Junior problems

J427. Find all complex numbers x, y, z which satisfy simultaneously the equations:

$$x + y + z = 1, \qquad x^3 + y^3 + z^3 = 1, \qquad x^2 + 2yz = 4.$$

Proposed by Mircea Becheanu, University of Bucharest, Romania

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy

$$x^3 + y^3 + (1 - x - y)^3 = 1 \iff (x + y)(x + y - 1 - xy) = 0 \tag{1}$$

If $y = -x$ we have $z = 1$ and from the third equation we get

$$x^2 - 2x - 4 = 0 \iff x = 1 \pm i\sqrt{5}$$

so here the solutions are

$$(x, y, z) = (1 - i\sqrt{5}, -1 + i\sqrt{5}, 1), \quad (1 + i\sqrt{5}, -1 - i\sqrt{5}, 1)$$

From (1) we have also

$$x + y + 1 - xy = 0 \iff x(1 - y) = 1 - y \tag{2}$$

If $y = 1$ we get $x = -z$ and from the third equation we have

$$x^2 - 2x = 4 \iff x = 1 \pm i\sqrt{5}$$

so the solutions we get are

$$(x, y, z) = (1 + i\sqrt{5}, 1, -1 - i\sqrt{5}), \quad (1 - i\sqrt{5}, 1, -1 + i\sqrt{5})$$

If in (2) we take $x = 1$ we get $y = -z$ and from the third equation

$$1 - 2y^2 = 4 \iff y = \pm i\sqrt{3/2}$$

so the solutions are

$$(x, y, z) = \left(1, i\sqrt{\frac{3}{2}}, -i\sqrt{\frac{3}{2}}\right), \quad \left((1, -i\sqrt{\frac{3}{2}}, i\sqrt{\frac{3}{2}}\right)$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Arkady Alt, San Jose, CA, USA; Kelvin Kim, St. George's School, RI, USA; Konstantinos Kritharidis, Evangelliki Model School of Smyrna, Athens, Greece; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Oana Prajitura, University of Pittsburgh, PA, USA; Naïm Mégarbané, UPMC, Paris, France; Soohyun Ahn, Middlesex School, Concord, MA, USA; Akash Singha Roy, Kolkata, India; Joehyun Kim, Fort Lee High School, NJ, USA; Erica Choi, Blair Academy, Blairstown, NJ, USA; Polyahedra, Polk State College, FL, USA; Joonsoo Lee, Dwight Englewood School, NJ, USA; Yejin Kim, The Taft School, Watertown, CT, USA; Adrienne Ko, Fieldston School, New York, NY, USA; Celine Lee, Chinese International School, Hong Kong; Bekhzod Kurbonboev, NUUz, Tashkent, Uzbekistan; Luca Ferrigno, Universitá degli studi di Tor Vergata, Roma, Italy; Nikos Kalapodis, Patras, Greece; Seo Yeong Kwag, Academy, Blairstown, NJ, USA; Pedro Acosta De Leon, Massachusetts Institute of Technology, Cambridge, MA, USA; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

J428. Solve the equation

$$2x[x] + 2\{x\} = 2017,$$

where $[a]$ denotes the greatest integer not greater than a and $\{a\}$ is the fractional part of a.

Proposed by Adrian Andreescu, Dallas, Texas

Solution by Oana Prajitura, University of Pittsburgh, PA, USA

$$2x[x] + 2\{x\} = 2017 \iff 2([x] + \{x\})[x] + 2\{x\} = 2017$$

$$\iff 2[x]^2 + 2[x]\{x\} + 2\{x\} = 2017 \iff 2\{x\}([x] + 1) = 2017 - 2[x]^2$$

If $[x] = -1$ the equation becomes $0 = 2019$, a contradiction. Thus $[x] \neq 1$ and so

$$\{x\} = \frac{2017 - 2[x]^2}{2([x] + 1)}$$

which implies that

$$0 \leq \frac{2017 - 2[x]^2}{2([x] + 1)} < 1.$$

If $[x] \geq 0$,

$$0 \leq \frac{2017 - 2[x]^2}{2([x] + 1)} \iff 2017 - 2[x]^2 \geq 0 \iff [x]^2 \leq 1008 \iff 0 \leq [x] \leq 31.$$

and

$$\frac{2017 - 2[x]^2}{2([x] + 1)} < 1 \iff 2017 - 2[x]^2 < 2[x] + 2 \iff 2[x]^2 + 2[x] - 2015 > 0$$

$$\iff [x] > \frac{-1 + \sqrt{4031}}{2} \iff [x] \geq 32.$$

Thus, in this case there is no solution. If $[x] \leq -2$,

$$0 \leq \frac{2017 - 2[x]^2}{2([x] + 1)} \iff 2017 - 2[x]^2 \leq 0 \iff [x]^2 \geq 1009 \iff [x] \leq -32.$$

and

$$\frac{2017 - 2[x]^2}{2([x] + 1)} < 1 \iff 2017 - 2[x]^2 > 2[x] + 2 \iff 2[x]^2 + 2[x] - 2015 < 0$$

$$\frac{-1 - \sqrt{4031}}{2} < x \leq -2 \iff -32 \leq [x] \leq -2$$

Thus $[x] = -32$ and

$$\{x\} = .5,$$

which gives us $x = -31.5$.

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J429. Let x, y be positive real numbers such that $x + y \le 1$. Prove that

$$\left(1 - \frac{1}{x^3}\right)\left(1 - \frac{1}{y^3}\right) \ge 49.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Using the given condition and AM-GM inequality we get

$$\begin{aligned} 1 \ge x + y \ \Leftrightarrow \ 1 \ge (x+y)^3 &= x^3 + y^3 + 3xy(x+y) \cdot 1 \\ &\ge x^3 + y^3 + 3xy(x+y)^4 \\ &\ge x^3 + y^3 + 3xy(2\sqrt{xy})^4 \\ &= x^3 + y^3 + 48x^3y^3, \end{aligned}$$

$$1 \ge x^3 + y^3 + 48x^3y^3 \ \Leftrightarrow \ (1 - x^3)(1 - y^3) \ge 49x^3y^3.$$

and we are done. Given equation equality holds only when $x = y = \frac{1}{2}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Naïm Mégarbané, UPMC, Paris, France; Soohyun Ahn, Middlesex School, Concord, MA, USA; Akash Singha Roy, Kolkata, India; Joehyun Kim, Fort Lee High School, NJ, USA; Erica Choi, Blair Academy, Blairstown, NJ, USA; Polyahedra, Polk State College, FL, USA; Joonsoo Lee, Dwight Englewood School, NJ, USA; Yejin Kim, The Taft School, Watertown, CT, USA; Adrienne Ko, Fieldston School, New York, NY, USA; Celine Lee, Chinese International School, Hong Kong; Bekhzod Kurbonboev, NUUz, Tashkent, Uzbekistan; Nikos Kalapodis, Patras, Greece; Daniel Cortild; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Arkady Alt, San Jose, CA, USA; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Konstantinos Kritharidis, Evangelliki Model School of Smyrna, Athens, Greece; Kunihiko Chikaya, Tokyo, Japan; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Fong Ho Leung, Hoi Ping Chamber of Commerce Secondary School, Hong Kong; Michael Tang, MN, USA; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

J430. In triangle ABC, $\angle C > 90°$ and $3a + \sqrt{15ab} + 5b = 7c$. Prove that $\angle C \leq 120°$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, USA

By the Cauchy-Schwarz inequality,

$$a^2 + b^2 + ab = \left(\frac{9}{49} + \frac{15}{49} + \frac{25}{49}\right)\left(a^2 + ab + b^2\right) \geq \left(\frac{3a}{7} + \frac{\sqrt{15ab}}{7} + \frac{5b}{7}\right)^2 = c^2.$$

Thus $\cos C = \frac{a^2+b^2-c^2}{2ab} \geq -\frac{1}{2}$, so $C \leq 120°$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Naïm Mégarbané, UPMC, Paris, France; Soohyun Ahn, Middlesex School, Concord, MA, USA; Akash Singha Roy, Kolkata, India; Joehyun Kim, Fort Lee High School, NJ, USA; Erica Choi, Blair Academy, Blairstown, NJ, USA; Yejin Kim, The Taft School, Watertown, CT, USA; Adrienne Ko, Fieldston School, New York, NY, USA; Celine Lee, Chinese International School, Hong Kong; Nikos Kalapodis, Patras, Greece; Seo Yeong Kwag, Academy, Blairstown, NJ, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Albert Stadler, Herrliberg, Switzerland.

