A WEIGHTED POWER LESSELS-PELLING INEQUALITY

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Abstract. Lessels-Pelling inequality states that in a triangle the sum of two angle bisectors and a medians is less or equal than the product between $\sqrt{3}$ and the semiperimeter. The purpose of this article is to find a powered and weighed version of this inequality (in the presence of some supplementary conditions), and to prove chains of inequalities which represent refinements of powered Lessels-Pelling inequality.

1. Introduction

Let ABC be a triangle. We shall denote the side lengths by $a = BC$, $b = CA$, $c = AB$ and the semiperimeter by $s = \dfrac{a+b+c}{2}$. The length of the median from A is denoted by m_a, while w_b, w_c denote the lengths of angle bisector from B and C.

A brief history of the Lessels-Lelling inequality is presented below.

In 1974, J. Garfunkel [1] conjectured on the basis of computer simulations the inequality $m_a + w_b + h_c \le s\sqrt{3}$, where h_c represent the length of the altitude from C. In 1976, C.S. Gardner [1] proved it using elementary transformations and derivatives. In 1977, G.S. Lessels and M.J. Pelling [2] used computer simulations to predict a stronger inequality

$$m_a + w_b + w_c \le s\sqrt{3}.$$

In 1980, B.E. Patuwo, R.S.D. Thomas and Chung-Lie Wang give in [3] a proof of this inequality. In [4] a new proof is given by C. Tănăsescu. In 1981, L. Panaitopol [5] finds an elementary solution of Lessels-Pelling inequality. M. Drăgan gives a simple proof and some refinements of Lessels-Pelling inequality in [6] and [7].

2. Main results

In what follows we use the notations:

$$x = \frac{a}{s},\ y = \frac{b}{s},\ z = \frac{c}{s},\ u = \sqrt{1-y},\ v = \sqrt{1-z},\ s_1 = u+v,\ p_1 = u \cdot v,$$
$$E = \sqrt{\frac{2(y^2+z^2)-x^2}{4}}\,,\ s_2 = \beta u + \gamma v,\ t = \frac{3\alpha^2 - 2\alpha + 1}{2}\,, \tag{2.1}$$

where numbers $\alpha, \beta, \gamma > 0$ satisfy $(\beta - \gamma)(b - c) \ge 0$ and $\alpha + \beta + \gamma = 1$.

Lemma 2.1. *The following inequalities hold:*

$$i) \qquad E \le \sqrt{1 - \frac{s_1^2}{2}}\,; \tag{2.2}$$

$$ii) \qquad E \le \sqrt{1 - \frac{2}{(1-\alpha)^2}\, s_2^2}\,. \tag{2.3}$$

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Proof. *i*) From (2.1) we have

$$E=\sqrt{\frac{2(2+u^4+v^4-2u^2-2v^2)-(u^2+v^2)^2}{4}}$$
$$=\sqrt{\frac{2[2+(u^2+v^2)^2-2p_1^2-2(u^2+v^2)]-(u^2+v^2)^2}{4}}$$
$$=\sqrt{\frac{4+2(s_1^2-2p_1)^2-4p_1^2-4(s_1^2-2p_1)-(s_1^2-2p_1)^2}{4}}$$
$$=\sqrt{\frac{4+s_1^4-4s_1^2+4p_1(2-s_1^2)}{4}}\le\sqrt{\frac{4-4s_1^2+2s_1^2}{4}}=\sqrt{1-\frac{s_1^2}{2}}.$$

ii) We have

$$s_2=\beta u+\gamma v\le\frac{1}{2}(\beta+\gamma)(u+v)=\frac{(1-\alpha)s_1}{2},\tag{2.4}$$

since by $(\beta-\gamma)(b-c)\ge 0$, it follows that $(\beta-\gamma)(u-v)\le 0$.
From (2.2) and (2.4) it results (2.3). □

Theorem 2.1 (The weighted power Lessels-Pelling inequality)**.** *In every triangle ABC, the following inequality holds:*

$$\alpha m_a+\beta w_b+\gamma w_c\le s\sqrt{t},\tag{2.5}$$

where numbers $\alpha,\beta,\gamma>0$ satisfy $(\beta-\gamma)(b-c)\ge 0$ and $\alpha+\beta+\gamma=1$.

