Junior problems

J433. Let a, b, c, x, y, z be real numbers such that $a^2 + b^2 + c^2 = x^2 + y^2 + z^2 = 1$. Prove that

$$|a(y-z) + b(z-x) + c(x-y)| \leq \sqrt{6(1 - ax - by - cz)}.$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Polyahedra, Polk State College, USA

By the Cauchy-Schwarz inequality,

$$\begin{aligned} & |a(y-z) + b(z-x) + c(x-y)| = |(a-x)(y-z) + (b-y)(z-x) + (c-z)(x-y)| \\ \leq \; & \sqrt{\left[(a-x)^2 + (b-y)^2 + (c-z)^2\right]\left[(y-z)^2 + (z-x)^2 + (x-y)^2\right]} \\ = \; & \sqrt{2(1 - ax - by - cz)\left[3 - (x+y+z)^2\right]} \leq \sqrt{6(1 - ax - by - cz)}. \end{aligned}$$

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Arkady Alt, San Jose, CA, USA; Naïm Mégarbané, UPMC, Paris, France.

J434. Solve in integers the equation

$$x^3 + y^3 = 7\max(x, y) + 7.$$

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Assume that $x \geq y$. Then we have

$$\begin{aligned} x^3 + y^3 = 7 \cdot \max(x, y) + 7 \;&\Leftrightarrow\; x^3 + y^3 = 7x + 7 \\ &\Leftrightarrow\; x^3 - 7x - 7 = y^3. \end{aligned}$$

If $x > 4$ then $(x-1)^3 < x^3 - 7x - 7 < x^3$.

If $x \leq -2$ then $x^3 < x^3 - 7x - 7 < (x+1)^3$.

Hence given equation has solution only on $-1 \leq x \leq 4$. Easy calculation shows that $x = 3$, $y = 1$.

Also solved by Polyahedra, Polk State College, USA; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

J435. Let $a \geq b \geq c > 0$ be real numbers. Prove that

$$2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq 3.$$

Proposed by Mircea Becheanu, Montreal, Canada

Solution by Nikos Kalapodis, Patras, Greece

By the AM-GM inequality we have $\frac{b}{a}+\frac{c}{b}+\frac{a}{c} \geq 3\sqrt[3]{\frac{b}{a}\cdot\frac{c}{b}\cdot\frac{a}{c}} = 3$ (1).
Since $a \geq b \geq c > 0$ we have that $(a-b)(b-c)(a-c) \geq 0$, which after expanding gives
$b^2c + c^2a + a^2b \geq a^2c + b^2a + c^2b$ and dividing by abc we get $\frac{b}{a}+\frac{c}{b}+\frac{a}{c} \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ (2).
Adding (1) and (2) we get $2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) \geq 3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ or $2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq 3$.
Equality holds iff $a=b=c$

Also solved by Polyahedra, Polk State College, USA; Naïm Mégarbané, UPMC, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Henry Ricardo, Westchester Area Math Circle; Kevin Soto Palacios, Huarmey, Perú; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.

J436. Let a, b, c be real numbers such that $a^4 + b^4 + c^4 = a + b + c$. Prove that

$$a^3 + b^3 + c^3 \leq abc + 2.$$

Proposed by Adrian Andreescu, University of Texas at Austin

Solution by Polyahedra, Polk State College, USA

If $a + b + c = 0$, then $a = b = c = 0$ and we are done. Thus we may assume that $a + b + c > 0$ and $a \geq b \geq c$. Then $a \geq |b|$, so $a^2(a - b)(a - c) \geq b^2(a - b)(b - c)$ and $c^2(a - c)(b - c) \geq 0$. Therefore,

$$\begin{aligned} 0 &\leq a^2(a-b)(a-c) - b^2(a-b)(b-c) + c^2(a-c)(b-c) \\ &= 2\left(a^4 + b^4 + c^4\right) + abc(a+b+c) - \left(a^3 + b^3 + c^3\right)(a+b+c) \\ &= (a+b+c)\left(2 + abc - a^3 - b^3 - c^3\right), \end{aligned}$$

Also solved by Arkady Alt, San Jose, CA, USA; Nikos Kalapodis, Patras, Greece; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Kevin Soto Palacios, Huarmey, Perú; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Titu Zvonaru, Comănești, Romania.

J437. Let a, b, c be real numbers such that $\left(a^2+2\right)\left(b^2+2\right)\left(c^2+2\right)=512$. Prove that

$|ab+bc+ca| \leq 18.$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, CA, USA

Since $\left(a^2+2\right)\left(b^2+2\right)\left(c^2+2\right)=(abc-2(a+b+c))^2+2(ab+bc+ca-2)^2$ then

$2(ab+bc+ca-2)^2 \leq 512 \iff |ab+bc+ca-2| \leq 16.$

Also we have $|ab+bc+ca-2|+2 \geq .\, |ab+bc+ca-2+2| = |ab+bc+ca|$

Hence, $|ab+bc+ca| \leq 18$.

Also solved by Polyahedra, Polk State College, USA; Naïm Mégarbané, UPMC, Paris, France; Nikos Kalapodis, Patras, Greece; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Nguyen Ngoc Tu, Ha Giang, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Kevin Soto Palacios, Huarmey, Perú; Titu Zvonaru, Comănești, Romania.

J438. (i) Find the greatest real number r such that

$$ab \geq r\left(1-\frac{1}{a}-\frac{1}{b}\right)$$

for all positive real numbers a and b.

(ii) Find the maximum of

$$xyz(2-x-y-z)$$

over all positive real numbers x, y, z.

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Polyahedra, Polk State College, USA

(i) By the AM-GM inequality, $\frac{1}{a}+\frac{1}{b} \geq \frac{2}{\sqrt{ab}}$, so

$$ab-27\left(1-\frac{1}{a}-\frac{1}{b}\right) \geq \frac{\left(\sqrt{ab}\right)^3-27\sqrt{ab}+54}{\sqrt{ab}} = \frac{\left(\sqrt{ab}+6\right)\left(\sqrt{ab}-3\right)^2}{\sqrt{ab}} \geq 0,$$

with equalities when $a=b=3$. Hence the greatest r is 27.

(ii) By the AM-GM inequality, $x+y+z \geq 3\sqrt[3]{xyz}$, so

$$\frac{1}{16}-xyz(2-x-y-z) \geq \frac{1}{16}-2xyz+3\sqrt[3]{(xyz)^4} = \left(\sqrt[3]{xyz}-\frac{1}{2}\right)^2\left[3\sqrt[3]{(xyz)^2}+\sqrt[3]{xyz}+\frac{1}{4}\right] \geq 0,$$

with equality when $x=y=z=\frac{1}{2}$. Hence the maximum is $\frac{1}{16}$.

