complex numbers and let $(a_n)_{n\geq 0}$ be the sequence defined by $a_0 = 2$, $a_1 = a$ and

$$a_n = aa_{n-1} + ba_{n-2},$$

for $n \geq 2$. Write a_n as a polynomial in a and b.

Proposed by Dorin Andrica and Grigore Călugăreanu, Romania

Solution by Daniel Lasaosa, Pamplona, Spain

Using standard techniques for the solution of recursive equations, we find that

$$a_n = \left(\frac{a+\sqrt{a^2+4b}}{2}\right)^n + \left(\frac{a-\sqrt{a^2+4b}}{2}\right)^n.$$

Using Newton's binomial formula twice, we find

$$a_n = 2\sum_{u=0}^{\lfloor \frac{n}{2}\rfloor} \binom{n}{2u} \frac{a^{n-2u}\left(a^2+4b\right)^u}{2^n} = 2\sum_{u=0}^{\lfloor \frac{n}{2}\rfloor} \binom{n}{2u} \frac{a^{n-2u}}{2^n} \sum_{v=0}^{u} \binom{u}{v} a^{2u-2v} 4^v b^v =$$

$$= \sum_{v=0}^{\lfloor \frac{n}{2}\rfloor} c_{n,v} a^{n-2v} b^v,$$

where

$$c_{n,v} = \frac{1}{2^{n-2v-1}} \sum_{u=v}^{\lfloor \frac{n}{2}\rfloor} \binom{n}{2u}\binom{u}{v},$$

and this is clearly a_n expressed as a polynomial in a, b.

Also solved by Joehyun Kim, Fort Lee High School, NJ, USA; Jiwon Park, St. Andrew's School, Middletown, DE, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Adnan Ali, NIT Silchar, Assam, India; Arkady Alt, San Jose, CA, USA; Michel Faleiros Martins, IMECC-Unicamp, Brazil; Moubinool Omarjee, Lycée Henri IV, Paris, France; Robert Bosch, USA; Albert Stadler, Herrliberg, Switzerland.

U423. Find the maximum and minimum of

$$f(x) = \sqrt{\sin^4 x + \cos^2 x + 1} + \sqrt{\cos^4 x + \sin^2 x + 1}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Robert Bosch, USA

Let us prove that

$$f(x) = \sqrt{\frac{\cos 4x + 15}{2}}.$$

Is well-known that $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$, so we need to show that

$$\sqrt{\sin^4 x + \cos^2 x + 1} + \sqrt{\cos^4 x - \cos^2 x + 2} = 2\sqrt{\cos^4 x - \cos^2 x + 2}.$$

Clearly, we used $\sin^2 x + \cos^2 x = 1$. This equation is equivalent to

$$\sqrt{\sin^4 x + \cos^2 x + 1} = \sqrt{\cos^4 x - \cos^2 x + 2}.$$

The proof is simple, squaring and using again $\sin^2 x = 1 - \cos^2 x$. Done. Finally since $-1 \le \cos 4x \le 1$ it follows that

$$\sqrt{7} \le f(x) \le 2\sqrt{2}.$$

Notice $f(\pi/4) = \sqrt{7}$ and $f(0) = 2\sqrt{2}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Jio Jeong, Seoul International School, South Korea; Joehyun Kim, Fort Lee High School, NJ, USA; Jiwon Park, St. Andrew's School, Middletown, DE, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Erica Choi, Blair Academy, Blairstown, NJ, USA; Nikos Kalapodis, Patras, Greece; Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Adnan Ali, NIT Silchar, Assam, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Nikolaos Evgenidis, Aristotle University of Thessaloniki, Greece; Gregory Toms, College at Brockport, SUNY, NY, USA; Kevin Soto Palacios Huarmey, Perú; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Megan Wren, College at Brockport, SUNY, NY, USA; Michel Faleiros Martins, IMECC-Unicamp, Brazil; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Pedro Acosta De Leon, MIT, MA, USA; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

U424. Let a be a real number such that $|a| > 2$. Prove that if $a^4 - 4a^2 + 2$ and $a^5 - 5a^3 + 5a$ are rational numbers, then a is a rational number as well.