J431. Let a, b, c, d, e be real numbers in the interval $[1, 2]$. Prove that

$$a^2 + b^2 + c^2 + d^2 + e^2 - 3abcde \le 2.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Albert Stadler, Herrliberg, Switzerland

We need to prove that

$$2 + 3(a+1)(b+1)(c+1)(d+1)(e+1) - (a+1)^2 - (b+1)^2 - (c+1)^2 - (d+1)^2 - (e+1)^2 \ge 0$$

if $0 \le a, b, c, d, e \le 1$. Indeed,

$$2 + 3(a+1)(b+1)(c+1)(d+1)(e+1) - (a+1)^2 - (b+1)^2 - (c+1)^2 - (d+1)^2 - (e+1)^2 \ge$$

$$\ge 3(a+b+c+d+e) - a^2 - b^2 - c^2 - d^2 - e^2 - 2(a+b+c+d+e) =$$

$$= a(1-a) + b(1-b) + c(1-c) + d(1-d) + e(1-e) \ge 0.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Naïm Mégarbané, UPMC, Paris, France; Soohyun Ahn, Middlesex School, Concord, MA, USA; Akash Singha Roy, Kolkata, India; Joehyun Kim, Fort Lee High School, NJ, USA; Erica Choi, Blair Academy, Blairstown, NJ, USA; Polyahedra, Polk State College, USA; Yejin Kim, The Taft School, Watertown, CT, USA; Adrienne Ko, Fieldston School, New York, NY, USA; Nikos Kalapodis, Patras, Greece; Chanyeol Paul Kim, Seoul International School, Seoul, South Korea; Fong Ho Leung, Hoi Ping Chamber of Commerce Secondary School, Hong Kong; Michael Tang, MN, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Arkady Alt, San Jose, CA, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Bekhzod Kurbonboev, NUUz, Tashkent, Uzbekistan; Joonsoo Lee, Dwight Englewood School, NJ, USA; Titu Zvonaru, Comănești, Romania.

J432. Let m and n be integers greater than 1. Prove that

$$\left(m^3 - 1\right)\left(n^3 - 1\right) \geq 3m^2n^2 + 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain
Consider the substitution $m = 1 + x$, and $n = 1 + y$. After some algebra the proposed inequality becomes

$$x^3y^3 + 3x^3y^2 + 3x^3y + 3x^2y^3 + 6x^2y^2 + 3x^2y - 3x^2 + 3xy^3 + 3xy^2 - 3xy - 6x - 3y^2 - 6y - 4 \geq 0$$

which it is equivalent to

$$(xy - 1)\left(x^2y^2 + 3x^2y + 3x^2 + 3xy^2 + 7xy + 6x + 3y^2 + 6y + 4\right) \geq 0$$

which it is true, since $x, y > 0$ are integers.

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S427. Solve in complex numbers the system of equations:

$$z + \frac{2017}{w} = 4 - i,$$

$$w + \frac{2018}{z} = 4 + i.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Joseph Currier, SUNY Binghamton, NY, USA

Solving for z in the first equation yields $z = 4 - i - \frac{2017}{w}$ and substituting that value into the second equation yields $w + \frac{2018}{w(4-i)-2017} = 4 + i$. Clearing fractions gives a quadratic equation in terms of w:

$$w^2(4 - i) - 16w + 2017(4 + i) = 0$$

The solutions to the above equation are $w = -9 + 44i$ and $w = \frac{217}{17} - \frac{732}{17}i$. Hence the solutions to the system of equations are

$$(w, z) = (-9 + 44i, 13 + 43i)$$

and

$$(w, z) = \left(\frac{217}{17} - \frac{732}{17}i, \frac{-149}{17} - \frac{749}{17}i\right)$$

Also solved by Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Kolkata, India; Joehyun Kim, Fort Lee High School, NJ, USA; Erica Choi, Blair Academy, Blairstown, NJ, USA; Yejin Kim, The Taft School, Watertown, CT, USA; Celine Lee, Chinese International School, Hong Kong; Seo Yeong Kwag, Academy, Blairstown, NJ, USA; Pedro Acosta De Leon, Massachusetts Institute of Technology, Cambridge, MA, USA; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Xingze Xu, Hangzhou Foreign Languages School A-level Centre, China; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.

S428. Let a,b,c be nonnegative real numbers, not all zero, such that $ab+bc+ca=a+b+c$. Prove that

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\leq\frac{5}{3}.$$

Proposed by An Zenping, Xianyang Normal University, China

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\leq\frac{5}{3}\;\Leftrightarrow\;a+b+c+5abc\geq 4 \tag{1}$$

Using Schur's inequality, we get

$$a^2+b^2+c^2+\frac{9abc}{a+b+c}\geq 2(ab+bc+ca).$$

Deduce that

$$(a+b+c)^2+\frac{9abc}{a+b+c}\geq 4(ab+bc+ca) \tag{2}$$

using $a+b+c=ab+bc+ca$ identities, we get

$$(2)\quad\Leftrightarrow\quad a+b+c+\frac{9abc}{(a+b+c)^2}\geq 4 \tag{3}$$

Other hand, we have

$$\begin{aligned}(a+b+c)^2&=a^2+b^2+c^2+2(ab+bc+ca)\\&\geq(ab+bc+ca)+2(ab+bc+ca)\\&=3(ab+bc+ca).\end{aligned}$$

and $a+b+c=ab+bc+ca$, hence we have $a+b+c\geq 3$.

Thus we get

$$(a+b+c)^2\geq +9\quad\Leftrightarrow\quad\frac{9}{(a+b+c)^2}\leq 1 \tag{4}$$

From (3) and (4), we get

$$a+b+c+abc\geq 4.$$

Using

$$a+b+c+5abc\geq a+b+c+abc$$

we have $a+b+c+5abc\geq 4$. Equality holds only when, $\{a,b,c\}=\{2,2,0\}$.

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S429. Let ABC be a triangle and let M be a point in its plane. Prove that for all positive real numbers x, y, z the following inequality holds

$$xMA^2 + yMB^2 + zMC^2 > \frac{yz}{2(y+z)}a^2 + \frac{zx}{2(z+x)}b^2 + \frac{xy}{2(x+y)}c^2$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina

Using Triangle inequality we have

$$MB + MC \geq BC = a\,(1)$$

Using Cauchy-Shwarz inequality we have

$$\left(\frac{1}{y} + \frac{1}{z}\right)(yMB^2 + zMC^2) \geq (MB + MC)^2 \geq^{(1)} a^2 \Rightarrow$$

$$yMB^2 + zMC^2 \geq (MB + MC)^2 \geq \frac{yz}{y+z}a^2$$

Similarly we obtain

$$zMC^2 + xMA^2 \geq (MC + MA)^2 \geq \frac{zx}{z+x}b^2$$

$$xMA^2 + yMB^2 \geq (MA + MB)^2 \geq \frac{xy}{x+y}c^2$$

Adding these three inequalities and dividing by 2 we obtain

$$xMA^2 + yMB^2 + zMC^2 \geq \frac{yz}{2(y+z)}a^2 + \frac{zx}{2(z+x)}b^2 + \frac{xy}{2(x+y)}c^2$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Kolkata, India; Joehyun Kim, Fort Lee High School, NJ, USA; Adrienne Ko, Fieldston School, New York, NY, USA; Celine Lee, Chinese International School, Hong Kong; Albert Stadler, Herrliberg, Switzerland; Kevin Soto Palacios, Huarmey, Perú; Titu Zvonaru, Comănești, Romania.

