# Searching for Homogeneity Across Multi-Variable Polynomials

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## 1 Theory

Every homogeneous polynomial $P(x,y)$ of degree $d$ can be written as

$$P(x,y)=\sum_{k=0}^{d}a_kx^ky^{d-k}.$$

That is, $P(tx,ty)=t^dP(x,y)$, and, if we take $t=\frac{1}{y}$ or $t=\frac{1}{x}$, we find that

$$P(x,y)=y^dP\left(\frac{x}{y},1\right)=x^dP\left(1,\frac{y}{x}\right).$$

The latter expression is of great importance. We can treat $P\left(\frac{x}{y},1\right)$ or $P\left(1,\frac{y}{x}\right)$ as a one-variable polynomial. Namely, $Q\left(\frac{x}{y}\right)$ or $Q\left(\frac{y}{x}\right)$. Furthermore, every polynomial $P(x,y)$ of degree $d$ can be written as a sum of its homogeneous parts $P_l(x,y)$ for some nonnegative integer $l\le d$,

$$P(x,y)=\sum_{l=0}^{d}P_l(x,y).$$

In the rest of the article, we provide problems about finding multi-variable polynomials.

### 1.1 Examples

**Problem 1.** (Navid Safaei) Find all homogeneous polynomials $P(x,y,z)$ such that $P(x,y,z)=1$ whenever $x^2+y^2+z^2=1$.

**Solution:** Let $d$ be the degree of polynomial $P$. Since $P(x,y,z)$ is homogeneous, we have

$$P(x,y,z)=k^dP\left(\frac{x}{k},\frac{y}{k},\frac{z}{k}\right),$$

for all nonzero real $k$. Moreover, assume $P(x,y,z)=ax^d+\cdots$. Since $P(1,0,0)=P(-1,0,0)=1$, we conclude that $d$ is even. Thus

$$P(x,y,z)=\left(\sqrt{x^2+y^2+z^2}\right)^d\cdot P\left(\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

Since $\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)^2+\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)^2=1,\quad P\left(\frac{x}{\sqrt{x^2+y^2+z^2}},\frac{y}{\sqrt{x^2+y^2+z^2}},\frac{z}{\sqrt{x^2+y^2+z^2}}\right)=1$. Hence

$$P(x,y,z)=\left(\sqrt{x^2+y^2+z^2}\right)^d=\left(x^2+y^2+z^2\right)^{\frac{d}{2}}.$$

**Problem 2.** Find all homogeneous polynomials $P(x, y)$ such that

$$P(x, x+y) + P(y, x+y) = 0.$$

**Solution:** Assume that, $P(x, y)$ is of degree $d$. Then

$$P(x, x+y) + P(y, x+y) = (x+y)^d P\left(\frac{x}{x+y}, 1\right) + (x+y)^d P\left(\frac{y}{x+y}, 1\right) = 0.$$

That is,

$$P\left(\frac{x}{x+y}, 1\right) + P\left(\frac{y}{x+y}, 1\right) = 0,$$

for all $x \neq -y$.
Let $P\left(\frac{x}{x+y}, 1\right) = Q\left(\frac{x}{x+y}\right)$. We find that $\left(\frac{y}{x+y}, 1\right) = Q\left(1 - \frac{x}{x+y}\right)$. Therefore

$$Q(t) + Q(1-t) = 0,$$

for all $t$. This implies that $Q(x)$ is of odd degree. Therefore

$$Q\left(\frac{1}{2} + t\right) = -Q\left(\frac{1}{2} - t\right),$$

Let us define $R(t) = Q\left(\frac{1}{2} + t\right)$; thus, $R(t) = -R(-t)$. This implies $R(t) = tS(t^2)$ for some polynomial $S(x)$; hence, $Q(t) = \left(t - \frac{1}{2}\right) S\left(t^2 - t + \frac{1}{4}\right) = \left(t - \frac{1}{2}\right) g\left(t^2 - t\right)$. Thereby, we can find that

$$Q\left(\frac{x}{x+y}\right) = \left(\frac{x}{x+y} - \frac{1}{2}\right) g\left(\frac{-xy}{(x+y)^2}\right) = \frac{x-y}{2(x+y)} g\left(\frac{-xy}{(x+y)^2}\right),$$

for some polynomial $g(x)$ of degree $\frac{d-1}{2}$. This shows that

$$P(x, x+y) = (x+y)^d Q\left(\frac{x}{x+y}\right) = (x+y)^d \cdot \frac{x-y}{2(x+y)} g\left(\frac{-xy}{(x+y)^2}\right) = \left(\frac{x-y}{2}\right) \cdot (x+y)^{d-1} \cdot g\left(\frac{-xy}{(x+y)^2}\right).$$