Proposed by Mircea Becheanu, University of Bucharest, Romania

First solution by the author

Denote $\alpha = a^4 - 4a^2 + 2$ and $\beta = a^5 - 5a^3 + 5a$. Consider the rational polynomials

$$P(X) = X^4 - 4X^2 + 2 - \alpha = X^4 - 4X^2 - (a^4 - 4a^2),$$

$$Q(X) = X^5 - 5X^3 + 5X - \beta.$$

The gcd of these polynomials is a polynomial with rational coefficients. It is clear that $P(a) = Q(a) = 0$, so they have common root a. We will show that only a is a common root. In this way, $\gcd(P, Q) = X - a$ is a rational polynomial, therefore it follows that a is a rational number. The polynomial $P(X)$ splits as

$$P(X) = (X - a)(X + a)(X^2 + a^2 - 4).$$

It has roots $-a$ and $z = i\sqrt{a^2 - 4}$. We have

$$Q(-a) = 0 \Longleftrightarrow a^4 - 5a^2 + 5 = 0 \Longleftrightarrow a^2 = \frac{5 \pm \sqrt{5}}{2},$$

which contradicts the hypothesis $|a| > 2$. Using the fact that $z^2 = 4 - a^2$ we obtain

$$Q(z) = 0 \Longleftrightarrow z^4 + az^3 + a^2z^2 + a^3z + a^4 - 5(z^2 + az + a^2) + 5 = 0 \Longleftrightarrow$$

$$z^2(z^2 + a^2) + az(z^2 + a^2) - 5(z^2 + a^2) - 5az + a^4 + 5 = 0 \Longleftrightarrow az = a^4 - 4a^2 + 1,$$

which is a contradiction, as $z \in \mathbb{C}$ and $z \notin \mathbb{R}$.

Second solution by Daniel Lasaosa, Pamplona, Spain

Note that

$$p^2 - 2 = a^{10} - 10a^8 + 35a^6 - 50a^4 + 25a^2 - 2 = \left(a^2 - 2\right)\left(q^2 - q - 1\right),$$

where $p = a^5 - 5a^3 + 5a$ and $q = a^4 - 4a^2 + 2$ are rational, or

$$a^2 = \frac{p^2 - 2}{q^2 - q - 1} + 2$$

is also rational. Note that we can perform this manipulation since $q^2 - q - 1 = 0$ results in

$$q = \frac{1 \pm \sqrt{5}}{2},$$

or q would not be rational, in contradiction with the problem statement. Moreover, $a^4 - 5a^2 + 5 = 0$ results in

$$a^2 = \frac{5 \pm \sqrt{5}}{2},$$

impossible since a^2 is rational. Hence $a^4 - 5a^2 + 5$ is a nonzero rational, and $a = \frac{p}{a^4 - 5a^2 + 5}$ is rational. The conclusion follows.

Also solved by Jio Jeong, Seoul International School, South Korea; Joehyun Kim, Fort Lee High School, NJ, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Erica Choi, Blair Academy, Blairstown, NJ, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Michel Faleiros Martins, IMECC-Unicamp, Brazil; Robert Bosch, USA and Jorge Erick, Brazil; Albert Stadler, Herrliberg, Switzerland.

U425. Let p be a prime number and let G be a group of order p^3. Define $\Gamma(G)$ the graph whose vertices are the noncentral conjugacy class sizes of G and two vertices are joined if and only if the two associated conjugacy class sizes are not coprime. Determine the structure of $\Gamma(G)$.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina

If G is an Abelian group then $Z(G) = G$ so $\Gamma(G)$ is a null graph. Let G be nonabelian.
If $p = 2$ then the group G is isomorphic to D_8 or to Q_8 both of which have three noncentral conjugacy classes all of size 2 , so $\Gamma(G)$ is a complete graph.
Let $p > 2$. Then G is isomorphic to exactly one of

$$Heis(\mathbb{Z}/(p)) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, a, b, c \in \mathbb{Z}/(p) \right\}$$

$$G_p = \left\{ \begin{pmatrix} 1+pm & b \\ 0 & 1 \end{pmatrix}, m, b \in \mathbb{Z}/(p^2) \right\}$$

both of which have all noncentral conjugacy class sizes divisible by p.
Therefore, $\Gamma(G)$ is a complete graph.

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India.

U426. Let $f:[0,1]\to\mathbb{R}$ be a continuous function and let $(x_n)_{n\geq 1}$ be the sequence defined by

$$x_n=\sum_{k=0}^{n}\cos\left(\frac{1}{\sqrt{n}}f\left(\frac{k}{n}\right)\right)-\alpha n^{\beta},$$

where α and β are real numbers. Evaluate $lim_{n\to\infty}x_n$.

