Junior Problems

J433. Let \( a, b, c, x, y, z \) be real numbers such that \( a^2 + b^2 + c^2 = x^2 + y^2 + z^2 = 1 \). Prove that

\[
|a(y - z) + b(z - x) + c(x - y)| \leq \sqrt{6(1 - ax - by - cz)}.
\]

Proposed by Titu Andreescu, University of Texas at Dallas

J434. Solve in integers the equation

\[
x^3 + y^3 = 7 \max(x, y) + 7.
\]

Proposed by Mihaela Berindeanu, Bucharest, România

J435. Let \( a \geq b \geq c > 0 \) be real numbers. Prove that

\[
2 \left( \frac{b}{a} + \frac{c}{b} + \frac{a}{c} \right) - \left( \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \right) \geq 3.
\]

Proposed by Mircea Becheanu, Montreal, Canada

J436. Let \( a, b, c \) be real numbers such that \( a^4 + b^4 + c^4 = a + b + c \). Prove that

\[
a^3 + b^3 + c^3 \leq abc + 2.
\]

Proposed by Adrian Andreescu, University of Texas at Austin

J437. Let \( a, b, c \) be real numbers such that \( (a^2 + 2)(b^2 + 2)(c^2 + 2) = 512 \). Prove that

\[
|ab + bc + ca| \leq 18.
\]

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J438. (i) Find the greatest real number \( r \) such that

\[
ab \geq r \left( 1 - \frac{1}{a} - \frac{1}{b} \right)
\]

for all positive real numbers \( a \) and \( b \).

(ii) Find the maximum of

\[
xyz(2 - x - y - z)
\]

over all positive real numbers \( x, y, z \).

Proposed by Titu Andreescu, University of Texas at Dallas
Senior Problems

S433. Let $a, b, c$ be real numbers such that $0 \leq a \leq b \leq c$ and $a + b + c = 1$. Prove that

$$\sqrt{\frac{2}{3}} \leq a\sqrt{a + b} + b\sqrt{b + c} + c\sqrt{c + a} \leq 1.$$

When do the equalities hold?

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S434. Let $a, b, c, x, y, z, w$ be real numbers such that $a^2 + b^2 + c^2 + d^2 = x^2 + y^2 + z^2 + w^2 = 1$. Prove that

$$ax + \sqrt{(b^2 + c^2)(y^2 + z^2)} + dw \leq 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas

S435. Let $a, b, c$ be positive real numbers such that $abc = 1$. Prove that

$$a^3 + b^3 + c^3 + \frac{8}{(a+b)(b+c)(c+a)} \geq 4.$$

Proposed by Alessandro Ventullo, Milan, Italy

S436. Prove that for all real numbers $a, b, c, d, e$,

$$2a^2 + b^2 + 3c^2 + d^2 + 2e^2 \geq 2(ab - bc - cd - de + ea).$$

Proposed by Titu Andreescu, University of Texas at Dallas

S437. Let $a, b, c$ be positive real numbers. Prove that

$$\frac{(4a + b + c)^2}{2a^2 + (b + c)^2} + \frac{(4b + c + a)^2}{2b^2 + (c + a)^2} + \frac{(4c + a + b)^2}{2c^2 + (a + b)^2} \leq 18.$$

Proposed by Marius Stănean, Zalău, România

S438. Let $ABC$ be an acute triangle. Determine all points $M$ inside the triangle such that the sum

$$\frac{AM}{AB \cdot AC} + \frac{BM}{BA \cdot BC} + \frac{CM}{CA \cdot CB}$$

is minimal.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România
U433. Let $x, y, z$ be positive real numbers such that $x + y + z = 3$. Prove that
\[ x^2 y^y z^z \geq 1. \]

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U434. Find all homogeneous polynomials $P(X, Y)$ such that
\[ P \left( x, \sqrt[3]{x^3 + y^3} \right) = P \left( y, \sqrt[3]{x^3 + y^3} \right), \]
for all real numbers $x$ and $y$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U435. Consider the sequence $(a_n)_{n \geq 1}$ defined by
\[ \left( 1 + \frac{1}{n} \right)^{n+a_n} = 1 + \frac{1}{1!} + \cdots + \frac{1}{n!}. \]

(i) Prove that $(a_n)_{n \geq 1}$ is convergent and $\lim_{n \to \infty} a_n = \frac{1}{2}$.
(ii) Evaluate $\lim_{n \to \infty} n \left( a_n - \frac{1}{2} \right)$.

Proposed by Dorin Andrica, Babeș-Bolyai University, Cluj-Napoca, România

U436. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that
\[ \int_0^1 xf(x) \left( x^2 + f(x)^2 \right) \, dx \geq \frac{2}{5}. \]
Prove that
\[ \int_0^1 \left( x^2 + \frac{1}{3} f(x)^2 \right)^2 \, dx \geq \frac{16}{45}. \]

Proposed by Titu Andreescu, University of Texas at Dallas

U437. Prove that for any $a > \frac{1}{e}$ the following inequality holds
\[ \int_{\frac{1+\ln(a+1)}{1+\ln a}}^{1+\ln(a+1)} x^2 \, dx \geq 1. \]

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U438. Prove that a positive integer $n$ can be represented by the quadratic form $x^2 + 7y^2$ if and only if

(i) $\vartheta_2(n) \neq 1$;
(ii) $\vartheta_p(n)$ is even for every prime number $p$, $p \equiv 3, 5, 6 \pmod{7}$.

Proposed by José Hernández Santiago, Morelia, Mexico
Olympiad Problems

O433. Let \( q, r, s \) be positive integers such that \( s^2 - s + 1 = 3qr \). Prove that \( q + r + 1 \) divides \( q^3 + r^3 - s^3 + 3qrs \).

*Proposed by Titu Andreescu, University of Texas at Dallas*

O434. Let \( a, b, c \) be positive real numbers such that \( a + b + c = 3 \). Prove that

\[
\frac{b^2}{\sqrt{2(a^4 + 1)}} + \frac{c^2}{\sqrt{2(b^4 + 1)}} + \frac{a^2}{\sqrt{2(c^4 + 1)}} \geq \frac{3}{2}
\]

*Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam*

O435. Let \( a, b, c \) positive numbers such that \( ab + bc + ca + 2abc = 1 \). Prove that

\[
\frac{1}{8a^2 + 1} + \frac{1}{8b^2 + 1} + \frac{1}{8c^2 + 1} \geq 1.
\]

*Proposed by An Zhenping, Xianyang Normal University, China*

O436. Prove that in any triangle \( ABC \) the following inequality holds:

\[
\frac{a^2}{\sin \frac{A}{2}} + \frac{b^2}{\sin \frac{B}{2}} + \frac{c^2}{\sin \frac{C}{2}} \geq \frac{8s^2}{3}.
\]

*Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia*

O437. Let \( a, b, c \) be the side-lengths of a triangle \( ABC \). Prove that

\[
\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \leq \frac{2s^2}{27r^2} + 1.
\]

*Proposed by Mircea Lascu and Titu Zvonaru, România*

O438. Let \( a, b, c \) be positive numbers such that

\[
(a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = \frac{49}{4}.
\]

Find all possible values of the expression

\[
E = \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}.
\]

*Proposed by Marius Stănean, Zalău, România*