Also solved by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Arkady Alt, San Jose, CA, USA; Frank Gamboa, Havana, Cuba; Titu Zvonaru, Comănești, Romania.

Senior problems

S433. Let a, b, c be real numbers such that $0 \le a \le b \le c$ and $a + b + c = 1$. Prove that

$$\sqrt{\frac{2}{3}} \le a\sqrt{a+b} + b\sqrt{b+c} + c\sqrt{c+a} \le 1$$

When do the equalities hold?

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Naïm Mégarbané, UPMC, Paris, France

With the hypothesis of the problem :

$$a\sqrt{a+b} + b\sqrt{b+c} + c\sqrt{c+a} \le a\sqrt{a+b+c} + b\sqrt{a+b+c} + c\sqrt{c+a+b} = a + b + c = 1$$

This show that the equality hold, if $a = b = 0$ and $c = 1$.

Now with the hypothesis of the problem, $3a \le 1 \iff a \le \frac{1}{3}$, so

$$a\sqrt{a+b} + b\sqrt{b+c} + c\sqrt{c+a} \ge (a+b+c)\sqrt{a+b} \ge \sqrt{2a} \ge \sqrt{\frac{2}{3}}.$$

This show that equality hold if and only if $a = b = c = \frac{1}{3}$.

Also solved by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Kevin Soto Palacios, Huarmey, Perú; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Paraskevi-Andrianna Maroutsou, Charters Sixth Form, Sunningdale, England, UK; Arkady Alt, San Jose, CA, USA.

S434. Let a,b,c,d,x,y,z,w be real numbers such that $a^2+b^2+c^2+d^2=x^2+y^2+z^2+w^2=1$. Prove that

$$ax+\sqrt{\left(b^2+c^2\right)\left(y^2+z^2\right)}+dw\le 1$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy

$$ax+\sqrt{(b^2+c^2)(y^2+z^2)}+dw\le 1 \iff \sqrt{(1-a^2-d^2)(1-x^2-w^2)}\le 1-ax-dw$$

Let $\vec{u}=(a,d)$, $\vec{v}=(x,w)$. Upon squaring the inequality we have

$$(1-|\vec{u}|^2)(1-|\vec{v}|^2)\le(1-\vec{u}\cdot\vec{v})^2$$

Letting $|\vec{u}|=p$, $|\vec{v}|=q$, we get

$$\sqrt{(1-p^2)(1-q^2)}\le 1-pq\cos\vartheta \iff \cos\vartheta\le\frac{1-\sqrt{(1-p^2)(1-q^2)}}{pq}$$

$$\frac{1-\sqrt{(1-p^2)(1-q^2)}}{pq}\ge 1 \iff p^2+q^2\ge 2pq$$

which evidently holds true thus concluding the proof.

Also solved by Arkady Alt, S;an Jose, CA, USA; Frank Gamboa, Havana, Cuba; Naïm Mégarbané, UPMC, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Kevin Soto Palacios, Huarmey, Perú.

S435. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$a^3 + b^3 + c^3 + \frac{8}{(a+b)(b+c)(c+a)} \geq 4.$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Adamopoulos Dionysios, 3rd High School, Pyrgos, Greece

We have

$$a^3 + b^3 + c^3 + \frac{8}{(a+b)(b+c)(c+a)} \geq 4 \Leftrightarrow$$

$$(a^3 + b^3 + c^3)[ab(a+b) + bc(b+c) + ca(c+a) + 2abc] + 8 \geq 4[ab(a+b) + bc(b+c) + ca(c+a) + 2abc] \Leftrightarrow$$

$$(a^3 + b^3 + c^3)[ab(a+b) + bc(b+c) + ca(c+a) + 2] + 8 \geq 4[ab(a+b) + bc(b+c) + ca(c+a) + 2] \Leftrightarrow$$

$$(a^3 + b^3 + c^3)[ab(a+b) + bc(b+c) + ca(c+a)] + 2(a^3 + b^3 + c^3) \geq 4[ab(a+b) + bc(b+c) + ca(c+a)]$$

From $AM - GM$ we know that $a^3 + b^3 + c^3 \geq 3abc = 3$

As a result

$$(a^3 + b^3 + c^3)[ab(a+b) + bc(b+c) + ca(c+a)] \geq 3[ab(a+b) + bc(b+c) + ca(c+a)]$$

So it suffices to prove that $2(a^3+b^3+c^3) \geq ab(a+b)+bc(b+c)+ca(c+a)$ which is true since $a^3+b^3 \geq ab(a+b)$ and so on.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Nikos Kalapodis, Patras, Greece; Naïm Mégarbané, UPMC, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Igli Mlloja; Kevin Soto Palacios, Huarmey, Perú; Konstantinos Metaxas, Athens, Greece; Nguyen Ngoc Tu, Ha Giang, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA.

S436. Prove that all real numbers a, b, c, d, e:

$$2a^2 + b^2 + 3c^2 + d^2 + 2e^2 \geq 2(ab - bc - cd - de + ea)$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Frank Rafael Gamboa, Havana, Cuba

Note that this inequality is equivalent to:

$$2a^2 - 2a(b + e) + (b^2 + 3c^2 + d^2 + 2e^2 + 2bc + 2cd + 2de) \geq 0$$

and this is a cuadratic equation at variable a; so its sufficient show that $\Delta_a \leq 0$, but:

$$\begin{aligned} \Delta_a &= 4(b + e)^2 - 8(b^2 + 3c^2 + d^2 + 2e^2 + 2bc + 2cd + 2de) \leq 0 \\ \Longleftrightarrow \quad & b^2 + 2b(2c - e) + 2(3c^2 + d^2 + 1,5e^2 + 2cd + 2de) \geq 0 \end{aligned}$$

but this is a cuadratic equation at variable b; so its sufficient show that $\Delta_b \leq 0$:

$$\begin{aligned} \Delta_b &= 4(2c - e)^2 - 8(3c^2 + d^2 + 1,5c^2 + 2cd + 2de) \leq 0 \\ \Longleftrightarrow \quad & c^2 + 2c(e + d) + (d^2 + e^2 + 2de) \geq 0 \\ \Longleftrightarrow \quad & c^2 + 2c(e + d) + (d + e)^2 \geq 0 \\ \Longleftrightarrow \quad & (c + d + e)^2 \geq 0 \end{aligned}$$

and this last inequality is true. Done.