Proposed by Dorin Andrica, Babes-Bolyai University, Cluj-Napoca, Romania

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Using Taylor's formula, we get

$$\begin{aligned} x_n &= \sum_{k=0}^{n}\left(1-\frac{1}{2n}f^2\left(\frac{k}{n}\right)+\frac{1}{24n^2}f^4\left(\frac{k}{n}\right)+O\left(\frac{1}{n^2}\right)\right)-\alpha n^{\beta} \\ &= (n+1)-\frac{1}{2n}\sum_{k=0}^{n}f^2\left(\frac{k}{n}\right)+\frac{1}{24n^2}\sum_{k=0}^{n}f^4\left(\frac{k}{n}\right)+O\left(\frac{1}{n}\right)-\alpha n^{\beta} \\ &\sim (n-\alpha n^{\beta})+1-\frac{1}{2}\int_0^1 f^2(x)dx+O\left(\frac{1}{n}\right) \end{aligned}$$

If

1) $\beta<1$, $\alpha\in\mathbb{R}$ then we have $\lim_{n\to\infty}x_n=+\infty$

2) $\beta=1$, $\alpha=1$ then we have $\lim_{n\to\infty}x_n=1-\frac{1}{2}\int_0^1 f^2(x)dx$

3) $\beta=1$, $\alpha>1$ then we have $\lim_{n\to\infty}x_n=-\infty$

4) $\beta=1$, $\alpha<1$ then we have $\lim_{n\to\infty}=+\infty$

5) $\beta>1$, $\alpha>0$ then we have $\lim_{n\to\infty}=-\infty$

6) $\beta>1$, $\alpha<0$ then we have $\lim_{n\to\infty}=+\infty$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Adnan Ali, NIT Silchar, Assam, India; Moubinool Omarjee, Lycée Henri IV, Paris, France; Albert Stadler, Herrliberg, Switzerland.

O421. Prove that for any real numbers a, b, c, d,

$$a^2+b^2+c^2+d^2+\sqrt{5}\min\left\{a^2,b^2,c^2,d^2\right\} \geq \left(\sqrt{5}-1\right)(ab+bc+cd+da).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy

Since the LHS is quadratic, we can assume $a, b, c, d \geq 0$.

The inequality is cyclic so we can also assume $d = \min\{a, b, c, d\}$.

First case $d = 0$

$$a^2+b^2+c^2+\sqrt{5}\min\{a^2,b^2,c^2\} \geq (\sqrt{5}-1)(ab+bc)$$

$$a^2+b^2+c^2-(\sqrt{5}-1)(ab+bc) = a^2+\frac{b^2}{2}+\frac{b^2}{2}+c^2-(\sqrt{5}-1)(ab+bc) \geq$$
$$\geq ab(\sqrt{2}-\sqrt{5}+1)+bc(\sqrt{2}-\sqrt{5}+1)$$

which is true so the inequality holds true.

Second case $d \neq 0$. The homogeneity allows us to take $d = 1$ and the inequality becomes

$$f(c) \doteq c^2-(\sqrt{5}-1)(b+1)c+a^2+b^2+1-(\sqrt{5}-1)(ab+a) \geq 0, \qquad a, b, c \geq 1$$

The derivative respect to c gives a minimum at $c = \dfrac{\sqrt{5}-1}{2}(1+b) \doteq \overline{c} > 1$ and

$$f(\overline{c}) = b^2(-1+\sqrt{5})+b(2a+2\sqrt{5}-2\sqrt{5}a-6)2a^2+2a-2\sqrt{5}a-1+3\sqrt{5}$$

The last expression (let's call it $g(b)$) has a minimum at $\overline{b} = \dfrac{a(\sqrt{5}-1)+3-\sqrt{5}}{\sqrt{5}-1} > 1$ and

$$g(\overline{b}) = \frac{1}{4}(3-\sqrt{5})(-2a+\sqrt{5}+3)^2 \geq 0$$

and this completes the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Joehyun Kim, Fort Lee High School, NJ, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Erica Choi, Blair Academy, Blairstown, NJ, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Adnan Ali, NIT Silchar, Assam, India; Arkady Alt, San Jose, CA, USA; Michel Faleiros Martins, IMECC-Unicamp, Brazil; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Robert Bosch, USA and Jorge Erick, Brazil; Albert Stadler, Herrliberg, Switzerland.

