Junior Problems

J427. Find all complex numbers x, y, z which satisfy simultaneously the equations:

$$x + y + z = 1, \;\; x^3 + y^3 + z^3 = 1, \;\; x^2 + 2yz = 4.$$

Proposed by Mircea Becheanu, University of Bucharest, Romania

J428. Solve the equation

$$2x[x] + 2\{x\} = 2017,$$

where $[a]$ denotes the greatest integer not greater than a and $\{a\}$ is the fractional part of a.

Proposed by Adrian Andreescu, Dallas, Texas

J429. Let x, y be positive real numbers such that $x + y \leq 1$. Prove that

$$\left(1 - \frac{1}{x^3}\right)\left(1 - \frac{1}{y^3}\right) \geq 49.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J430. In triangle ABC, $\angle C > 90^\circ$ and $3a + \sqrt{15ab} + 5b = 7c$. Prove that $\angle C \leq 120^\circ$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J431. Let a, b, c, d, e be real numbers in the interval $[1, 2]$. Prove that

$$a^2 + b^2 + c^2 + d^2 + e^2 - 3abcde \leq 2.$$

Proposed by An Zhenping, Xianyang Normal University, China

J432. Let m and n be integers greater than 1. Prove that

$$(m^3 - 1)(n^3 - 1) \geq 3m^2n^2 + 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Senior Problems

S427. Solve in complex numbers the system of equations:

$$z + \frac{2017}{w} = 4 - i$$

$$w + \frac{2018}{z} = 4 + i.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA.

S428. Let a, b, c be nonnegative real numbers, not all zero, such that $ab + bc + ca = a + b + c$. Prove that

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \leq \frac{5}{3}$$

Proposed by An Zenping, Xianyang Normal University, China

S429. Let ABC be a triangle and let M be a point in its plane. Prove that for all positive real numbers x, y, z the following inequality holds

$$xMA^2 + yMB^2 + zMC^2 > \frac{yz}{2(y+z)}a^2 + \frac{zx}{2(z+x)}b^2 + \frac{xy}{2(x+y)}c^2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S430. Prove that

$$\sin\frac{\pi}{2n} \geq \frac{1}{n}\ ,$$

for all positive integers n.

Proposed by Florin Rotaru, Focşani, Romania

S431. Let a, b, c be positive numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} \geq \frac{3}{4}$$

Proposed by Konstantinos Metaxas, Athens, Greece

S432. Let d be an open half-disk of diameter AB and h be the half-plane defined by the line AB and containing d. Let X be a point on d and let Y and Z be points in h on the semicircles of diameters AX and BX, respectively. Prove that

$$AY \cdot BZ + XY \cdot XZ \leq AX^2 - AX \cdot BX + BX^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Undergraduate Problems

U427. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \mathbf{1}_{(0,1/y)}(x) \cdot \mathbf{1}_{(0,1)}(y) \cdot y,$$

where $\mathbf{1}$ is the characteristic function. Evaluate

$$\int_{\mathbb{R}^2} f(x,y)\ dx\ dy.$$

Proposed by Alessandro Ventulo, Milan, Italy

U428. Let a, b, c positive real numbers such that $a + b + c = 1$. Prove that

$$(1 + a^2b^2)^c(1 + b^2c^2)^a(1 + c^2a^2)^b \geq 1 + 9a^2b^2c^2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U429. Let $n \geq 2$ be an integer and let A be an $n \times n$ real matrix in which exactly $(n-1)^2$ entries are zero. Prove that if B is an $n \times n$ matrix with all entries nonzero numbers, then BA can not be a nonsingular diagonal matrix.

Proposed by Alessandro Ventullo, Milan, Italy

U430. Let A and B be 3×3 matrices with complex numbers entries, such that

$$(AB - BA)^2 = AB - BA.$$

Prove that $AB = BA$.

Proposed by Florin Stănescu, Găeşti, Romania

U431. Evaluate

$$\lim_{t \to 0} \frac{1}{t} \int_0^t \sqrt{1 + e^x}\ dx \quad \text{and} \quad \lim_{t \to 0} \frac{1}{t} \int_0^t e^{e^x}\ dx$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U432. For every point $P(x,y,z)$ on the unit sphere, consider the points $Q(y,z,x)$ and $R(z,x,y)$. For every point A on the sphere, denote $\angle(AOP) = p$, $\angle(AOQ) = q$ and $\angle(AOR) = r$. Prove that

$$|\cos q - \cos r| \leq 2\sqrt{3} \sin \frac{p}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Olympiad Problems

O427. Let ABC be a triangle and m_a, m_b, m_c be the lengths of its medians. Prove that

$$\sqrt{3}\,(am_a + bm_b + cm_c) \leq 2s^2.$$

Proposed by Dragoljub Miloševič, Gornji Milanovac, Serbia

O428. Determine all positive integers n for which the equation

$$x^2 + y^2 = n(x - y)$$

is solvable in positive integers. Solve the equation

$$x^2 + y^2 = 2017(x - y).$$

Proposed by Dorin Andrica, Cluj-Napoca, Romania and
Vlad Crişan, Göttingen, Germany

O429. Let ABC be a non-obtuse triangle. Prove that

$$m_a m_b + m_b m_c + m_c m_a \leq (a^2 + b^2 + c^2)\left(\frac{5}{8} + \frac{r}{4R}\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O430. Find the number of positive integers $n \leq 10^6$ such that 5 divides $\binom{2n}{n}$.

Proposed by Enrique Trevinio, Lake Forest College, USA

O431. Let a, b, c, d be positive real numbers such that $a + b + c + d = 3$. Prove that

$$a^2 + b^2 + c^2 + d^2 + \frac{64}{27}abcd \geq 3.$$

Proposed by An Zhenping, Xianyang Normal University, China

O432. Let $ABCDEF$ be a cyclic hexagon which contains an inscribed circle. Denote by $\omega_A, \omega_B, \omega_C, \omega_D, \omega_E$ and ω_F the inscribed circle in the triangle FAB, ABC, BCD, CDE, DEF and EFA, respectively. Let ℓ_{AB} be the external common tangent of ω_A and ω_B, other than the line AB; lines $\ell_{BC}, \ell_{CD}, \ell_{DE}, \ell_{EF}$ and ℓ_{FA} are defined analogously. Let A_1 be the intersection of the lines ℓ_{FA} and ℓ_{AB}, B_1 the intersection of the lines ℓ_{AB} and ℓ_{BC}; points C_1, D_1, E_1 and F_1 are defined analogously. Suppose that $A_1B_1C_1D_1E_1F_1$ is a convex hexagon. Prove that its diagonals A_1D_1, B_1E_1 and C_1F_1 are concurrent.

Proposed by Nairi Sedrakian, Yerevan, Armenia