# Junior Problems

**J433.** Let $a$, $b$, $c$, $x$, $y$, $z$ be real numbers such that $a^2+b^2+c^2=x^2+y^2+z^2=1$. Prove that

$$|a(y-z)+b(z-x)+c(x-y)| \le \sqrt{6(1-ax-by-cz)}.$$

*Proposed by Titu Andreescu, University of Texas at Dallas*

**J434.** Solve in integers the equation

$$x^3+y^3=7\max(x,\, y)+7.$$

*Proposed by Mihaela Berindeanu, Bucharest, România*

**J435.** Let $a \ge b \ge c > 0$ be real numbers. Prove that

$$2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge 3.$$

*Proposed by Mircea Becheanu, Montreal, Canada*

**J436.** Let $a$, $b$, $c$ be real numbers such that $a^4+b^4+c^4=a+b+c$. Prove that

$$a^3+b^3+c^3 \le abc+2.$$

*Proposed by Adrian Andreescu, University of Texas at Austin*

**J437.** Let $a$, $b$, $c$ be real numbers such that $\left(a^2+2\right)\left(b^2+2\right)\left(c^2+2\right)=512$. Prove that

$$|ab+bc+ca| \le 18.$$

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

**J438.** (i) Find the greatest real number $r$ such that

$$ab \ge r\left(1-\frac{1}{a}-\frac{1}{b}\right)$$

for all positive real numbers $a$ and $b$.

(ii) Find the maximum of

$$xyz(2-x-y-z)$$

over all positive real numbers $x$, $y$, $z$.

*Proposed by Titu Andreescu, University of Texas at Dallas*

# Senior Problems

**S433.** Let $a$, $b$, $c$ be real numbers such that $0 \leq a \leq b \leq c$ and $a+b+c=1$. Prove that

$$\sqrt{\frac{2}{3}} \leq a\sqrt{a+b}+b\sqrt{b+c}+c\sqrt{c+a} \leq 1.$$

When do the equalities hold?

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

**S434.** Let $a$, $b$, $c$, $d$, $x$, $y$, $z$, $w$ be real numbers such that $a^2+b^2+c^2+d^2=x^2+y^2+z^2+w^2=1$. Prove that

$$ax+\sqrt{\left(b^2+c^2\right)\left(y^2+z^2\right)}+dw \leq 1.$$

*Proposed by Titu Andreescu, University of Texas at Dallas*

**S435.** Let $a$, $b$, $c$ be positive real numbers such that $abc=1$. Prove that

$$a^3+b^3+c^3+\frac{8}{(a+b)(b+c)(c+a)} \geq 4.$$

*Proposed by Alessandro Ventullo, Milan, Italy*

**S436.** Prove that for all real numbers $a$, $b$, $c$, $d$, $e$,

$$2a^2+b^2+3c^2+d^2+2e^2 \geq 2(ab-bc-cd-de+ea).$$

*Proposed by Titu Andreescu, University of Texas at Dallas*

**S437.** Let $a,b,c$ be positive real numbers. Prove that

$$\frac{(4a+b+c)^2}{2a^2+(b+c)^2}+\frac{(4b+c+a)^2}{2b^2+(c+a)^2}+\frac{(4c+a+b)^2}{2c^2+(a+b)^2} \leq 18.$$

*Proposed by Marius Stănean, Zalău, România*

**S438.** Let $ABC$ be an acute triangle. Determine all points $M$ inside the triangle such that the sum

$$\frac{AM}{AB\cdot AC}+\frac{BM}{BA\cdot BC}+\frac{CM}{CA\cdot CB}$$

is minimal.

*Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România*

# Undergraduate Problems

**U433.** Let $x$, $y$, $z$ be positive real numbers such that $x+y+z=3$. Prove that

$$x^x y^y z^z \geq 1.$$

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

**U434.** Find all homogeneous polynomials $P(X,\, Y)$ such that

$$P\left(x,\ \sqrt[3]{x^3+y^3}\right) = P\left(y,\ \sqrt[3]{x^3+y^3}\right),$$

for all real numbers $x$ and $y$.

*Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran*

**U435.** Consider the sequence $(a_n)_{n\geq 1}$ defined by

$$\left(1+\frac{1}{n}\right)^{n+a_n} = 1+\frac{1}{1!}+\cdots+\frac{1}{n!}.$$

(i) Prove that $(a_n)_{n\geq 1}$ is convergent and $\lim\limits_{n\to\infty} a_n = \dfrac{1}{2}$.

(ii) Evaluate

$$\lim_{n\to\infty} n\left(a_n - \frac{1}{2}\right).$$

*Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România*

**U436.** Let $f : [0,\, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 x f(x)\left(x^2+f(x)^2\right) dx \geq \frac{2}{5}.$$

Prove that

$$\int_0^1 \left(x^2+\frac{1}{3}f(x)^2\right)^2 dx \geq \frac{16}{45}.$$

*Proposed by Titu Andreescu, University of Texas at Dallas*

**U437.** Prove that for any $a > \frac{1}{e}$ the following inequality holds

$$\int\limits_{1+\ln a}^{1+\ln(a+1)} x^x dx \geq 1.$$

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

**U438.** Prove that a positive integer $n$ can be represented by the quadratic form $x^2+7y^2$ if an only if

(i) $\vartheta_2(n) \neq 1$;

(ii) $\vartheta_p(n)$ is even for every prime number $p$, $p \equiv 3, 5, 6 \pmod 7$.

*Proposed by José Hernández Santiago, Morelia, Mexico*

# Olympiad Problems

**O433.** Let $q$, $r$, $s$ be positive integers such that $s^2 - s + 1 = 3qr$. Prove that $q + r + 1$ divides $q^3 + r^3 - s^3 + 3qrs$.

*Proposed by Titu Andreescu, University of Texas at Dallas*

**O434.** Let $a$, $b$, $c$ be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{b^2}{\sqrt{2(a^4+1)}} + \frac{c^2}{\sqrt{2(b^4+1)}} + \frac{a^2}{\sqrt{2(c^4+1)}} \geq \frac{3}{2}$$

*Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam*

**O435.** Let $a$, $b$, $c$ positive numbers such that $ab + bc + ca + 2abc = 1$. Prove that

$$\frac{1}{8a^2+1} + \frac{1}{8b^2+1} + \frac{1}{8c^2+1} \geq 1.$$

*Proposed by An Zhenping, Xianyang Normal University, China*

**O436.** Prove that in any triangle $ABC$ the following inequality holds:

$$\frac{a^2}{\sin\frac{A}{2}} + \frac{b^2}{\sin\frac{B}{2}} + \frac{c^2}{\sin\frac{C}{2}} \geq \frac{8}{3}s^2.$$

*Proposed by Dragoljub Miloševič, Gornji Milanovac, Serbia*

**O437.** Let $a$, $b$, $c$ be the side-lengths of a triangle $ABC$. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \leq \frac{2s^2}{27r^2} + 1.$$

*Proposed by Mircea Lascu and Titu Zvonaru, România*

**O438.** Let $a$, $b$, $c$ be positive numbers such that

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = \frac{49}{4}.$$

Find all possible values of the expression

$$E = \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}.$$

*Proposed by Marius Stănean, Zalău, România*