Also solved by Arkady Alt, San Jose, CA, USA; Naïm Mégarbané, UPMC, Paris, France; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Moubinool Omarjee, Lycée Henri IV, Paris, France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Titu Zvonaru, Comănești, Romania.

S437. Let a, b, c be positive real numbers. Prove that

$$\frac{(4a+b+c)^2}{2a^2+(b+c)^2}+\frac{(4b+c+a)^2}{2b^2+(c+a)^2}+\frac{(4c+a+b)^2}{2c^2+(a+b)^2}\le 18$$

Proposed by Marius Stănean, Zalău, Romania

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Given inequality is homogeneous, hence we can assume that $a+b+c=3$. Then we have

$$\begin{aligned}(1) \;\Leftrightarrow\; & \frac{(a+1)^2}{a^2-2a+3}+\frac{(b+1)^2}{b^2-2b+3}+\frac{(c+1)^2}{c^2-2c+3}\le 6\\ \Leftrightarrow\; & \sum\left(3-\frac{(a+1)^2}{a^2-2a+3}\right)\ge 3\\ \Leftrightarrow\; & \sum\frac{a^2-4a+4}{a^2-2a+3}\ge\frac{3}{2}.\end{aligned}$$

Using Cauchy-Schwartz's inequality, we get

$$\left(\sum(a^2-2a+3)(a^2-4a+4)\right)\left(\sum\frac{a^2-4a+4}{a^2-2a+3}\right)\ge\left(\sum(a^2-4a+4)\right)^2$$

$$\Leftrightarrow\; \sum\frac{a^2-4a+4}{a^2-2a+3}\ge\frac{\left(\sum(a^2-4a+4)\right)^2}{\sum(a^2-2a+3)(a^2-4a+4)}.$$

Thus we have

$$2\left(\sum a^2\right)^2\ge 3\sum(a^2-2a+3)(a^2-4a+4)$$

i.e

$$4\sum a^4+3\sum a^2bc\ge 3\sum a^2b^2+2\sum a^3b+2\sum a^3c \tag{2}$$

We need to prove (2), then our problem is solved.

Applying Schur's inequalty, we get

$$\begin{aligned}\sum a^2(a-b)(a-c) \;\Leftrightarrow\; & \sum a^4+\sum a^2bc\ge\sum a^3b+\sum a^3c\\ \Leftrightarrow\; & 3\sum a^4+3\sum a^2bc\ge 3\sum a^3b+3\sum a^3c\end{aligned} \tag{3}$$

By the AM-GM inequality, we have

$$\sum a^4+ab^3+ab^3\ge\sum 3a^2b^2 \tag{4}$$

Then we have (3) and (4), give us (2). Problem is solved.

Also solved by Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy.

S438. Let ABC be an acute triangle. Determine all points M inside the triangle such that the sum

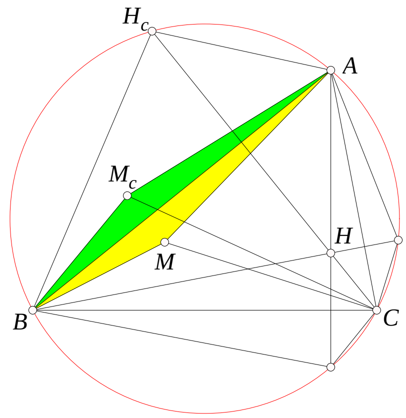
$$\frac{AM}{AB \cdot AC} + \frac{BM}{BC \cdot BA} + \frac{CM}{CA \cdot CB}$$

is minimal.

Proposed by Dorin Andrica, Babeș-Bolyai University, Cluj-Napoca, România

Solution by Li Zhou, Polk State College, USA

We show that the minimum (of $1/R$) is attained only at the orthocenter H of $\triangle ABC$. Let M_c be the reflection of M across AB. Let $x = AM = AM_c$, $y = BM = BM_c$, and $z = CM$.



Applying Ptolemy's inequality to AM_cBC we get $ax+by \geq c \cdot CM_c$, with equality if and only if M_c is on the circumcircle of $\triangle ABC$. Let T, T_a, T_b, and T_c be the areas of the triangles ABC, BCM, CAM, and ABM, respectively. Then $c \cdot CM_c \geq 2(T + T_c)$, with equality if and only if $CM_c \perp AB$. Hence, $ax + by \geq 2(T + T_c)$, with equality if and only if $M = H$. Adding this with the other two analogous inequalities we get

$$\frac{x}{bc} + \frac{y}{ca} + \frac{z}{ca} \geq \frac{3T + T_c + T_a + T_b}{abc} = \frac{4T}{abc} = \frac{1}{R},$$

with equality if and only if $M = H$.

Also solved by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina.

Undergraduate problems

U433. Let x, y, z be positive real numbers such that $x + y + z = 3$. Prove that

$$x^x y^y z^z \geq 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

First solution by Henry Ricardo, Westchester Area Math Circle

The weighted geometric mean-harmonic mean inequality gives us

$$(x^x y^y z^z)^{\frac{1}{x+y+z}} \geq \frac{x+y+z}{\frac{x}{x}+\frac{y}{y}+\frac{z}{z}} = \frac{x+y+z}{3} = 1,$$

from which the desired inequality follows.

Second solution by Henry Ricardo, Westchester Area Math Circle

The function $f(t) = t \ln t$ is convex on $(0, \infty)$: $f''(t) = 1/t > 0$. Therefore Jensen's inequality yields $f((x+y+z)/3) \leq (f(x)+f(y)+f(z))/3$, or

$$0 = f(1) \leq \frac{1}{3}(x \ln x + y \ln y + z \ln z) = \frac{\ln(x^x y^y z^z)}{3}.$$

Multiplying by 3 and exponentiating, we get $1 \leq x^x y^y z^z$.

Also solved by Nikos Kalapodis, Patras, Greece; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Adamopoulos Dionysios, 3rd High School, Pyrgos, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arpon Basu, AECS-4, Mumbai, India; Frank Gamboa, Havana, Cuba; Moubinool Omarjee, Lycée Henri IV, Paris, France; Naïm Mégarbané, UPMC, Paris, France; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Paul Revenant, Lycée du Parc, Lyon, France; Kevin Soto Palacios, Huarmey, Perú; Thiago Landim de Souza Leão, Federal University of Pernambuco, Recife, Brazil; Arkady Alt, San Jose, CA, USA.