Then, setting $y \to y - x$, we find that

$$P(x, y) = \left(\frac{2x-y}{2}\right) \cdot (y)^{d-1} \cdot g\left(\frac{x^2 - xy}{y^2}\right).$$

**Problem 3.** Let $a, b > 0$. Find all homogeneous polynomials $P(x, y)$ such that

$$P(x+a, y+b) = P(x, y).$$

**Solution:** Assume that $P(x, y)$ is of degree $d$. Without loss of generality, we can write $P(x, y) = y^d P\left(\frac{x}{y}, 1\right)$. Assume $Q\left(\frac{x}{y}\right) = P\left(\frac{x}{y}, 1\right)$. Then, we can find

$$(y+b)^d Q\left(\frac{x+a}{y+b}\right) = y^d Q\left(\frac{x}{y}\right).$$

Consider the equation $\frac{x+a}{y+b} = \frac{x}{y}$. We find that $\frac{x}{y} = \frac{a}{b}$. Set $x = a$ and $y = b$. Then

$$(a+b)^d Q\left(\frac{a}{b}\right) = b^d Q\left(\frac{a}{b}\right).$$

Since $b \neq 0$, assume $Q\left(\frac{a}{b}\right) \neq 0$ which implies $(a+b)^d = b^d$ or $\left(1 + \frac{a}{b}\right)^d = 1$, a contradiction. Hence $Q\left(\frac{a}{b}\right) = 0$. Write $Q(x) = (bx - a)^k R(x)$, where $R\left(\frac{a}{b}\right) \neq 0$. We now rewrite the original equation to obtain

$$(y+b)^d \cdot \left(\frac{bx - ay}{y+b}\right)^k \cdot R\left(\frac{x+a}{y+b}\right) = \left(\frac{bx - ay}{y}\right)^k \cdot y^d R\left(\frac{x}{y}\right).$$

Then

$$(y+b)^{d-k}R\left(\frac{x+a}{y+b}\right)=y^{d-k}R\left(\frac{x}{y}\right).$$

If $d\neq k$, $R\left(\frac{a}{b}\right)$ must be zero, a contradiction. Thus, $d=k$ and $R(x)$ must be constant. Hence

$$Q(x)=C(bx-a)^d.$$

Then, $P(x,y)=y^dQ\left(\frac{x}{y}\right)=C(bx-ay)^d$.

**Problem 4.** Let $a,b\neq(0,0)$ find all polynomials $P(x,y)$ such that

$$P(x+a,y+b)=P(x,y).$$

**Solution:** It is easy to observe that if $P_1(x,y)$ and $P_2(x,y)$ satisfy the above equality, then their linear combinations do so as well. All homogeneous polynomials that satisfy the above equality are of the form $C(bx-ay)^d$; thus their linear combinations are of the form $R(bx-ay)$, for some polynomial $R(x)$. Furthermore, we show that polynomial $P(x,y)$ is indeed a polynomial of $bx-ay$.

For this reason, we divide the polynomial $P(x,y)$ by $bx-ay$ with respect to $x$. We obtain

$$P(x,y)=(bx-ay)\,Q(x,y)+R(y).$$

Taking $x=y=0$, we find that $R(0)=P(0,0)$, as the previous problem. We have $\frac{x}{y}=\frac{a}{b}$, $b\neq 0$. From the original equality,

$$P(a,b)=P(2a,2b)=\cdots=P(ka,kb).$$

Then $R(b)=R(2b)=\cdots=R(kb)=\cdots$. This means that $R(y)$ must be constant. Denoting $R(y)=R(0)=P(0,0)$,