S430. Prove that

$$\sin \frac{\pi}{2n} \geq \frac{1}{n},$$

for all positive integers n.

Proposed by Florin Rotaru, Focşani, Romania

Solution by Arkady Alt, San Jose, CA, USA

Note that $f(x) := \dfrac{\sin x}{x}$ is decreasing function on $(0, \pi/2]$.

Indeed, $f'(x) = \dfrac{x \cos x - \sin x}{x^2} < 0$ for any $x \in (0, \pi/2]$.Hence, $f(x) \geq f(\pi/2) = \dfrac{\sin \frac{\pi}{2}}{\frac{\pi}{2}} \iff$

$$\frac{\sin \frac{\pi}{2n}}{\frac{\pi}{2n}} \geq \frac{1}{\frac{\pi}{2}} \iff \sin \frac{\pi}{2n} \geq \frac{\pi}{2n} \cdot \frac{2}{\pi} = \frac{1}{n}$$

for any $x \in (0, \pi/2]$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Kolkata, India; Joehyun Kim, Fort Lee High School, NJ, USA; Erica Choi, Blair Academy, Blairstown, NJ, USA; Yejin Kim, The Taft School, Watertown, CT, USA; Adrienne Ko, Fieldston School, New York, NY, USA; Bekhzod Kurbonboev, NUUz, Tashkent, Uzbekistan; Pedro Acosta De Leon, Massachusetts Institute of Technology, Cambridge, MA, USA; Chanyeol Paul Kim, Seoul International School, Seoul, South Korea; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Albert Stadler, Herrliberg, Switzerland; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Joel Schlosberg, Bayside, NY, USA; José Hernández Santiago, México; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Titu Zvonaru, Comăneşti, Romania.

S431. Let a, b, c be positive numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} \geq \frac{3}{4}.$$

Proposed by Konstantinos Metaxas, Athens, Greece

Solution by Daniel Lasaosa, Pamplona, Spain

Define $s = a+b+c$ and $p = abc$, or after multiplying both sides of the proposed equation by $4(1+a)^2(1+b)^2(1+c)^2$ and rearranging terms, results in the equivalent inequality

$$0 \leq 5s^2 + 16s - 14ps - 3p^2 - 48p = (5s + p - 16)(s - 3p).$$

Now, by the AM-GM inequality we have $ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2}$ and $a + b + c \geq 3\sqrt[3]{abc}$, and using that $ab + bc + ca = 3$, we have

$$3s = (ab + bc + ca)(a + b + c) \geq 9abc,$$

or $s \geq 3p$ with equality iff $a = b = c = 1$. It then suffices to show that $5s + p \geq 16$, where by the scalar product inequality we have $s^2 - 6 = a^2 + b^2 + c^2 \geq 3$, or $s \geq 3$, and by previous results we have $\sqrt[3]{a^2b^2c^2} \leq 1$, or $abc \leq 1$. Now, denoting $s = 3 + \Delta$ for some nonnegative real Δ, note that

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)\left(a^2 + b^2 + c^2 - ab - bc - ca\right) =$$

$$= s\left(s^2 - 3(ab + bc + ca)\right) = 18\Delta + 9\Delta^2 + \Delta^3,$$

and by Schur's inequality,

$$0 \leq a(a - b)(a - c) + b(b - c)(b - a) + c(c - a)(c - b) =$$

$$= a^3 + b^3 + c^3 + 6abc - (a + b + c)(ab + bc + ca),$$

or

$$3p = a^3 + b^3 + c^3 - 18\Delta - 9\Delta^2 - \Delta^3 \geq 3s - 6p - 18\Delta - 9\Delta^2 - \Delta^3,$$

and finally

$$p \geq 1 - \frac{15}{9}\Delta - \Delta^2 - \frac{\Delta^3}{9}.$$

Now, if $\Delta \leq 1$, then $p \geq 1 - \Delta\left(\frac{15}{9} + 1 + \frac{1}{9}\right) \geq 1 - 3\Delta$, and $5s + p \geq 16 + 2\Delta \geq 16$, with equality iff $\Delta = 0$, whereas if $\Delta > 1$, then $s > 4$ and $5s + p \geq 5s > 20 > 16$, and the inequality holds strictly. The conclusion follows, equality holds iff $\Delta = 0$, or iff $a = b = c = 1$.

Also solved by Joehyun Kim, Fort Lee High School, NJ, USA; Akash Singha Roy, Kolkata, India; Yejin Kim, The Taft School, Watertown, CT, USA; Adrienne Ko, Fieldston School, New York, NY, USA; Celine Lee, Chinese International School, Hong Kong; Nikos Kalapodis, Patras, Greece; Seo Yeong Kwag, Academy, Blairstown, NJ, USA; Albert Stadler, Herrliberg, Switzerland; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Arkady Alt, San Jose, CA, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Ioan Viorel Codreanu, Satulung, Maramureș, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Paraskevi-Andrianna Maroutsou, Charters Sixth Form, Sunningdale, England, UK; Giurgi Vasile, C.N. Dragos-Voda, Sighetu Marmatiei, Romania; Titu Zvonaru, Comănești, Romania.

S432. Let d be an open half-disk of diameter AB and h be the half-plane defined by the line AB and containing d. Let X be a point on d and let Y and Z be points in h on the semicircles of diameters AX and BX, respectively. Prove that

$$AY \cdot BZ + XY \cdot XZ \le AX^2 - AX \cdot BX + BX^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, CA, USA

Let $a := AX, b := BX, y := AY, z := BZ, u := XY, v := XZ$.

Then inequality becomes

$$yz + uv \le a^2 + b^2 - ab.$$

Since $y^2 + u^2 = a^2$ and $z^2 + v^2 = b^2$ (because $\angle AYX = \angle BZX = 90°$) then by Cauchy Inequality

$$ab = \sqrt{y^2 + u^2} \cdot \sqrt{z^2 + v^2} \ge yz + uv$$

and also

$$a^2 + b^2 - ab \ge ab.$$

Hence, $a^2 + b^2 - ab \ge ab \ge yz + uv$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Kolkata, India; Joehyun Kim, Fort Lee High School, NJ, USA; Erica Choi, Blair Academy, Blairstown, NJ, USA; Adrienne Ko, Fieldston School, New York, NY, USA; Celine Lee, Chinese International School, Hong Kong; Seo Yeong Kwag, Academy, Blairstown, NJ, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Titu Zvonaru, Comănești, Romania.

U427. Let $f:\mathbb{R}^2\to\mathbb{R}$ be the function defined by

$$f(x,y)=\mathbf{1}_{\left(0,\frac{1}{y}\right)}(x)\cdot\mathbf{1}_{(0,1)}(y)\cdot y,$$

where $\mathbf{1}$ is the characteristic function. Evaluate

$$\int_{\mathbb{R}^2} f(x,y)dxdy.$$

Proposed by Alessandro Ventulo, Milan, Italy

Solution by Henry Ricardo, Westchester Area Math Circle, NY, USA

We have

$$f(x,y)=\mathbf{1}_{(0,1/y)}(x)\cdot\mathbf{1}_{(0,1)}(y)\cdot y=\left\{\begin{array}{ll} 1 & \text{if } y\in(0,1) \text{ and } 0<x<1/y \\ 0 & \text{if } y\notin(0,1) \text{ or } x\notin(0,1/y)\end{array}\right\}\cdot y.$$

Therefore

$$\int_{\mathbb{R}^2} f(x,y)\;dx\;dy=\int_0^1\int_0^{1/y} y\;dx\;dy=\int_0^1 y\left(\int_0^{1/y}1\;dx\right)dy=\int_0^1 y\cdot\frac{1}{y}\;dy=1.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Kolkata, India; Joehyun Kim, Fort Lee High School, NJ, USA; Yejin Kim, The Taft School, Watertown, CT, USA; Albert Stadler, Herrliberg, Switzerland; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Naïm Mégarbané, UPMC, Paris, France.