O422. Let $P(x)$ be a polynomial with integer coefficients which has an integer root. Prove that if p and q are distinct odd primes such that $P(p) = p < 2q - 1$ and $P(q) = q < 2p - 1$, then p and q are twin primes.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Albert Stadler, Herrliberg, Switzerland

We add an additional assumption that $r \neq 0$ is the integer zero of the polynomial. Then, the polynomial can then be written as

$$P(x) = (x - r)Q(x),$$

where $Q(x)$ is a polynomial with integer coefficients. Then

$$P(p) = (p - r)Q(p) = p$$

$$P(q) = (q - r)Q(q) = q$$

and we conclude that $p - r \in \{\pm 1, \pm p\}$ and $q - r \in \{\pm 1, \pm q\}$, since p and q are prime.

Therefore $r \in \{p - 1, p + 1, 2p\} \cap \{q - 1, q + 1, 2q\}$. We conclude that either $p - 1 = q + 1$ or $p + 1 = q - 1$, since $p < 2q - 1$ and $q < 2p - 1$. This means that p and q are twin primes.

Also solved by Joehyun Kim, Fort Lee High School, NJ, USA; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Erica Choi, Blair Academy, Blairstown, NJ, USA; Prajnanaswaroopa S, Bangalore, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Adnan Ali, NIT Silchar, Assam, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Joel Schlosberg, Bayside, NY, USA.

O423. Prove that in any triangle ABC,

$$\sqrt{\frac{1}{{r_b}^2}+\frac{1}{r_c}+1}+\sqrt{\frac{1}{{r_c}^2}+\frac{1}{r_b}+1}\geq 2\sqrt{\frac{1}{{h_a}^2}+\frac{1}{h_a}+1}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Evgenidis Nikolaos, Aristotle University of Thessaloniki, Greece

It is well known that $S=r_b(s-b)=r_c(s-c)=\frac{1}{2}ah_a$.
Using these, the given inequality can be written as

$$\sqrt{\frac{(s-b)^2}{S^2}+\frac{s-c}{S}+1}+\sqrt{\frac{(s-c)^2}{S^2}+\frac{s-b}{S}+1}\geq\sqrt{\frac{a^2}{S^2}+\frac{2a}{S}+4}.$$

For convenience, let $x=s-b, y=s-c$. Then, it is easy to see that $x+y=a$. Hence, after some algebra, we need to prove

$$\sqrt{x^2+Sy+S^2}+\sqrt{y^2+Sy+S^2}\geq\sqrt{(x+y)^2+2S(x+y)+4S^2}.$$

Squaring both sides of the last inequality and simplifying, it suffices to show that

$$2\sqrt{(x^2+Sy+S^2)(y^2+Sx+S^2)}\geq 2xy+S(x+y)+2S^2.$$

Squaring both sides of this and doing the maths, it is easy to get that we need to prove

$$3S^2(x-y)^2+4S[x^3+y^3-xy(x+y)]\geq 0.$$

But this holds because $x^3+y^3-xy(x+y)=(x+y)(x^2-xy+y^2)-xy(x+y)=(x+y)(x-y)^2\geq 0$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Arkady Alt, San Jose, CA, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

O424. For a positive integer n, we define $f(n)$ to be the number of 2's that appear (as digits) after writing the numbers $1, 2, \ldots, n$ in their decimal expansion. For example, $f(22) = 6$ because 2 appears once in the numbers $2, 12, 20, 21$ and it appears twice in the number 22. Prove that there are finitely many numbers n such that $f(n) = n$.

Proposed by Enrique Trevinio, Lake Forest College, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Allowing zeros to the left, the first 10^k numbers are all possible ordered strings of k digits from 0 to 9, where each digit appears exactly as many times as any other. Since these numbers contain in total $k \cdot 10^k$ digits, digit 2 occurs exactly $k \cdot 10^{k-1}$ times, ie $f(n) \geq k \cdot 10^{k-1}$ for all $n \geq 10^k$. Thus for $k \geq 100$, if $10^k \leq n < 10^{k+1}$, we have $f(n) \geq k \cdot 10^{k-1} \geq 10^{k+1} > n$, and all numbers n such that $f(n) = n$ are necessarily less than 10^{100}. The conclusion follows.

Also solved by Leonard Arkhanhelskyi, Hofstra University, NY, USA; Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy; Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Adnan Ali, NIT Silchar, Assam, India; Joel Schlosberg, Bayside, NY, USA; Albert Stadler, Herrliberg, Switzerland.

O425. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + abc = 4$ and let k be a nonnegative real number. Prove that

$$a + b + c + \sqrt{k\left(k - 1 + \frac{a^2 + b^2 + c^2}{3}\right)} \le k + 3.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by Robert Bosch, USA

fI $a^2 + b^2 + c^2 + abc = 4$ then there exists A, B, C angles of an acute-angled triangle such that $a = 2\cos A,\ b = 2\cos B,\ c = 2\cos C$. See the book "*Problems from the Book*" by Titu Andreescu and Gabriel Dospinescu for detailed information.