U434. Find all homogeneous polynomials $P(X,Y)$ such that

$$P(x, \sqrt[3]{x^3+y^3}) = P(y, \sqrt[3]{x^3+y^3}),$$

for all real numbers x and y.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Let d be the degree of the polynomial $P(x,y)$ and since $P(x,y) = y^d P\left(\frac{x}{y}, 1\right)$ the equality can be rewritten as:

$$P\left(\frac{x}{\sqrt[3]{x^3+y^3}}, 1\right) = P\left(\frac{y}{\sqrt[3]{x^3+y^3}}, 1\right)$$

Thus assume $P(t,1) = Q(t)$ for some polynomial Q and since $\left(\frac{x}{\sqrt[3]{x^3+y^3}}\right)^3 + \left(\frac{y}{\sqrt[3]{x^3+y^3}}\right)^3 = 1$ we can reduce the problem to finding all polynomials $Q(x)$ such that: $Q(a) = Q(b)$ whenever $a^3 + b^3 = 1$. Now set $\omega a, \omega^2 a$ instead of a where ω is non-unit 3-th root of unity we find that:

$$Q(a) = Q(\omega a) = Q(\omega^2 a)$$

Thus $Q(x) = R(x^3)$ for some polynomial $R(x)$, then: $R(a^3) = R(b^3) = R(1-a^3)$ which leads to the equation:

$$R(x) = R(1-x)$$

We deduce that $R(x) = T(x^2 - x)$ for some polynomial $T(x)$ thus:

$$Q(x) = T(x^6 - x^3)$$

Whence

$$P(x,1) = T(x^6 - x^3)$$

Implies that: $P\left(\frac{x}{y}, 1\right) = T\left(\left(\frac{x}{y}\right)^6 - \left(\frac{x}{y}\right)^3\right)$. Then $P(x,y) = y^d P\left(\frac{x}{y}, 1\right) = y^d T\left(\frac{x^6 - x^3 y^3}{y^6}\right)$.

U435. Consider the sequence $(a_n)_{n\ge 1}$ defined by

$$\left(1+\frac{1}{n}\right)^{n+a_n}=1+\frac{1}{1!}+\cdots+\frac{1}{n!}$$

(i) Prove that $(a_n)_{n\ge 1}$ is convergent and $\lim_{n\to\infty} a_n=\frac{1}{2}$.

(ii) Evaluate $\lim_{n\to\infty} n\left(a_n-\frac{1}{2}\right)$.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy

(i) Let $S_n=\sum_{k=0}^{n}\frac{1}{k!}\to e$. We have

$$n\ln(1+\frac{1}{n})+a_n\ln(1+\frac{1}{n})=\ln S_n \iff a_n=\frac{\ln S_n-n\ln(1+\frac{1}{n})}{\ln(1+\frac{1}{n})}$$

We adopt Cesaro–Stolz theorem together with $\ln(1+x)=x-x^2/2+x^3/3+o(x^3)$ and search the limit

$$\lim_{n\to\infty}\frac{\ln S_{n+1}-\ln S_n-(n+1)\ln(1+\frac{1}{n+1})+n\ln(1+\frac{1}{n})}{\ln(1+\frac{1}{n+1})-\ln(1+\frac{1}{n})}=$$

$$\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{S_n(n+1)!}\right)-1+\frac{1}{2(n+1)}+-\frac{1}{3(n+1)^2}+o(\frac{1}{n^2})+1-\frac{1}{2n}+\frac{1}{3n^2}+o(\frac{1}{n^2})}{\ln\left(1-\frac{1}{(n+1)^2}\right)}$$

$$\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{S_n(n+1)!}\right)}{\ln\left(1-\frac{1}{(n+1)^2}\right)}=-\lim_{n\to\infty}\frac{(n+1)^2}{S_n(n+1)!}=0$$

$$\lim_{n\to\infty}\frac{\frac{1}{2(n+1)}-\frac{1}{2n}}{\ln\left(1-\frac{1}{(n+1)^2}\right)}=\lim_{n\to\infty}\frac{-\frac{1}{(n+1)^2}}{\ln\left(1-\frac{1}{(n+1)^2}\right)}\lim_{n\to\infty}\frac{-(n+1)^2}{-2n(n+1)}=1\cdot\frac{1}{2}=\frac{1}{2}$$

$$\lim_{n\to\infty}\frac{\frac{-1}{3(n+1)^2}+\frac{1}{3n^2}}{\ln\left(1-\frac{1}{(n+1)^2}\right)}=\lim_{n\to\infty}\frac{-\frac{1}{(n+1)^2}}{\ln\left(1-\frac{1}{(n+1)^2}\right)}\lim_{n\to\infty}\frac{-(n+1)^2(2n+1)}{3n^2(n+1)^2}=0$$

$$\lim_{n\to\infty}\frac{o(\frac{1}{n^2})}{\ln\left(1-\frac{1}{(n+1)^2}\right)}=0$$

(ii) Cesaro-Stolz again after writing

$$\frac{1}{2}\lim_{n\to\infty}\frac{2a_n-1}{\frac{1}{n}}$$

so we pass to the limit

$$\begin{aligned}
&\frac{1}{2}\lim_{n\to\infty}\frac{2a_{n+1}-2a_n}{\frac{1}{n+1}-\frac{1}{n}}=\\
&\lim_{n\to\infty}n(n+1)\left[\frac{\ln S_n-n\ln(1+\frac{1}{n})}{\ln(1+\frac{1}{n})}-\frac{\ln S_{n+1}-(n+1)\ln(1+\frac{1}{n+1})}{\ln(1+\frac{1}{n+1})}\right]=\\
&\lim_{n\to\infty}n(n+1)\left[\frac{\ln S_n}{\ln(1+\frac{1}{n})}-\frac{\ln S_{n+1}}{\ln(1+\frac{1}{n+1})}+1\right]=\\
&\lim_{n\to\infty}n(n+1)\left[\frac{\ln S_n}{\ln(1+\frac{1}{n})}-\frac{\ln S_n}{\ln(1+\frac{1}{n+1})}-\frac{\ln(1+\frac{1}{S_n(n+1)!})}{\ln(1+\frac{1}{n+1})}+1\right];
\end{aligned}$$