U428. Let a, b, c positive real numbers such that $a+b+c=1$. Prove that

$$\left(1+a^2b^2\right)^c\left(1+b^2c^2\right)^a\left(1+c^2a^2\right)^b \geq 1+9a^2b^2c^2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain

Taking natural logarithms on both sides, the proposed inequality is equivalent to

$$a\ln\left(1+b^2c^2\right)+b\ln\left(1+c^2a^2\right)+c\ln\left(a^2+b^2\right) \geq \ln\left(1+9a^2b^2c^2\right).$$

Consider now function $f(x)=\ln\left(1+x^2\right)$, whose first and second derivatives are respectively

$$f'(x)=\frac{2x}{1+x^2}, \qquad\qquad f''(x)=\frac{2\left(1-x^2\right)}{1+x^2}.$$

Now, since $a+b+c=1$ and a, b, c are positive, we have $a, b, c<1$, or $ab, bc, ca<1$, and function $f(x)$ is strictly convex in an interval which contains ab, bc, ca. It follows from Jensen's inequality that

$$a\ln\left(1+b^2c^2\right)+b\ln\left(1+c^2a^2\right)+c\ln\left(a^2+b^2\right)=af(bc)+bf(ca)+cf(ab) \geq$$

$$\geq (a+b+c)f\left(\frac{3abc}{a+b+c}\right)=f(3abc)=\ln\left(1+9a^2b^2c^2\right),$$

where we have used that $a+b+c=1$, and where equality holds iff $ab=bc=ca$, ie iff $a=b=c$. The conclusion follows, equality holds iff $a=b=c=\frac{1}{3}$.

Also solved by Albert Stadler, Herrliberg, Switzerland; Yejin Kim, The Taft School, Watertown, CT, USA; Akash Singha Roy, Kolkata, India; Joehyun Kim, Fort Lee High School, NJ, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy.

U429. Let $n \geq 2$ be an integer and let A be an $n \times n$ real matrix in which exactly $(n-1)^2$ entries are zero. Prove that if B is an $n \times n$ matrix with all entries nonzero numbers, then BA can not be a nonsingular diagonal matrix.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina

Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$, the by the condition $b_{ij} \neq 0, \forall i, j$

Since there are n^2 entries at A and $(n-1)^2$ of them are zero, there are

$$n^2 - (n-1)^2 = 2n - 1$$

nonzero entries at A.If all the columns have at least two nonzero elements then the total number of nonzero elements is at least $2n > 2n - 1$ which is contradiction.Hence there is a column which contains at most one nonzero element, let it be A_j,the $j-$th column of A. If all the entries of A_j are zero then all the entries of $j-$th column of the prodact BA are zeros, hence the matrix BA is singular.Suppose A_j has one nonzero entry, say a_{ij}.Then the $j-$th column of the product BA is $\begin{pmatrix} b_{1i} \cdot a_{ij} \\ b_{2i} \cdot a_{ij} \\ \vdots \\ b_{ni} \cdot a_{ij} \end{pmatrix}$.Since by the condition we have

$$\prod_{k=1}^{n} b_{ki} \neq 0$$

and the matrix is diagonal, so at least two of the $b_{1i} \cdot a_{ij}, b_{2i} \cdot a_{ij}, ..., b_{ni} \cdot a_{ij}$ are zero then the entry a_{ij} equals to zero, so rezulting product BA is a singular matrix.

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Kolkata, India; Deep Ghoshal, Indian Statistical Institute, Kolkata, India.

U430. Let A and B be 3×3 matrices with complex numbers entries, such that

$$(AB - BA)^2 = AB - BA.$$

Prove that $AB = BA$.

Proposed by Florin Stănescu, Găeşti, Romania

Solution by Albert Stadler, Herrliberg, Switzerland

We note that the trace of $AB - BA$ equals $tr(AB - BA) = tr(AB) - tr(BA) = 0$.

We will prove a slightly more general result: Let C be a 3×3 matrix with complex number entries and trace 0 such that $C^2 = C$. Then $C = 0$.

If λ is an eigenvalue of C and x an associated eigenvector then $C^2 x = \lambda^2 x = Cx = \lambda x$. So $\lambda \in \{0, 1\}$. $tr(C) = 0$ equals the sum of the eigenvalues of C. Therefore $\lambda = 0$ is the only eigenvalue of C. C is similar to an upper triangular matrix whose diagonal entries are the eigenvalues of C. Hence there are complex numbers r, s, t such that C is similar to

$$C \sim \begin{pmatrix} 0 & r & t \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix}$$

Then

$$C^2 \sim \begin{pmatrix} 0 & 0 & rs \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

However $C^2 = C$, so $r = s = 0, t = rs = 0$. We conclude that $C = 0$.

Also solved by Pop Ovidiu Florin, C.N. Dragos-Voda, Sighetu Marmatiei, Ramania; Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Kolkata, India; Joehyun Kim, Fort Lee High School, NJ, USA; Yejin Kim, The Taft School, Watertown, CT, USA; Seo Yeong Kwag, Academy, Blairstown, NJ, USA; Fong Ho Leung, Hoi Ping Chamber of Commerce Secondary School, Hong Kong; Michael Tang, MN, USA; P.V.Swaminathan, Smart Minds Academy, Chennai, India; Naïm Mégarbané, UPMC, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Moubinool Omarjee, Lycée Henri IV, Paris, France.

U431. Evaluate

$$\lim_{t\to 0}\frac{1}{t}\int_0^t \sqrt{1+e^x}dx \qquad \text{and} \qquad \lim_{t\to 0}\frac{1}{t}\int_0^t e^{e^x}dx.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Henry Ricardo, Westchester Area Math Circle, NY, USA

Since $\lim_{t\to 0}\int_0^t \sqrt{1+e^x}dx = 0$, L'Hospital's Rule and the Fundamental Theorem of Calculus give us

$$\lim_{t\to 0}\frac{\int_0^t \sqrt{1+e^x}dx}{t} = \lim_{t\to 0}\frac{\sqrt{1+e^t}}{1} = \sqrt{2}.$$

Similarly, we see that

$$\lim_{t\to 0}\frac{\int_0^t e^{e^x}dx}{t} = \lim_{t\to 0}\frac{e^{e^t}}{1} = e.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Kolkata, India; Joehyun Kim, Fort Lee High School, NJ, USA; Erica Choi, Blair Academy, Blairstown, NJ, USA; Yejin Kim, The Taft School, Watertown, CT, USA; Bekhzod Kurbonboev, NUUz, Tashkent, Uzbekistan; Luca Ferrigno, Universitá degli studi di Tor Vergata, Roma, Italy; Albert Stadler, Herrliberg, Switzerland; Naïm Mégarbané, UPMC, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Arkady Alt, San Jose, CA, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Moubinool Omarjee, Lycée Henri IV, Paris, France; Narayanan P, Vivekananda College, Chennai, India; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Giurgi Vasile, C.N. Dragos-Voda, Sighetu Marmatiei, Romania; Xingze Xu, Hangzhou Foreign Languages School A-level Centre, China.