$$P(x,y)=(bx-ay)\,Q(x,y)+P(0,0).$$

Continuing the same procedure with the above polynomial,

$$Q(x+a,y+b)=Q(x,y),$$

since $Q(x,y)$ is of a lesser degree than $P(x,y)$. Simple induction tells us that $Q(x,y)=S(bx-ay)$, for some polynomial $S(x)$. Then

$$P(x,y)=(bx-ay)\,S\,(bx-ay)+P(0,0)=R(bx-ay).$$

**Problem 5.** (Saint Petersburg-1998) Find all polynomials $P(x,y)$ such that

$$P(x,y)=P(x+y,y-x).$$

**Solution:** Let $P(x,y)=\sum_{l=0}^{d}P_l(x,y)$, where $P_l(x,y)$ are homogenous polynomial of degree $d$. Note that

$$P(x+y,y-x)=\sum_{l=0}^{d}P_l(x+y,y-x).$$

Let $P_l(x,y)=\sum_{k=0}^{l}a_kx^ky^{l-k}$. Then

$$P_l(x+y,y-x)=\sum_{k=0}^{l}a_k(x+y)^k(y-x)^{l-k},$$

implying

$$\sum_{k=0}^{l}a_k(x+y)^k(y-x)^{l-k}=\sum_{k=0}^{l}a_k\left(\sum_{i=0}^{k}\binom{k}{i}x^iy^{k-i}\right)\left(\sum_{j=0}^{k}\binom{l-k}{j}x^jy^{l-k-j}\right).$$

The general term of the last expression is $x^{i+j}y^{l-i-j}$, a monomial of degree $l$. It follows that the degree of $P(x+y, y-x)$ remains unchanged. Hence the parts of the same degree in the both sides must be equal. Thus

$$P_l(x+y, y-x) = P_l(x, y).$$

Without loss of generality, we can assume that $P(x, y)$ is homogeneous. Now

$$y^d P\left(\frac{x}{y}, 1\right) = (y-x)^d P\left(\frac{x+y}{y-x}, 1\right).$$

Assume that $P\left(\frac{x}{y}, 1\right) = Q\left(\frac{x}{y}\right)$. Then

$$y^d Q\left(\frac{x}{y}\right) = (y-x)^d Q\left(\frac{x+y}{y-x}\right)$$

Taking $x = iy$, we find that $y^d Q(i) = y^d (1-i)^d Q(i)$ for some $d \neq 0$. We have $Q(i) = 0$. By the same argument, $Q(-i) = 0$. Let $Q(x) = (x-i)^r (x+i)^s R(x)$, where $R(\pm i) \neq 0$. Assuming $r > s$, $Q(x) = (x^2+1)^s (x-i)^{r-s} R(x)$. Now

$$y^d \left(\frac{x^2+y^2}{y^2}\right)^s \cdot \left(\frac{x-iy}{y}\right)^{r-s} R\left(\frac{x}{y}\right) = (y-x)^d \cdot \left(\frac{2\left(x^2+y^2\right)}{(y-x)^2}\right)^s \cdot \left(\frac{x-iy}{y-x}\right)^{r-s} \cdot (i+1)^{r-s} R\left(\frac{x+y}{y-x}\right).$$

That is,

$$y^{d-r-s} \cdot R\left(\frac{x}{y}\right) = 2^s (i+1)^{r-s} \cdot (y-x)^{d-r-s} R\left(\frac{x+y}{y-x}\right).$$

We set $x = i$ and $y = 1$ and find that $R(i) = 2^s (1-i)^{d-2s} \cdot i^{r-s} R(i)$. Since $R(i) \neq 0$,

$$2^s (1-i)^{d-2s} \cdot i^{r-s} = 1.$$

Setting, $x = -i$ and $y = 1$, it follows that $2^s (1-i)^{d-2s} = 1$. But taking the conjugate, we have $2^s (1+i)^{d-2s} = 1$, which, after multiplying, becomes

$$2^{2s} \cdot 2^{d-2s} = 2^d = 1.$$

Hence $d = 0$, implying $P(x) = C$.