Proof. Since $w_b\le\sqrt{s(s-b)}$ and $w_c\le\sqrt{s(s-c)}$, in order to prove (2.5), it is sufficient to show that

$$\alpha m_a+\beta\sqrt{s(s-b)}+\gamma\sqrt{s(s-c)}\le s\sqrt{t}.\tag{2.6}$$

By (2.1), the inequality (2.6) may be written as

$$\alpha\sqrt{\frac{2(y^2+z^2)-x^2}{4}}+\beta\sqrt{1-y}+\gamma\sqrt{1-z}\le\sqrt{t},$$

or

$$\alpha E+s_2\le\sqrt{t}.\tag{2.7}$$

From (2.2) and (2.4) we have that

$$\alpha E+s_2\le\alpha\sqrt{\frac{2-s_1^2}{2}}+\frac{1-\alpha}{2}\,s_1.$$

It results that in order to prove (2.7), it is sufficient to show that

$$\alpha\sqrt{\frac{2-s_1^2}{2}}+\frac{1-\alpha}{2}\,s_1\le\sqrt{t}.\tag{2.8}$$

Note that $s_1=\sqrt{1-y}+\sqrt{1-z}\le\sqrt{2(2-y-z)}=\sqrt{2x}=\sqrt{\frac{2a}{s}}<\sqrt{2}$.

Consider the function $f:\left(0,\sqrt{2}\right)\to\mathbb{R}$, $f(s_1)=\alpha\sqrt{\frac{2-s_1^2}{2}}+\frac{(1-\alpha)s_1}{2}$. The derivative is $f'(s_1)=\frac{-\alpha s_1}{\sqrt{4-2s_1^2}}+\frac{1-\alpha}{2}$. Clearly, the equation $f'(s_1)=0$ has the root $s_0=\frac{2(1-\alpha)}{\sqrt{6\alpha^2-4\alpha+2}}$.

Since $f'(0) \geq 0$, $f'\left(\sqrt{2}\right) \leq 0$, it results that s_0 is a maximum point for f. It follows that

$$\begin{aligned} f(s_1) \leq f(s_0) &= \alpha\sqrt{\frac{1}{2}\left[2 - \frac{4(1-\alpha)^2}{6\alpha^2 - 4\alpha + 2}\right]} + \frac{(1-\alpha)^2}{\sqrt{6\alpha^2 - 4\alpha + 2}} \\ &= \alpha\sqrt{\frac{4\alpha^2}{6\alpha^2 - 4\alpha + 2}} + \frac{(1-\alpha)^2}{\sqrt{6\alpha^2 - 4\alpha + 2}} \\ &= \sqrt{\frac{3\alpha^2 - 2\alpha + 1}{2}} = \sqrt{t}, \end{aligned}$$

which proves (2.8).

□

The following results are direct consequences of Theorem 2.1.

Corollary 2.1. *In every triangle ABC, the following inequality holds*

$$\alpha m_a + \beta w_b + \gamma w_c \leq s\sqrt{\frac{2\alpha^2 + (\beta + \gamma)^2}{2}}, \tag{2.9}$$

where α, β, γ are positive numbers such that $(b - c)(\beta - \gamma) \geq 0$.

Proof. Take $\alpha \to \dfrac{\alpha}{\alpha+\beta+\gamma}$, $\beta \to \dfrac{\beta}{\alpha+\beta+\gamma}$, $\gamma \to \dfrac{\gamma}{\alpha+\beta+\gamma}$ in (2.5). □

Corollary 2.2. *In every triangle ABC, the following inequality holds*

$$m_a w_a + m_c w_b + m_b w_c \leq s\sqrt{\frac{2w_a^2 + (m_b + m_c)^2}{2}}.$$

Proof. Since $(b - c)(m_c - m_b) \geq 0$, if we take $\alpha = w_a$, $\beta = m_c$, $\gamma = m_b$ in (2.9), we obtain the inequality from the statement. □

Corollary 2.3. *In every triangle ABC, the following inequality holds*

$$m_a m_b m_c + w_a w_b m_c + w_a m_b w_c \leq s\sqrt{m_b^2 m_c^2 + \frac{w_a^2(m_b + m_c)^2}{2}}. \tag{2.10}$$

Proof. Since $(b - c)\left(\dfrac{1}{m_b} - \dfrac{1}{m_c}\right) \geq 0$, taking $\alpha = \dfrac{1}{w_a}$, $\beta = \dfrac{1}{m_b}$, $\gamma = \dfrac{1}{m_c}$ in (2.9), we obtain the inequality from the statement. □

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