U432. For every point $P(x,y,z)$ on the unit sphere, consider the points $Q(y,z,x)$ and $R(z,x,y)$. For every point A on the sphere, denote $\angle(AOP) = p$, $\angle(AOQ) = q$ and $\angle(AOR) = r$. Prove that

$$|\cos q - \cos r| \le 2\sqrt{3}\sin\frac{p}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

By the Cosine Law, $AQ^2 = OA^2 + OQ^2 - 2OA \cdot OQ\cos q = 2 - 2\cos q$, or $\cos q = 1 - \frac{AQ^2}{2}$, and similarly $\cos r = 1 - \frac{AR^2}{2}$. Moreover, since $OP = OA = 1$, triangle AOP is isosceles at O, and $\sin\frac{p}{2} = OA\sin\frac{p}{2} = \frac{AP}{2}$. The proposed inequality rewrites as

$$AR^2 - AQ^2 \le 2\sqrt{3}AP.$$

Note now that, under this form, we may perform a variable change since the problem has been stated in terms of distances. Now, in order to perform a correct variable change, note that

$$PQ^2 = QR^2 = RP^2 = (x-y)^2 + (y-z)^2 + (z-x)^2,$$

or we may choose a coordinate system with origin at the center of the unit sphere, such that $P \equiv (1,0,0)$, the equation $y = 0$ bisects segment QR, or $Q \equiv (d,\ell,h)$ and $R \equiv (d,-\ell,h)$, where $2\ell = PQ = QR = RP$. Note further that $d^2 + \ell^2 + h^2 = 1$ because Q, R are on the unit sphere, whereas

$$4\ell^2 = PQ^2 = RP^2 = (1-d)^2 + \ell^2 + h^2 = 2 - 2d, \qquad d = 1 - 2\ell^2, \qquad h = \ell\sqrt{3-4\ell^2}.$$

Now, the expression found for h suggests that $\ell \le \frac{\sqrt{3}}{2}$, which indeed holds, and equality means that the circumcircle of PQR is an equator of the unit sphere, which is consistent with $d = -\frac{1}{2}$ and $h = 0$. Having this in mind, note further that A is any point on this unit sphere, with coordinates $A \equiv (u,v,w)$ such that $u^2 + v^2 + w^2 = 1$, or

$$AR^2 - AQ^2 = (v+\ell)^2 - (v-\ell)^2 = 4v\ell, \qquad AP^2 = (1-u)^2 + v^2 + w^2 = 2 - 2u.$$

The proposed inequality then rewrites as

$$2v\ell \le \sqrt{6(1-u)},$$

which holds trivially if $v \le 0$. If $v \ge 0$, we may square both sides, resulting in the equivalent inequality $4v^2\ell^2 \le 6(1-u)$, where since $4\ell^2 \le 3$, it suffices to show that

$$v^2 \le 2(1-u) = 1 - 2u + u^2 + v^2 + w^2 = (1-u)^2 + w^2 + v^2,$$

trivially true and with equality iff $u = 1$ and $w = 0$, and consequently $v = 0$. Note further that, regardless of the value of ℓ, if $u = 1$ and $v = w = 0$, then $A = P$, resulting in $AP = 0$, and since $PQ = RP$, also in $AQ = AR$, or equality holds regardless of ℓ. The conclusion follows, equality holds iff $A = P$, in which case $p = 0$ and $q = r$.

Also solved by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Akash Singha Roy, Kolkata, India.

O427. Let ABC be a triangle and m_a, m_b, m_c be the lengths of its medians. Prove that

$$\sqrt{3}\,(am_a + bm_b + cm_c) \le 2s^2.$$

Proposed by Dragoljub Miloševič, Gornji Milanovac, Serbia

First solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy

The inequality is

$$\sqrt{3}(a+b+c)\sum_{\text{cyc}} \frac{a}{(a+b+c)}\sqrt{\frac{2b^2+2c^2-a^2}{4}} \le \frac{(a+b+c)^2}{2}$$

The concavity of $\sqrt{x}$ yields

$$\sqrt{3}(a+b+c)\sum_{\text{cyc}} \frac{a}{(a+b+c)}\sqrt{\frac{2b^2+2c^2-a^2}{4}} \le \sqrt{3}(a+b+c)\sqrt{\sum_{\text{cyc}} \frac{a}{(a+b+c)}\frac{2b^2+2c^2-a^2}{4}}$$

so it suffices to prove

$$\sqrt{3}(a+b+c)\sqrt{\sum_{\text{cyc}} \frac{a}{(a+b+c)}\frac{2b^2+2c^2-a^2}{4}} \le \frac{(a+b+c)^2}{2}$$

or squaring

$$3\sum_{\text{cyc}} a(2b^2+2c^2-a^2) \le (a+b+c)^3$$

Now let's set $a = y+z,\ b = x+z,\ c = x+y,\ x, y, z \ge 0$
and the inequality becomes

$$\sum_{\text{sym}} (x^3 + 3x^2y) \ge 24xyz$$

and this is clearly AGM.

Second solution by Daniel Lasaosa, Pamplona, Spain

Note first that ${m_a}^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4} = \frac{a^2+b^2+c^2}{2} - \frac{3a^2}{4}$. Define therefore

$$f(x) = \sqrt{K - x^2},$$

where $x \le \sqrt{K}$. Note that for $K = \frac{a^2+b^2+c^2}{2}$, we have $m_a = f\left(\frac{\sqrt{3}a}{2}\right)$, and similarly for b, c. Now, the first and second derivatives of $f(x)$ are

$$f'(x) = -\frac{x}{\sqrt{K - x^2}}, \qquad f''(x) = -\frac{K}{\left(\sqrt{K - x^2}\right)^3} < 0,$$

or f is strictly concave, and by Jensen's inequality,

$$\sqrt{3}\left(am_a + bm_b + cm_c\right) \le \sqrt{3}(a+b+c)f\left(\frac{\sqrt{3}\left(a^2+b^2+c^2\right)}{2(a+b+c)}\right) = \sqrt{12Ks^2 - 9K^2},$$

with equality iff $a = b = c$, or it suffices to show that

$$\sqrt{12Ks^2 - 9K^2} \le 2s^2, \qquad 0 \le 4s^4 - 12Ks^2 + 9K^2 = \left(2s^2 - 3K\right)^2,$$

clearly true. Note that condition $a = b = c$, necessary for equality in Jensen's inequality, also results in $3K = 2s^2 = \frac{9a^2}{2}$. The conclusion follows, equality holds iff ABC is equilateral.

Also solved by Albert Stadler, Herrliberg, Switzerland; Akash Singha Roy, Kolkata, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Arkady Alt, San Jose, CA, USA; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Scott H. Brown, Auburn University Montgomery, Montgomery, AL, USA.

O428. Determine all positive integers n for which the equation

$$x^2 + y^2 = n(x - y)$$

is solvable in positive integers. Solve the equation

$$x^2 + y^2 = 2017(x - y).$$

Proposed by Dorin Andrica, Cluj-Napoca, Romania and Vlad Crişan, Göttingen, Germany

Solution by Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua

$$x^2 + y^2 = n(x - y) \Leftrightarrow 4x^2 - 4nx + n^2 + 4y^2 + 4ny + n^2 = 2n^2$$

Then if $u = 2x - n$ and $v = 2y + n$, the original equation is equivalent to solve the Pythagorean equation: $u^2 + v^2 = 2n^2$.

If $\gcd(u, 2) = 2$, then $u = 2u_1$ and $4u_1^2 + v^2 = 2n^2$ implies that $v = 2v_1 \Rightarrow 2(u_1^2 + v_1^2) = n^2 \Rightarrow n = 2n_1 \Rightarrow u_1^2 + v_1^2 = 2n_1^2$, and this equation is equivalent to the original. So we can assume that u and v are odds.

Now consider $\mathbb{Z}[i]$, let $z = u + vi$, then $z\overline{z} = N(z) = u^2 + v^2 = 2n^2 = (1 + i)(1 - i)n^2$. By unique factorization in $\mathbb{Z}[i]$ implies that $1 + i|z$ or $1 - i|z$. If $1 + i|z$, then $w = \frac{z}{1+i} = \frac{z(1-i)}{2} = \frac{u+v}{2} + \frac{v-u}{2}i$. Then we have $\left(\frac{u+v}{2}\right)^2 + \left(\frac{v-u}{2}\right)^2 = N(w) = n^2$.

Then n^2 is the sum of two squares because $\frac{u+v}{2}$ and $\frac{|v-u|}{2}$ are integers, then solving this equation in positive integers we get by the Pythagorean equation that $n = a^2 + b^2$ where $a > b$. We can also get $\frac{u+v}{2}$ and $\frac{|v-u|}{2}$ from $\{2ab, a^2 - b^2\}$.