$$\lim_{n\to\infty}n(n+1)\frac{\ln(1+\frac{1}{S_n(n+1)!})}{\ln(1+\frac{1}{n+1})}=\lim_{n\to\infty}\frac{n(n+1)^2}{S_n(n+1)!}\frac{\frac{1}{n+1}}{\ln(1+\frac{1}{n+1})}\frac{\ln(1+\frac{1}{S_n(n+1)!})}{\frac{1}{S_n(n+1)!}}=0$$

$$\begin{aligned}
&n(n+1)\left[\frac{\ln S_n}{\ln(1+\frac{1}{n})}-\frac{\ln S_n}{\ln(1+\frac{1}{n+1})}+1\right]=\\
&n(n+1)\left[\frac{\ln S_n}{\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+O(\frac{1}{n^5})}+\right.\\
&\qquad\left.-\frac{\ln S_n}{\frac{1}{n+1}-\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}-\frac{1}{4(n+1)^4}+O(\frac{1}{n^5})}+1\right]=\\
&n(n+1)\left[\frac{n\ln S_n}{1-\left(\frac{1}{2n}-\frac{1}{3n^2}+\frac{1}{4n^3}+O(\frac{1}{n^4})\right)}+\right.\\
&\qquad\left.-\frac{(n+1)\ln S_n}{1-\left(\frac{1}{2(n+1)}-\frac{1}{3(n+1)^2}+\frac{1}{4(n+1)^3}+O(\frac{1}{n^4})\right)}+1\right]=
\end{aligned}$$

The expansion $1/(1-x)=1+x+x^2+x^3+x^4+O(x^5)$ yields

$$\begin{aligned}
&n(n+1)\left[\ln S_n\left(\frac{1}{2}-\frac{1}{12n}-\frac{1}{12n^2}+O(\frac{1}{n^3})\right)+\right.\\
&\qquad\left.-\ln S_n\left(\frac{1}{2}-\frac{1}{12(n+1)}-\frac{1}{12(n+1)^2}+O(\frac{1}{(n+1)^3})\right)\right]=\\
&=\frac{-\ln S_n}{12}-\frac{n(n+1)(2n+1)\ln S_n}{12n^2(n+1)^2}+O(\frac{1}{n^2})\to\frac{-1}{12}
\end{aligned}$$

Also solved by Moubinool Omarjee, Lycée Henri IV, Paris, France; Thiago Landim de Souza Leão, Federal University of Pernambuco, Recife, Brazil; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA.

U436. Let $f:[0,1]\to\mathbb{R}$ be a continuous function such that

$$\int_0^1 xf(x)(x^2+(f(x))^2)dx \geq \frac{2}{5}$$

Prove that

$$\int_0^1 \left(x^2+\frac{1}{3}(f(x))^2\right)^2 dx \geq \frac{16}{45}$$

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia
Applying given condition, we have

$$\begin{aligned}
0 &\leq \int_0^1 (f^2(x)+x^2-2xf(x))^2dx \\
&= \int_0^1 (f^4(x)+x^4+4x^2f^2(x)+2x^2f^2(x)-4xf^3(x)-4x^3f(x)dx \\
&= \int_0^1 f^4(x)+9x^4+6x^2f^2(x)dx - \int_0^1 8x^4dx - 4\int_0^1 (xf^3(x)+x^3f(x))dx \\
&\leq 9\int_0^1 \left(x^2+\frac{1}{3}f^2(x)\right)^2 dx - \frac{8}{5} - \frac{8}{5} \\
&\quad \Leftrightarrow \int_0^1 (x^2+\frac{1}{3}f^2(x))^2dx \geq \frac{16}{45}.
\end{aligned}$$

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy.

U437. Prove that for any $a > \frac{1}{e}$ the following inequality holds

$$\int_{1+\ln a}^{1+\ln(1+a)} x^x dx \geq 1$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Consider the function

$$f(x) = \frac{x^x}{e^{x-1}}, \;\; x > 0$$

Then we have

$$f'(x) = \frac{x^x \log x}{e^{x-1}}.$$

Hence we have $f(x)$ function is decreasing on $(0,1)$ and increasing $(1,+\infty)$. Thus we have

$$\forall x: \;\; f(x) \geq f(1) = 1$$

Using Mean Value theorem, we get

$$\begin{aligned}\int_{1+\log a}^{1+\log(1+a)} x^x dx &= \int_{1+\log a}^{1+\log(a+1)} \frac{x^x}{e^{x-1}} \cdot e^{x-1} dx \\ &= f(c) \cdot \int_{1+\log a}^{1+\log(a+1)} e^{x-1} dx = f(c).\end{aligned}$$

where exist $c \in (1+\log a, 1+\log(a+1))$. Remembering $a > \frac{1}{e}$, then we have $c > 0$. Hence we get $f(c) \geq 1$. Thus we have

$$\int_{1+\log a}^{1+\log(a+1)} x^x dx \geq 1.$$

Also solved by Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy.

U438. Prove that a positive integer n can be represented by the quadratic form x^2+7y^2 if and only if

(i) $v(n) \neq 1$

(ii) $v_p(n)$ is even for every prime number $p, p \equiv 3,5,6 \pmod 7$

Proposed by José Hernández Santiago, Morelia, México

Solution by the author

Proposition: A natural number n is of the form x^2+7y^2 for some $x,y \in \mathbb{Z}$ iff $v_2(n) \neq 1$ and, for every prime number $p \equiv 3,5,6 \pmod 7$, $v_p(n)$ is even.

Our proof of this proposition leans on the following four preliminary results:

Lemma. The product of two numbers of the form x^2+7y^2 is equal to another number of the same form.

Proof: If $a,b,c,d \in \mathbb{R}$, then

$$(a+i\sqrt{7}b)(c+i\sqrt{7}d) = (ac-7bd)+i\sqrt{7}(ad+bc).$$

Therefore,

$$\begin{aligned}(ac-7bd)^2+7(ad+bc)^2 &= |(ac-7bd)+i\sqrt{7}(ad+bc)|^2\\ &= |(a+i\sqrt{7}b)(c+i\sqrt{7}d)|^2\\ &= |a+i\sqrt{7}b|^2|c+i\sqrt{7}d|^2\\ &= (a^2+7b^2)(c^2+7d^2).\end{aligned}$$

□

Theorem: Let p be an odd prime number and D a natural number such that $p \nmid D$ and $\left(\frac{-D}{p}\right)=1$. Then, there exists $(k,x,y) \in \mathbb{Z}^3$ such that $0<k\le D$, $0<x,y<\sqrt{p}$, and $x^2+Dy^2=kp$.