**Problem 6.** Find all polynomials $P(x, y)$ such that

$$2P(x, y) = P(x+y, y-x).$$

**Solution:** By the same argument, we can assume that $P(x, y)$ is homogeneous. We have

$$2y^d Q\left(\frac{x}{y}\right) = (x-y)^d Q\left(\frac{x+y}{y-x}\right).$$

Take $\frac{x}{y} = t$ then $2Q(t) = (t-1)^d \cdot Q\left(\frac{t+1}{1-t}\right)$. Now take $t = 0, -1$ we find that

$$2Q(-1) = (-2)^d Q(0), \quad 2Q(0) = (-1)^d Q(-1).$$

Then $2Q(-1) = 2^{d-1} Q(-1)$. Assuming $d \neq 2$, we have $Q(0) = Q(-1) = 0$. Now set $Q(t) = t^k (t+1)^s R(t)$, where $R(0), R(-1) \neq 0$. Then

$$R(t) = 2^{s-1} (t-1)^{d-s-k} R\left(\frac{t+1}{1-t}\right).$$

If $s > k$, take $t = 0$, then we reach $R(0) = 0$, a contradiction. Thus $s \leq k$ and

$$t^{k-s} R(t) = 2^{s-1} (t-1)^{d-s-k} R\left(\frac{t+1}{1-t}\right).$$

Taking $t = 0$ we find that $R(-1) = 0$, a contradiction. Hence $k = s$ and

$$R(t) = 2^{s-1} (t-1)^{d-2s} R\left(\frac{t+1}{1-t}\right).$$

Again, set $t = 0, -1$. We find that

$$R(0) = 2^{d-2}R(0).$$

Then $d = 2$. Now setting $Q(t) = at^2 + bt + c$, we find that

$$2at^2 + 2bt + 2c = a(t+1)^2 + b\left(t^2 - 1\right) + c(t-1)^2.$$

Checking the coefficients of both sides, we arrive at $a = b + c$. Then

$$Q(t) = (b+c)t^2 + bt + c,$$

implying

$$P(x,y) = y^2 Q\left(\frac{x}{y}\right) = (b+c)x^2 + bxy + cy^2.$$

**Problem 7.** (A.Golovanov-Tuymada-2014) Find all polynomials $P(x,y)$ with real coefficients satisfying

$$P(x+2y,\ x+y) = P(x,y).$$

**Solution:** By the same argument above, we can assume that $P(x,y)$ is homogeneous. Therefore

$$(x+y)^d Q\left(\frac{x+2y}{x+y}\right) = y^d Q\left(\frac{x}{y}\right).$$

Taking $\frac{x}{y} = t$,

$$(1+t)^d Q\left(\frac{t+2}{t+1}\right) = Q(t).$$

Considering $t = \frac{t+2}{t+1}$, $t = \pm\sqrt{2}$. Thus

$$\left(1 \pm \sqrt{2}\right)^d \cdot Q\left(\pm\sqrt{2}\right) = Q\left(\pm\sqrt{2}\right).$$

Hence, $Q\left(\pm\sqrt{2}\right) = 0$. That is, $Q(x) = \left(x - \sqrt{2}\right)^r \left(x + \sqrt{2}\right)^s R(x)$, when $R\left(\pm\sqrt{2}\right) \neq 0$. By the same method used in Problem 7 we find that $r = s$. Then $Q(x) = \left(x^2 - 2\right)^s R(x)$, that is

$$(1+t)^{d-2s} R\left(\frac{t+2}{t+1}\right) = (-1)^s R(t).$$

If $d \neq 2s$, we find that $R\left(\pm\sqrt{2}\right) = 0$, a contradiction. That is, $d = 2s$. Then

$$R\left(\frac{t+2}{t+1}\right) = (-1)^s R(t).$$

Now if $s$ is odd, $R\left(\pm\sqrt{2}\right) = 0$, which leads to a contradiction. This implies that $s$ is even; set $s = 2k$. Then $R(x) = C\left(x^2 - 2\right)^{2k}$. This shows that

$$P(x,y) = Cy^{2s}\left(\frac{x^2 - 2y^2}{y^2}\right)^{2k} = C\left(x^2 - 2y^2\right)^{2k},$$

when $P(x,y) = C\left(\left(x^2 - 2y^2\right)^2\right)^k$. Therefore,

$$P(x,y) = \sum C_k \left(\left(x^2 - 2y^2\right)^2\right)^k = T\left(\left(x^2 - 2y^2\right)^2\right).$$