Finally the equation $x^2 + y^2 = 2017(x - y)$ has solution, because $2017 = 44^2 + 9^2$ (this pair is unique), then $\frac{u+v}{2}$ and $\frac{|v-u|}{2}$ take the values of $2 \cdot 44 \cdot 9$ or $44^2 - 9^2$. Using that $\frac{u+v}{2} = x + y$ and $\frac{|v-u|}{2} = |y - x + n|$. We have four cases:

$$\begin{cases} x + y & = 1855 \\ y - x + 2017 & = 792 \end{cases} \Rightarrow (x, y) = (1540, 315)$$

$$\begin{cases} x + y & = 1855 \\ -y + x - 2017 & = 792 \end{cases} \Rightarrow (x, y) = (2332, -477)$$

$$\begin{cases} x + y & = 792 \\ y - x + 2017 & = 1855 \end{cases} \Rightarrow (x, y) = (477, 315)$$

$$\begin{cases} x + y & = 792 \\ -y + x - 2017 & = 1855 \end{cases} \Rightarrow (x, y) = (2332, -1540)$$

And the positive integers solutions are $(x, y) \in \{(1540, 315), (477, 315)\}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Luca Ferrigno, Universitá degli studi di Tor Vergata, Roma, Italy; Akash Singha Roy, Kolkata, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Albert Stadler, Herrliberg, Switzerland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

O429. Let ABC be non-obtuse triangle.Prove that

$$m_am_b + m_bm_c + m_cm_a \le (a^2+b^2+c^2)\left(\frac{5}{8}+\frac{r}{4R}\right)$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina

Since the triangle is nonobtuse we have $a^2+b^2-c^2 \ge 0$.Then we have

$$(a^2+b^2)[(a+b)^2-c^2] - 2ab(2a^2+2b^2-c^2) = (a-b)^2(a^2+b^2-c^2) \ge 0 \Rightarrow$$

$$\sqrt{(a^2+b^2)\cdot\frac{(a+b)^2-c^2}{2ab}} \ge \sqrt{2a^2+2b^2-c^2} \tag{1}$$

Using formulas

$$m_c = \frac{1}{2}\sqrt{2a^2+2b^2-c^2}$$

$$\frac{r}{R} = \cos A + \cos B + \cos C - 1 = -1 + \frac{1}{2}\sum_{cyc}\frac{a^2+b^2-c^2}{ab}$$

we have

$$m_am_b + m_bm_c + m_cm_a \le (a^2+b^2+c^2)\left(\frac{5}{8}+\frac{r}{4R}\right) \Leftrightarrow$$

$$\frac{1}{4}\sum_{cyc}\sqrt{(2a^2+2b^2-c^2)(2b^2+2c^2-a^2)} \le (a^2+b^2+c^2)\left(\frac{5}{8}-\frac{1}{4}+\frac{1}{8}\sum_{cyc}\frac{a^2+b^2-c^2}{ab}\right) \Leftrightarrow$$

$$2\sum_{cyc}\sqrt{(2a^2+2b^2-c^2)(2b^2+2c^2-a^2)} \le (a^2+b^2+c^2)\left(3+\sum_{cyc}\frac{a^2+b^2-c^2}{ab}\right) \Leftrightarrow$$

$$2\sum_{cyc}\sqrt{(2a^2+2b^2-c^2)(2b^2+2c^2-a^2)} \le (a^2+b^2+c^2)\cdot\sum_{cyc}\frac{a^2+b^2+ab-c^2}{ab} \Leftrightarrow$$

$$\sum_{cyc}(2a^2+2b^2-c^2) + 2\sum_{cyc}\sqrt{(2a^2+2b^2-c^2)(2b^2+2c^2-a^2)} \le (a^2+b^2+c^2)\cdot\sum_{cyc}\frac{a^2+b^2+ab-c^2}{ab} + 3\sum_{cyc}a^2 \Leftrightarrow$$

$$\left(\sum_{cyc}\sqrt{2a^2+2b^2-c^2}\right)^2 \le \left(a^2+b^2+c^2\right)\cdot\left(3+\sum_{cyc}\frac{a^2+b^2+ab-c^2}{ab}\right) \Leftrightarrow$$

$$(\sum_{cyc}\sqrt{2a^2+2b^2-c^2})^2 \le (a^2+b^2+c^2)\cdot\sum_{cyc}\frac{(a+b)^2-c^2}{ab}$$

Since $a+b>c$ using Cauchy-Shwarz inequality we have

$$(a^2+b^2+c^2)\cdot\sum_{cyc}\frac{(a+b)^2-c^2}{ab} = \frac{1}{2}\sum_{cyc}(a^2+b^2)\cdot\sum_{cyc}\frac{(a+b)^2-c^2}{ab} =$$

$$= \sum_{cyc}(a^2+b^2)\cdot\sum_{cyc}\frac{(a+b)^2-c^2}{2ab} \ge \left(\sum_{cyc}\sqrt{(a^2+b^2)\cdot\frac{(a+b)^2-c^2}{2ab}}\right)^2 \ge^{(1)} \left(\sum_{cyc}\sqrt{2a^2+2b^2-c^2}\right)^2$$

Equality holds if and only if $a=b=c$

Also solved by Albert Stadler, Herrliberg, Switzerland; Kevin Soto Palacios, Huarmey, Perú; Akash Singha Roy, Kolkata, India; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Scott H. Brown, Auburn University Montgomery, Montgomery, AL, USA.

O430. Find the number of positive integers $n \le 10^6$ such that 5 divides $\binom{2n}{n}$.

Proposed by Enrique Trevinio, Lake Forest College, USA

Solution by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina
Lemma:
Let $x, y \in \mathbb{R}$, then we have

$$\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x + y \rfloor$$

Proof:
W.L.O.G let $0 \le x, y < 1$.Then $\{x\} = x, \{y\} = y$

$$\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x + y \rfloor \Leftrightarrow \{x\} + \{y\} \ge \{x + y\}$$

If $x + y < 1$ then

$$\{x\} + \{y\} = x + y = \{x + y\}$$

so equality holds.
Let $x + y \ge 1$ then

$$\lfloor x + y \rfloor = 1 \Leftrightarrow x + y = 1 + \{x + y\} \Rightarrow \{x\} + \{y\} = 1 + \{x + y\} > \{x + y\}$$

so the inequality holds, with equality only if $x + y < 1$. ■

As a corolary we have

$$2\lfloor t \rfloor \le \lfloor 2t \rfloor \tag{1}$$

With equality only if $0 < \{x\} < \frac{1}{2}$.
Let $e_p(n)$ be the greatest exponent of p dividing n.Then $e_p(n!) = \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor$, so

$$e_5(\binom{2n}{n}) = e_5(\frac{(2n)!}{(n!)^2}) = \sum_{k=1}^{\infty} (\lfloor \frac{2n}{5^k} \rfloor - 2\lfloor \frac{n}{5^k} \rfloor)$$

Using (1) we have

$$\lfloor \frac{2n}{5^k} \rfloor - 2\lfloor \frac{n}{5^k} \rfloor \ge 0 \Rightarrow \sum_{k=1}^{\infty} (\lfloor \frac{2n}{5^k} \rfloor - 2\lfloor \frac{n}{5^k} \rfloor) \ge 0$$

Equality holds if and only if

$$\lfloor \frac{2n}{5^k} \rfloor - 2\lfloor \frac{n}{5^k} \rfloor = 0, \forall k \ge 1 \Rightarrow 0 < \{\frac{n}{5^k}\} < \frac{1}{2}, \forall k \ge 1$$

Let $\binom{2n}{n}$ not divisible by 10.Then $e_5(\binom{2n}{n}) = 0$.Then we have

$$e_{10}(\binom{2n}{n}) = \sum_{k=1}^{\infty} (\lfloor \frac{2n}{5^k} \rfloor - 2\lfloor \frac{n}{5^k} \rfloor) = 0$$