Proof: (d'après A. Thue) Let s be an integer which satisfies the congruence $s^2 \equiv -D \pmod p$ and let us denote the set $\{0,1,\ldots,\lfloor\sqrt{p}\rfloor\}$ by $\mathcal{S}$. Then, a simple cardinality argument allows us to ascertain the existence of two elements of the set $\{t-su: (t,u)\in\mathcal{S}\times\mathcal{S}\}$ that are congruent modulo p. Let us suppose that these two elements correspond to the two (distinct) pairs $(t,u),(v,w)\in\mathcal{S}\times\mathcal{S}$. Then, if $x := |t-v|$ and $y := |u-w|$, it follows that $(x,y)\in\mathcal{S}\times\mathcal{S}$ and $x \equiv \pm sy \pmod p$. Since x,y are not simultaneously equal to zero we have that

$$0 < x^2+Dy^2 < p+Dp = (1+D)p. \tag{1}$$

On the other hand, it is the case that

$$x^2+Dy^2 \equiv s^2y^2+Dy^2 \equiv 0 \pmod p. \tag{2}$$

It follows from (1) and (2) that $x^2+Dy^2=kp$ for some $k \in (0,D]\cap\mathbb{N}$ and we are done. □

Corollary 1: Let $p \neq 2,7$ be a prime number. Then, $p=x^2+7y^2$ for some $x,y\in\mathbb{Z}$ iff $\left(\frac{-7}{p}\right)=1$.

Proof: [⇒] If $p=x^2+7y^2$ for some $x,y\in\mathbb{Z}$, then $p\nmid y$. Then, the thesis in question follows in this case after multiplying both sides of the congruence $x^2\equiv -7y^2 \pmod p$ by the square of an integer z such that $yz\equiv 1 \pmod p$.

[⇐] If $p\neq 2,7$ and $\left(\frac{-7}{p}\right)=1$, then the *theorem* above guarantees the existence of $(k,x,y)\in\mathbb{Z}^3$ such that $0<k\le 7$, $0<x,y<\sqrt{p}$, and $x^2+7y^2=kp$. We claim that from $x^2+7y^2=kp$ and the condition $0<k\le 7$, it can be deduced that $p=X^2+7Y^2$ for some $X,Y\in\mathbb{Z}$:

The case $k=2$ is impossible, otherwise the left-hand side of $x^2+7y^2=2p$ would be divisible by 4. The case $k=3$ is also impossible, otherwise the left-hand side of $x^2+7y^2=3p$ would be a multiple of 9: this would imply in turn that $p=3$, which is absurd because $\left(\frac{-7}{3}\right)\neq 1$.

If $k = 4$ and $x^2 + 7y^2 = 4p$, then x, y are integers of the same parity. If both of them are odd, then the left-hand side of $x^2 + 7y^2 = 4p$ is divisible by 8 (which is absurd given that $p \neq 2$). If $x = 2u$ and $y = 2v$ for some $u, v \in \mathbb{Z}$, then $p = \frac{x^2+7y^2}{4} = \frac{4u^2+7(4v^2)}{4} = u^2 + 7v^2$. In the case $k = 5$, a necessary condition for the equality $x^2 + 7y^2 = 5p$ to hold is that $x = 5u$ and $y = 5v$ for some $u, v \in \mathbb{Z}$; since this would imply that $5 \mid p$, this case can be discarded too (-7 is not a quadratic residue modulo 5). Finally, in the case $k = 6$, a necessary condition for the equality $x^2 + 7y^2 = 6p$ to hold is that $x = 3u$ and $y = 3v$ for some $u, v \in \mathbb{Z}$; since this would imply that $3 \mid p$, this case can also be discarded (-7 is not a quadratic residue modulo 3). □

Corollary 2: Let p be an odd prime number. Then, $p = x^2 + 7y^2$ for some $x, y \in \mathbb{Z}$ iff $p = 7$ or $p \equiv 1, 2, 4 \pmod 7$.

Proof. It is a straightforward consequence of the previous *corollary.* □

We proceed to demonstrate the *proposition* now:

[$\Rightarrow$] If $n = x^2 + 7y^2$ for some $x, y \in \mathbb{Z}$ and $2 \mid n$, then x and y are integers of the same parity: if both of them are even, $v_2(n) \geq 2$; if both of them are odd, $v_2(n) \geq 3$. Now, let us suppose that p is a prime divisor of n which is congruent to 3, 5 or 6 modulo 7. In order to reach a contradiction, let us assume that $v_p(n)$ is odd. If $d = \gcd(x, y)$, then $n = d^2(x_0^2 + 7y_0^2)$ for some coprime integers x_0, y_0. From this equality and the assumption we made on $v_p(n)$, it follows that $p \mid x_0^2 + 7y_0^2$ which clearly conflicts with the fact that -7 is not a quadratic residue modulo p.

[$\Leftarrow$] For every $\alpha \in \mathbb{Z}^+ \setminus \{1\}$, 2^α is a number of the form $x^2 + 7y^2$. The required conclusion follows from *corollary 2* and the *lemma.*

O433. Let q, r, s be positive integers such that $s^2 - s + 1 = 3qr$. Prove that $q + r + 1$ divides $q^3 + r^3 - s^3 + 3qrs$.

Proposed by Titu Andreescu, University of Texas at Dallas

Solution by Frank Rafael Gamboa, Havana, Cuba

We know that $s^2 - s + 1 = 3qr$, then $s^3 = 3qrs + 3qr - 1$. We deduce that:

$$A = q^3 + r^3 - s^3 + 3qrs = q^3 + r^3 - 3qr + 1$$

Consider the polynomial:

$$f(x) = x^3 - 3xr + (r^3 + 1)$$

Its clear that $f(q) = A$, and $f(-r-1) = 0$. This implies that exist $g \in \mathbb{Z}[x]$ such that $f(x) = (x + r + 1)g(x)$; in particular, $A = f(q) = (q + r + 1)g(q)$, and the conclusion follow.

Also solved by Arpon Basu, AECS-4, Mumbai, India; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Anish Ray, India; Jamal Gadirov, Baku Engineering University, Azerbaidjan; Thiago Landim de Souza Leão, Federal University of Pernambuco, Recife, Brazil; Titu Zvonaru, Comănești, Romania.