**Problem 8.** Find all polynomials $P(x, y)$ such that

$$P\left(x^2, y^2\right) = P\left(\frac{(x+y)^2}{2}, \frac{(x-y)^2}{2}\right).$$

**Solution:** Assume that $P(x, y)$ is homogeneous. We have

$$y^{2d} Q\left(\frac{x^2}{y^2}\right) = \frac{(y-x)^{2d}}{2^d} \cdot Q\left(\left(\frac{x+y}{y-x}\right)^2\right).$$

Take $\frac{x}{y} = t$ , then

$$Q\left(t^2\right) = \frac{(t-1)^{2d}}{2^d} \cdot Q\left(\left(\frac{t+1}{t-1}\right)^2\right)$$

At first, take $t = i, -i$, Then $Q(-1) = \frac{(\pm 2i)^d}{2^d} Q(-1) = (\pm i)^d Q(-1)$. Now if $d$ is odd, we have $Q(-1) = 0$.
Set $\frac{1}{t}$ as instead of $t$ in the above equation. Then

$$Q\left(t^2\right) = t^{2d} Q\left(\frac{1}{t^2}\right).$$

Moreover $Q\left(t^2\right) = t^d R\left(t + \frac{1}{t}\right)$, for some polynomial $R(x)$. Now assume that $d$ is even. We then have

$$R\left(t + \frac{1}{t}\right) = \frac{1}{2^d} \cdot \left(\frac{t^2-1}{t}\right)^d R\left(2\left(\frac{t^2+1}{t^2-1}\right)\right).$$

That is $R\left(-t - \frac{1}{t}\right) = R\left(t + \frac{1}{t}\right)$. Hence, $R(x)$ is an even polynomial, and one can verify that $R(x) = S(x^2)$. Thus,

$$Q\left(t^2\right) = t^d S\left(\left(t + \frac{1}{t}\right)^2\right).$$

Since $\deg S(x) = \frac{d}{2}$,

$$S\left(\left(t + \frac{1}{t}\right)^2\right) = \frac{1}{2^d} \cdot \left(\frac{t^2-1}{t}\right)^d R\left(4\left(\frac{t^2+1}{t^2-1}\right)^2\right).$$

Assuming $2a = \frac{2t}{t^2+1}$ and $2b = \frac{t^2-1}{t^2+1}$, we know that $a^2 + b^2 = \frac{1}{4}$. Then

$$a^d S\left(\frac{1}{a^2}\right) = b^d S\left(\frac{1}{b^2}\right).$$

Defining $x^d S\left(\frac{1}{x^2}\right) = T\left(x^2\right)$, we have $T\left(a^2\right) = T\left(b^2\right) = T\left(\frac{1}{4} - a^2\right)$. It follows that $T(x)$ is of even degree, that is, $T(x) = A\left(x^2 - \frac{x}{4}\right)$. Hence,

$$S\left(\frac{1}{x^2}\right) = \frac{A\left(x^4 - \frac{x^2}{4}\right)}{x^d}.$$

Therefore,

$$S\left(x^2\right) = x^d A\left(\frac{1}{x^4} - \frac{1}{4x^2}\right).$$

It follows that $Q\left(x^2\right) = x^d S\left(\left(x + \frac{1}{x}\right)^2\right) = x^d \cdot \left(x + \frac{1}{x}\right)^d A\left(-\frac{\left(x - \frac{1}{x}\right)^2}{4\left(x + \frac{1}{x}\right)^4}\right) = \left(x^2+1\right)^d A\left(-\frac{x^2\left(x^2-1\right)^2}{\left(x^2+1\right)^4}\right)$, implying

$$P\left(x^2, y^2\right) = y^{2d} Q\left(\frac{x^2}{y^2}\right).$$

Then, $P(x,y) = (x+y)^d A\left(-\frac{xy(x-y)^2}{(x+y)^4}\right) = B\left(xy(x-y)^2\right)$, where $\deg A(x) = \deg B(x) = \frac{d}{4}$.

Now if $d$ is odd, we find that $Q(-1) = 0$. Write $Q(x) = (x+1)^r Q_1(x)$, then $Q_1(-1) \neq 0$. Since $Q\left(t^2\right) = t^{2d} Q\left(\frac{1}{t^2}\right)$, we find that $Q_1\left(t^2\right) = t^{2(d-r)} Q_1\left(\frac{1}{t^2}\right)$.