Moving $a + b + c$ to the right side and squaring the inequality to be proved becomes

$$yk \le 2(3 - x)k + (3 - x)^2,$$

where

$$\begin{aligned} x &= 2\cos A + 2\cos B + 2\cos C, \\ y &= -1 + \frac{4}{3}\left(\cos^2 A + \cos^2 B + \cos^2 C\right). \end{aligned}$$

Now, it is natural to think $y \le 2(3 - x)$ and consider $(3 - x)^2$ as an extra positive factor. Notice that we have equality for $a = b = c = 1$, that is to say, for the equilateral triangle, in this case $x = 3$, and $(3 - x)^2 = 0$.

It only remains to show the following inequality

$$4\left(\cos A + \cos B + \cos C\right) + \frac{4}{3}\left(\cos^2 A + \cos^2 B + \cos^2 C\right) \le 7,$$

for an acute triangle. Denote the left side by $f(A, B, C)$. We shall prove that $f(A, B, C) \le f(A, B, B)$, that is to say for the isosceles triangle. By symmetry, assume without loss of generality $A \le B \le C$. We need to show

$$4(\cos B + \cos C) + \frac{4}{3}(\cos^2 B + \cos^2 C) \le 8\cos B + \frac{8}{3}\cos^2 B,$$

which after some elementary transformations becomes

$$(\cos B - \cos C)(3 + \cos B + \cos C) \ge 0,$$

clearly true because the function $\cos x$ is decreasing on $\left(0, \frac{\pi}{2}\right)$.

Now let us prove $f(A, B, B) = f(\pi - 2B, B, B) \le 7$. Say,

$$\frac{8}{3}\cos^2 B + 8\cos B + \frac{4}{3}\cos^2 2B - 4\cos 2B \le 7.$$

Using the formula $\cos 2x = 2\cos^2 x - 1$ the inequality becomes

$$\frac{16}{3}x^4 - \frac{32}{3}x^2 + 8x + \frac{16}{3} \le 7,$$

where $x = \cos B$, and then $0 < x < \frac{\sqrt{2}}{2}$ since $\frac{\pi}{4} < B < \frac{\pi}{2}$.

Factoring yields to

$$\frac{1}{3}(4x^2+4x-5)(2x-1)^2 \le 0,$$

and finally

$$4x^2+4x-5=(2x+1)^2-6<0,$$

because $\frac{\sqrt{2}}{2}<\frac{\sqrt{6}-1}{2}$ and we are done.

Also solved by Akash Singha Roy, Hariyana Vidya Mandir, Salt Lake, Kolkata, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Arkady Alt, San Jose, CA, USA; Ashley Case, SUNY Brockport, NY, USA; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Michel Faleiros Martins, IMECC-Unicamp, Brazil; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

O426. Let a, b, c be positive numbers such that

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}=1.$$

Prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\le\frac{a+b+c}{2}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Michel Faleiros Martins, IMECC-Unicamp, Brazil

Let's solve the problem substituting symmetric expressions by elementary polynomials:

$$p=a+b+c>0,\quad q=ab+bc+ca>0,\quad r=abc>0$$

Thus we have

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}=1\quad\Rightarrow\quad q+r=4$$

Using AMGM inequality we have

$$1=\frac{ab+bc+ca+abc}{4}\ge\sqrt[4]{a^3b^3c^3}\quad\Rightarrow\quad abc\le 1$$

So, the condition $q+r=4$ implies that $0<r\le 1$ and $3\le q<4$. The original inequality can be simplified to

$$2(a^2+b^2+c^2)+6(ab+bc+ca)\le(a+b+c)(a+b)(b+c)(c+a)\quad\Leftrightarrow\quad p^2(q-2)+p(q-4)-2q\ge 0$$

Solving for p we get $p\le-\frac{2q-4}{q-2}<0$ or $p\ge\frac{q}{q-2}$. However, $(a-b)^2+(b-c)^2+(c-a)^2\ge 0$ implies that $p^2\ge 3q$. Thus,

$$p^2\ge 3q\ge 9\ge\left(1+\frac{2}{q-2}\right)^2=\frac{q^2}{(q-2)^2}\quad\Rightarrow\quad p\ge\frac{q}{q-2}.$$

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