So we have

$$\{\frac{n}{5^k}\} < \frac{1}{2}, \forall k \ge 1$$

Let

$$n = \sum_{k=0}^{m} c_k \cdot 5^k, c_k \in \{0,1,2,3,4\}$$

Now we have

$$\frac{1}{2} > \{\frac{n}{5}\} = \frac{c_0}{5} \Rightarrow c_0 < \frac{5}{2} \Rightarrow c_0 \in \{0,1,2\}$$

$$\frac{1}{2} > \{\frac{n}{5^2}\} = \frac{5c_1 + c_0}{25} \Rightarrow c_1 < \frac{25 - 2c_0}{10} \Rightarrow c_1 \in \{0,1,2\}$$

$$\frac{1}{2} > \{\frac{n}{5^3}\} = \frac{25c_2 + 5c_1 + c_0}{125} \Rightarrow c_2 < \frac{125 - 10c_1 - 2c_0}{50} \Rightarrow c_2 \in \{0,1,2\}$$

$$\vdots$$

$$\frac{1}{2} > \{\frac{n}{5^m}\} = \frac{5^{m-1}c_{m-1} + ... + c_0}{5^m} \Rightarrow c_{m-1} < \frac{5^m - 2\sum_{k=0}^{m-2} c_k \cdot 5^k}{2 \cdot 5^{m-1}} \Rightarrow c_{m-1} \in \{0,1,2\}$$

$$\frac{1}{2} > \{\frac{n}{5^{m+1}}\} = \frac{5^m c_m + ... + c_0}{5^{m+1}} \Rightarrow c_m < \frac{5^{m+1} - 2\sum_{k=0}^{m-1} c_k \cdot 5^k}{2 \cdot 5^m} \Rightarrow c_m \in \{0,1,2\}$$

So, positive integer n , such that $\binom{2n}{n}$ is not divisible by 5 is of the form

$$n = \sum_{k=0}^{m} c_k \cdot 5^k, c_k \in \{0,1,2\}$$

Since

$$10^6 = 2 \cdot 5^8 + 2 \cdot 5^7 + 4 \cdot 5^6 + 0 \cdot 5^5 + 0 \cdot 5^4 + 0 \cdot 5^3 + 0 \cdot 5^2 + 0 \cdot 5^1 + 0 \cdot 5^0$$

All the numbers of the form

$$n = \sum_{k=0}^{8} c_k \cdot 5^k, c_k \in \{0,1,2\}$$

are included, and excluding zero the total number of such is $3^8 - 1$.
Hence the total number of n such that $\binom{2n}{n}$ is divisible by 5 is

$$10^6 - 3^8 + 1 = 993440$$

Also solved by Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Kolkata, India.

O431. Let a, b, c, d be positive real numbers such that $a + b + c + d = 3$. Prove that

$$a^2 + b^2 + c^2 + d^2 + \frac{64}{27}abcd \geq 3.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina

Using Schurs inequality we have

$$\begin{matrix} a^3 + b^3 + c^3 + 3abc \geq a^2(b + c) + b^2(c + a) + c^2(a + b) \\ b^3 + c^3 + d^3 + 3bcd \geq b^2(c + d) + c^2(d + b) + d^2(b + c) \\ a^3 + c^3 + d^3 + 3acd \geq a^2(c + d) + c^2(d + a) + d^2(a + c) \\ a^3 + b^3 + d^3 + 3abd \geq a^2(b + d) + b^2(d + a) + d^2(a + b) \end{matrix} \Rightarrow$$

$$\begin{matrix} a^3d + b^3d + c^3d + 3abcd \geq a^2d(b + c) + b^2d(c + a) + c^2d(a + b) \\ b^3a + c^3a + d^3a + 3abcd \geq b^2a(c + d) + c^2a(d + b) + d^2a(b + c) \\ a^3b + c^3b + d^3b + 3abcd \geq a^2b(c + d) + c^2b(d + a) + d^2b(a + c) \\ a^3c + b^3c + d^3c + 3abcd \geq a^2c(b + d) + b^2c(d + a) + d^2c(a + b) \end{matrix} \Rightarrow$$

$$\sum_{cyc} a^3(b + c + d) + 12abcd \geq 2\sum_{cyc} a^2(bc + cd + db) \Leftrightarrow$$

$$5\sum_{cyc} a^3(b + c + d) + 60abcd \geq 10\sum_{cyc} a^2(bc + cd + db) \tag{1}$$

where the cyclic sum runs over the set $\{a, b, c, d\}$.

Also using Schurs inequality we have

$$\begin{matrix} a^4 + b^4 + c^4 + abc(a + b + c) \geq a^3(b + c) + b^3(c + a) + c^3(a + b) \\ b^4 + c^4 + d^4 + bcd(b + c + d) \geq b^3(c + d) + c^3(d + b) + d^3(b + c) \\ a^4 + c^4 + d^4 + acd(a + c + d) \geq a^3(c + d) + c^3(d + a) + d^3(a + c) \\ a^4 + b^4 + d^4 + abd(a + b + d) \geq a^3(b + d) + b^3(d + a) + d^3(a + b) \end{matrix} \Rightarrow$$

$$3\sum_{cyc} a^4 + \sum_{cyc} a^2(bc + cd + db) \geq 2\sum_{cyc} a^3(b + c + d) \tag{2}$$

Adding (1) and (2) we have

$$3\sum_{cyc} a^4 + 3\sum_{cyc} a^3(b + c + d) + 60abcd \geq 9\sum_{cyc} a^2(bc + cd + db) \Leftrightarrow$$

$$2\sum_{cyc} a^4 + 2\sum_{cyc} a^3(b + c + d) + 40abcd \geq 6\sum_{cyc} a^2(bc + cd + db) \Leftrightarrow$$

$$3[\sum_{cyc} a^4 + 2\sum_{cyc} a^3(b + c + d) + 2\sum_{cyc} a^2(bc + cd + db) + \sum_{cyc} a^2(b^2 + c^2 + d^2)] + 64abcd \geq$$

$$\geq \sum_{cyc} a^4 + 4\sum_{cyc} a^3(b + c + d) + 12\sum_{cyc} a^2(bc + cd + db) + 3\sum_{cyc} a^2(b^2 + c^2 + d^2) + 24abcd \Leftrightarrow$$

$$3\sum_{cyc} a^2 \cdot (\sum_{cyc} a)^2 + 64abcd \geq (\sum_{cyc} a)^4 \Leftrightarrow$$

$$27\sum_{cyc} a^2 + 64abcd \geq 81 \Leftrightarrow \sum_{cyc} a^2 + \frac{64}{27}abcd \geq 3$$

Equality holds if and only if $a = b = c = d = \dfrac{3}{4}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Albert Stadler, Herrliberg, Switzerland; Akash Singha Roy, Kolkata, India; Arkady Alt, San Jose, CA, USA; Ioan Viorel Codreanu, Satulung, Maramureș, Romania; Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy.

O432. Let $ABCDEF$ be a cyclic hexagon which contains an inscribed circle. Denote by $\omega_A, \omega_B, \omega_C, \omega_D, \omega_E$ and ω_F the inscribed circle in the triangle FAB, ABC, BCD, CDE, DEF and EFA, respectively. Let ℓ_{AB} be the external common tangent of ω_A and ω_B, other than the line AB; lines ℓ_{BC},ℓ_{CD},ℓ_{DE},ℓ_{EF} and ℓ_{FA} are defined analogously. Let A_1 be the intersection of te lines ℓ_{FA} and ℓ_{AB}, B_1 the intersection of the lines ℓ_{AB} and ℓ_{BC}; points C_1, D_1, E_1 and F_1 are defined analogously. Suppose that $A_1B_1C_1D_1E_1F_1$ is a convex hexagon. Prove that its diagonals A_1D_1, B_1E_1 and C_1F_1 are concurrent.