Recall that $Q_1(-1) \neq 0$; therefore $d - 2r$ must be even (otherwise, observe the case when $t = i$ in the latter equality). Then $r$ must be odd and $Q_1(x)$ is of even degree, satisfying the same equality. When $d$ is odd, we have

$$Q\left(x^2\right) = \left(x^2+1\right)^r \cdot \left(x^2+1\right)^{d-r} A\left(-\frac{x^2(x^2-1)^2}{(x^2+1)^4}\right),$$

where $\deg A(x) = \frac{d-r}{4}$. Hence $P(x,y) = (x+y)^r \cdot (x+y)^{d-r} A\left(-\frac{xy(x-y)^2}{(x+y)^4}\right) = (x+y)^r B\left(xy(x-y)^2\right)$, where $\deg A(x) = \deg B(x) = \frac{d-r}{4}$. Thus

$$P(x,y) = \sum c_{d-k}(x+y)^r B_{d-k}\left(xy(x-y)^2\right) + \sum c_s B_s\left(xy(x-y)^2\right) = G\left(x+y, xy(x-y)^2\right).$$

It is easy to verify that all such polynomials satisfy the statement of the problem.

**Problem 9.** (Saint Petersburg) Let $P(x,y)$ be a polynomial with real coefficients such that there is a function $f$ where

$$P(x,y) = f(x+y) - f(x) - f(y).$$

Prove that, there is a polynomial $Q(x)$ such that $f(t) = Q(t)$ for infinitely many real numbers, $t$.

**Solution:** It is clear that $P(x,y) = P(y,x)$. Moreover,

$$P(x+z,y) + P(x,z) = P(x+y,z) + P(x,y) = f(x+y+z) - f(x) - f(y) - f(z).$$

Now consider the equality $P(x+z,y) + P(x,z) = P(x+y,z) + P(x,y)$. We assume that $P(x,y)$ is homogeneous and symmetric. Write

$$P(x,y) = \sum_{k=0}^{d} a_k x^k y^{d-k},$$

where $a_k = a_{d-k}$. Consider the coefficients of $y^d$ in both sides to find that $a_0 = a_d = 0$. Furthermore, comparing the coefficients of $x^{d-a-b} y^a z^b$,

$$a_{d-a}\binom{d-a}{b} = a_{d-b}\binom{d-b}{a}.$$

Then $\frac{a_{d-a}}{\binom{d}{a}} = \frac{a_{d-b}}{\binom{d}{b}}$. That is, $a_{d-b} = \frac{\binom{d}{b}}{\binom{d}{a}} \cdot a_{d-a}$. Then $P(x,y) = C\left((x+y)^d - x^d - y^d\right)$.

Furthermore, $P(x,y) = \sum c_n\left((x+y)^n - x^n - y^n\right)$. Now define

$$g(x) = f(x) - \sum c_n x^n.$$

One can easily find that $g(x+y) = g(x) + g(y)$. For all rational numbers, $g(r) = g(1)r$. Thus $f(r) = g(1)r + \sum c_n r^n$.

**Problem 10.** (American Mathematical Monthly) Let $P(x,y)$ be a homogeneous polynomial of degree $d$ such that there are polynomials $R(t)$ and $Q(t)$ for which

$$P(R(t), Q(t)) = C,$$

where $C$ is a constant. Prove that $P(x,y) = (bx - ay)^d$, for some real numbers $a$ and $b$.

**Solution:** Let $d > 0$. Assume that $(t) = a_0 + a_1 t + \cdots + a_r t^r$ and $Q(t) = b_0 + b_1 t + \cdots + b_q t^q$. It is clear that $q = r$. The coefficient of $t^{rd}$ in $P(R(t), Q(t))$ is $P(a_r, b_r)$, which must be zero. Then $P(a_r, b_r) = 0$. Thus $P(x,y)$ is divisible by $b_r x - y a_r$. This implies that $(x,y) = (b_r x - y a_r) P_1(x,y)$. We know that

$$P_1(R(t), Q(t))$$

must be a non-zero constant. The conclusion follows by induction.

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