O434. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{b^2}{\sqrt{2(a^4+1)}}+\frac{c^2}{\sqrt{2(b^4+1)}}+\frac{a^2}{\sqrt{2(c^4+1)}} \geq \frac{3}{2}$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy
The concavity of $\sqrt{x}$ yields

$$\sqrt{1+x} \leq \sqrt{2}+\frac{x-1}{2\sqrt{2}}$$

thus it suffices to prove

$$\frac{b^2}{\sqrt{2}(\sqrt{2}+\frac{a^4-1}{2\sqrt{2}})}+\frac{c^2}{\sqrt{2}(\sqrt{2}+\frac{b^4-1}{2\sqrt{2}})}+\frac{a^2}{\sqrt{2}(\sqrt{2}+\frac{c^4-1}{2\sqrt{2}})} \geq \frac{3}{2}$$

that is
$$\frac{4b^2}{a^4+3}+\frac{4c^2}{b^4+3}+\frac{4a^2}{c^4+3} \geq 3 \iff \frac{4b^6}{b^4a^4+3b^4}+\frac{4c^6}{c^4b^4+3c^4}+\frac{4a^6}{a^4c^4+3a^4} \geq 3$$
Cauchy–Schwarz reversed yields

$$\frac{4b^6}{b^4a^4+3b^4}+\frac{4c^6}{c^4b^4+3c^4}+\frac{4a^6}{a^4c^4+3a^4} \geq \frac{4(a^3+b^3+c^3)^2}{(ab)^4+(bc)^4+(ca)^4+3(a^4+b^4+c^4)}$$

so we come to
$$\frac{4(a^3+b^3+c^3)^2}{(ab)^4+(bc)^4+(ca)^4+3(a^4+b^4+c^4)} \geq 3$$

and homogenizing

$$4(a^3+b^3+c^3)^2\frac{(a+b+c)^2}{9}-3(a^4b^4+b^4c^4+c^4a^4)-9(a^4+b^4+c^4)\frac{(a+b+c)^4}{81} \geq 0 \qquad (1)$$

This last inequality is symmetric so we set

$$a+b+c=3u, \quad ab+bc+ca=3v^2, \quad abc=w^3$$

$$a^3+b^3+c^3=27u^3-27uv^2+3w^3, \quad a^4+b^4+c^4=81u^4-108u^2v^2+18v^4+12uw^3$$
$$(ab)^4+(bc)^4+(ca)^4=\Big(\sum_{\text{cyc}} a^2b^2\Big)^2-2(abc)^2\sum_{\text{cyc}} a^2=$$
$$=(9v^4-6uw^3)^2-2w^3(9u^2-6v^2)$$

Plugging in (1) we get

$$-36v^2w^6 - 18u^2w^6 + 540u^5w^3 + 324uv^2w^3 - 648u^3v^2w^3 - 4860u^6v^2 - 243v^4 +$$
$$+2754u^4v^4 + 2187u^8 \geq 0$$

This is a concave parabola in w^3 and the inequality holds true if and only if it holds for the extreme values of w^3. The standard theory states that once fixed the values of u and v^2, the extreme values of w^3 occur when $c = 0$ (or cyclic) or $c = b$ (or cyclic).

If $c = 0$ we get

$$3a^8 + 4a^7b - 2a^6b^2 + 4a^5b^3 - 13a^4b^4 + 4a^3b^5 - 2b^6a^2 + 4b^7a + 3b^8 \geq 0$$

and this is clearly true by $4a^3b^5 - 2b^6a^2 + 4b^7a \geq 0$ via the AGM $4a^3b^5 + 4b^7a \geq 8b^6a^2$.

If $b = c$ we get

$$(3a^6 + 14a^5b + 17a^4b^2 + 4a^3b^3 - 17a^2b^4 + 10ab^5 + 5b^6)(a-b)^2 \geq 0$$

since

$$4a^3b^3 + 10ab^5 + 5b^6 + 17a^4b^2 \geq (2\sqrt{85} + 4\sqrt{10})b^4a^2 > 17b^4a^2$$

and this concludes the proof

Also solved by Kevin Soto Palacios, Huarmey, Perú; Anish Ray, India.

O435. Let a, b, c positive numbers such that $ab + bc + ca + 2abc = 1$. Prove that

$$\frac{1}{8a^2+1} + \frac{1}{8b^2+1} + \frac{1}{8c^2+1} \geq 1$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy

$$ab + bc + ca + 2abc = 1 \iff \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 2 = \frac{1}{abc}$$

$x = 1/a$, $y = 1/b$, $c = 1/z$ yields

$$x + y + z + 2 = xyz \implies \sum_{\text{cyc}} \frac{x^2}{8+x^2} \geq 1$$

$$x + y + z + 2 = xyz \iff x = \frac{\beta+\gamma}{\alpha},\ y = \frac{\alpha+\gamma}{\beta},\ z = \frac{\alpha+\beta}{\gamma}$$

Indeed

$$x + y + z = \frac{\sum_{\text{sym}} \beta^2\gamma}{\alpha\beta\gamma}, \quad xyz = \frac{2\alpha\beta\gamma + \sum_{\text{sym}} \beta^2\gamma}{\alpha\beta\gamma}$$

whence the equality. The inequality becomes

$$\sum_{\text{cyc}} \frac{(\beta+\gamma)^2}{8\alpha^2 + (\beta+\gamma)^2} \geq 1,\ \alpha, \beta, \gamma > 0$$

Let's assume $\alpha + \beta + \gamma = 1$ by homogeneity. The inequality becomes

$$\sum_{\text{cyc}} \frac{(1-\alpha)^2}{8\alpha^2 + (1-\alpha)^2} = \sum_{\text{cyc}} \frac{(1-\alpha)^2}{9\alpha^2 - 2\alpha + 1} \geq 1$$

Let

$$f(x) = \frac{(1-x)^2}{9x^2 - 2x + 1}, \quad f''(x) = 16\frac{(27x^2 - 18x^3 - 1)}{(9x^2 + 1 - 2x)^3}$$

The second derivative changes sign one time in the interval $[0, 1]$. Indeed $9x^2 + 1 - 2x > 0$ while $27x^2 - 18x^3 - 1$ passes from positive to negative values.

The standard theory states that the minimum of $f(\alpha) + f(\beta) + f(\gamma)$ occurs when $\alpha = \beta$ and then $\gamma = 1 - 2\alpha$ so we search the minimum of

$$2f(\alpha) + f(1-2\alpha) - 1 = \frac{2(1-3\alpha^2)(-1+3\alpha)^2}{(9\alpha^2 - 2\alpha + 1)(9\alpha^2 - 8\alpha + 2)} \geq 0,\ 0 \leq \alpha \leq 1/2$$

so the inequality holds true.