Proposed by Nairi Sedrakian, Yerevan, Armenia

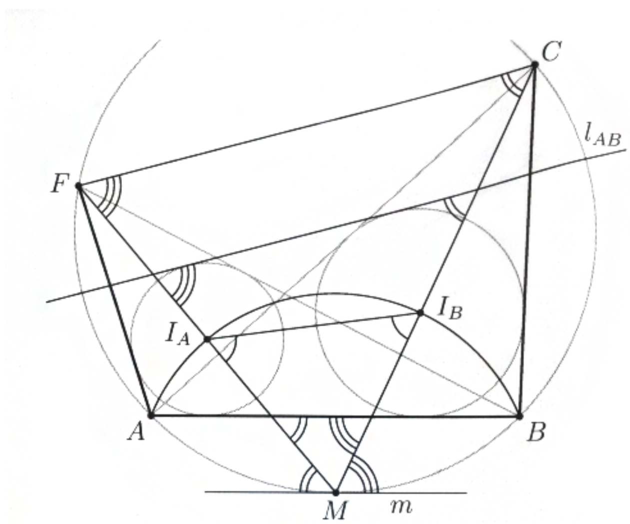
Solution by the author

We will prove that the hexagon $A_1B_1C_1D_1E_1F_1$ is centrally symmetric and therefore the main diagonals A_1D_1, B_1E_1 and C_1F_1 pass though the symmetry center of the hexagon. We denote the centers of $\omega_A, \omega_B, \omega_C, \omega_D, \omega_E$ and ω_F by I_A, I_B, I_C, I_D, I_E and I_F respectively.

Claim 1:

$\ell_{AB}||CF||\ell_{DE}, \ell_{BC}||AD||\ell_{EF}$, and $\ell_{CD}||BE||\ell_{FA}$. *Proof:*

By the symmetry it suffices to prove that $\ell_{AB}||CF$. Let M be the midpoint of the arc AB of the circumcircle not containing C and F, and let m be the tangent to the circumcircle at M, which is parallel to AB.



Then we have

$$\angle(FM, CF) = \angle(m, CM) = \angle(AB, CM). \tag{1}$$

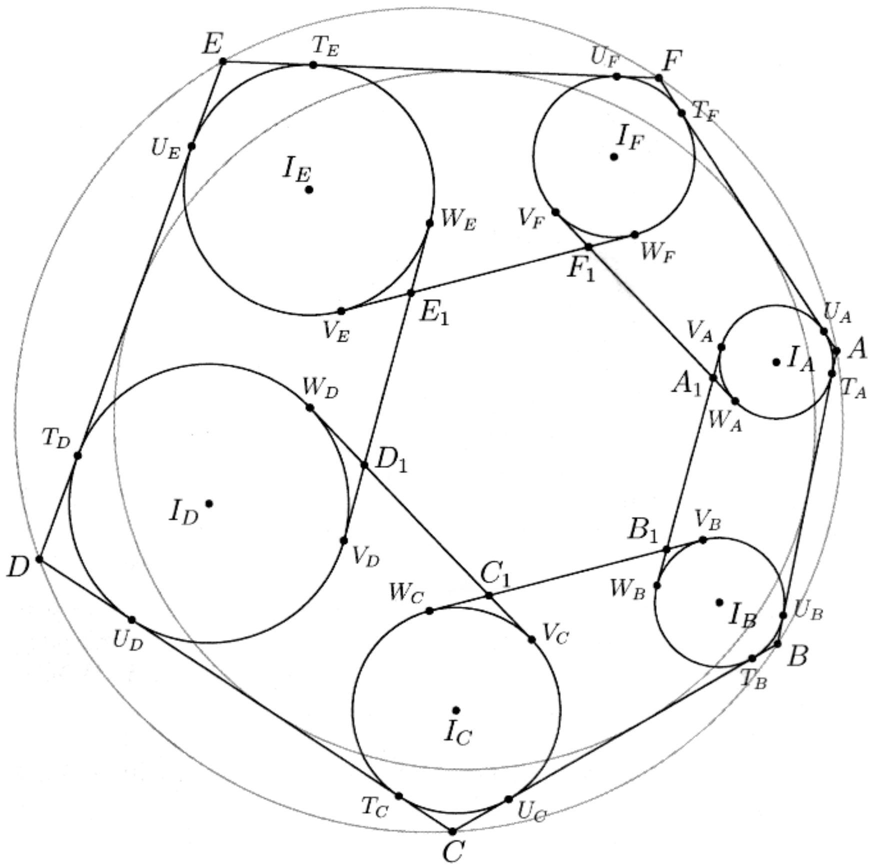
It is well known that the incenter I_A of the triangle FAB satisfies $MI_A = MA = MB$; similarly $MI_B = MA = MB$, so $MI_A = MI_B$ and therefore $\angle(I_AI_B, CM) = \angle(FM, I_AI_B)$. The lines AB and ℓ_{AB} are symmetric about the line I_AI_B, so

$$\angle(AB, CM) = \angle(AB, I_AI_B) + \angle(I_AI_B, CM) = \angle(I_AI_B, \ell_{AB}) + \angle(FM, I_AI_B) = \angle(FM, \ell_{AB}).$$

This equation combined with (1) yields $\angle(FM, CF) = \angle(FM, \ell_{AB})$, so indeed $CF||\ell_{AB}$. ■

Claim 2:
$A_1B_1 + C_1D_1 + E_1F_1 = B_1C_1 + D_1E_1 + F_1A_1$.

Proof:
Let T_A, U_A, V_A, and W_A be the points where ω_A touches the lines AB, FA, ℓ_{AB}, and ℓ_{FA}, respectively, and define the points $T_B, \ldots, W_F$ analogously.



Since the hexagon $ABCDEF$ is tangential, we have

$$AB + CD + EF = BC + DE + FA \tag{2}$$

Furthermore we have

$$AT_A = AU_A, \ldots, FT_F = FU_F \text{ and } A_1V_A = A_1W_A, \ldots, F_1V_F = F_1W_F, \tag{3}$$

because these pairs of segments are tangents drawn to the circles $\omega_A, \ldots, \omega_F$.

Finally, from the symmetry about the lines $I_AI_B, \ldots, I_FI_A$, we can wee that

$$T_AU_B = V_AW_B, \ldots, T_FU_A = V_FW_A. \tag{4}$$

By combining (3) and (4),

$$A_1B_1 = V_AW_B - A_1V_A - B_1W_B = T_AU_B - A_1V_A - B_1W_B$$

$$= (AB - AT_A - BU_B) - A_1V_A - B_1W_B$$

$$= AB - AT_A - BT_B - A_1V_A - B_1V_B.$$

Analogously,

$$B_1C_1 = BC - BT_B - CT_C - B_1V_B - C_1V_C,$$

$$\dots$$

$$F_1A_1 = FA - FT_F - AT_A - F_1V_F - A_1V_A.$$

Now the clain can be achieved by plugging these formulae into (2) and cancelling identical terms. ∎

Claim 3:
$\overrightarrow{A_1B_1} = \overrightarrow{E_1D_1}, \overrightarrow{B_1C_1} = \overrightarrow{F_1E_1}$, and $\overrightarrow{C_1D_1} = \overrightarrow{A_1F_1}$.
Proof:
Let X, Y, and Z be those points for which the quadrilaterals $F_1A_1B_1X$, $B_1C_1D_1Y$, and $D_1E_1F_1Z$ are parallelograms. By *Claim 1* we fave $F_1X \| A_1B_1 \| E_1D_1 \| F_1Z$, so the points F_1, X, Z are collinear; it can be seen similarly that B_1, X, Y are collinear and so are D_1, Y, Z. We will show that the points X, Y, Z coincide.

The points X, Y, Z either coincide or form a triangle. Suppose that XYZ is a triagnle with the same orientation as the haxagon $A_1B_1C_1D_1E_1F_1$. Then

$$F_1A_1 + B_1C_1 + D_1E_1 = XB_1 + YD_1 + ZF_1 >$$
$$YB_1 + ZD_1 + XF_1 = C_1D_1 + E_1F_1 + A_1B_1,$$

contradicting *Claim 2*.

If XYZ is a triangle with the opposite orientation from the hexagon, we get another contradiction

$$F_1A_1 + B_1C_1 + D_1E_1 <$$
$$C_1D_1 + E_1F_1 + A_1B_1$$

∎

Claim 3 shows that the hexagon $A_1B_1C_1D_1E_1F_1$ is indeed centrally symmetric, as requred. Done!