Also solved by Kevin Soto Palacios, Huarmey, Perú; Nikos Kalapodis, Patras, Greece; Arpon Basu, AECS-4, Mumbai, India; Frank Gamboa, Havana, Cuba; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Adamopoulos Dionysios, 3rd High School, Pyrgos, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.

O436. Prove that in any triangle ABC the following inequality holds:

$$\frac{a^2}{\sin\frac{A}{2}}+\frac{b^2}{\sin\frac{B}{2}}+\frac{c^2}{\sin\frac{C}{2}}\geq\frac{8}{3}s^2$$

Proposed by Dragoljub Miloševič, Gornji Milanovac, Serbia

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

We will prove the desired inequality using the Engel's form (or Titu's lemma) of the Cauchy-Schwarz inequality and observing that $f(x)=\sin\frac{x}{2}$ is a concave function in the interval $(0,\pi)$. The analytic criterion for concavity of a function is that its second derivative is negative. Indeed, $f'(x)=\frac{1}{2}\cos\frac{x}{2}$ and $f''(x)=-\frac{1}{4}\sin\frac{x}{2}<0$ for $0<x<\pi$.

Thus, by Jensen's inequality,

$$\frac{a^2}{\sin\frac{A}{2}}+\frac{b^2}{\sin\frac{B}{2}}+\frac{c^2}{\sin\frac{C}{2}}\geq\frac{(a+b+c)^2}{\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}}\geq\frac{(a+b+c)^2}{3\cdot\sin\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}}=\frac{(2s)^2}{3\sin\frac{\pi}{6}}=\frac{8}{3}s^2,$$

with equality if and only if $A=B=C$.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Kevin Soto Palacios, Huarmey, Perú; Nikos Kalapodis, Patras, Greece; Arpon Basu, AECS-4, Mumbai, India; Frank Gamboa, Havana, Cuba; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania.

O437. Let a, b, c the side-lengths of a triangle ABC. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \leq \frac{2s^2}{27r^2}+1$$

Proposed by Mircea Lascu and Titu Zvonaru, România

Solution by Arkady Alt, San Jose, CA, USA

Using well known cyclic inequality $abc+a^2b+b^2c+c^2a \leq \dfrac{4\left(a+b+c\right)^3}{27}$ we obtain

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{ab^2+a^2c+bc^2}{abc} \leq \frac{4\left(a+b+c\right)^3}{27abc}-1.$$

Thus, suffice to prove inequality

$$\frac{4\left(a+b+c\right)^3}{27abc} \leq \frac{2s^2}{27r^2}+2 \iff \frac{4\cdot 8s^3}{27\cdot 4Rrs} \leq \frac{2s^2}{27r^2}+2 \iff \frac{4s^2}{27Rr} \leq \frac{s^2}{27r^2}+1.$$

Since $16Rr-5r^2 \leq s^2 \leq 4R^2+4Rr+3r^2$ (Gerretsn's Inequalities) and $R \geq 2r$(Eulers Inequality) we have

$$\frac{s^2}{27r^2}+1-\frac{4s^2}{27Rr}=\frac{s^2}{27r^2}-\frac{2s^2}{27Rr}+1-\frac{2s^2}{27Rr}=$$

$$\frac{s^2\left(R-2r\right)}{27Rr^2}-\left(\frac{2s^2}{27Rr}-1\right) \geq$$

$$\frac{\left(16Rr-5r^2\right)\left(R-2r\right)}{27Rr^2}-\frac{2\left(4R^2+4Rr+3r^2\right)-27Rr}{27Rr}=\frac{\left(16Rr-5r^2\right)\left(R-2r\right)}{27Rr^2}-\frac{\left(R-2r\right)\left(8R-3r\right)}{27Rr}=$$

$$\frac{R-2r}{27Rr^2}\left(16Rr-5r^2-r\left(8R-3r\right)\right)=\frac{2r\left(R-2r\right)\left(4R-r\right)}{27Rr^2} \geq 0.$$

Also solved by Kevin Soto Palacios, Huarmey, Perú; Nermin Hodžic, Dobošnica, Bosnia and Herzegovina; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy.

O438. Let a, b, c be positive numbers such that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{49}{4}$$

Find all possible values of the expression

$$E=\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}$$

Proposed by Marius Stanean, Zalau, România

Solution by Nermin Hodžic, Dobošnica, Bosnia and Herzegovina

Let $\frac{a}{b}=x, \frac{b}{c}=y, \frac{c}{a}=z$, then $xyz=1$ and

$$(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})=\frac{49}{4} \Leftrightarrow$$

$$x+y+z+xy+yz+zx=\frac{37}{4}$$

Now we have

$$(x-y)^2(y-z)^2(z-x)^2 \geq 0 \Leftrightarrow$$

$$\sum_{cyc} x^2y^2(x^2+y^2)-2xyz\sum_{cyc} x^3+2xyz\sum_{cyc} xy(x+y)-2\sum_{cyc} x^3y^3-6x^2y^2z^2 \geq 0 \Leftrightarrow$$

$$4[(x+y+z)^2-3(xy+yz+zx)]^3 \geq [27xyz+2(x+y+z)^3-9(x+y+z)(xy+yz+zx)]^2$$

Let $x+y+z=t$ then $xy+yz+zx=\frac{37}{4}-t$, so

$$4[(x+y+z)^2-3(xy+yz+zx)]^3 \geq [27xyz+2(x+y+z)^3-9(x+y+z)(xy+yz+zx)]^2 \Leftrightarrow$$

$$4[t^2-3(\frac{37}{4}-t)]^3 \geq [27+2t^3-9t(\frac{37}{4}-t)]^2 \Leftrightarrow$$

$$(4t-17)(t-5)(4t^2-37t-601) \geq 0$$

since x, y, z are positive then $0<t<\frac{37}{4}$, so we have $\frac{17}{4} \leq t \leq 5$.

Since

$$E=\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}=(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})^2-2(\frac{b}{a}+\frac{c}{b}+\frac{a}{c})=$$

$$=(x+y+z)^2-2(\frac{37}{4}-x-y-z)=t^2+2t-\frac{37}{2}=$$

$$=(t+1)^2-\frac{39}{2}$$

then we have

$$\frac{129}{16} \leq E \leq \frac{33}{2}$$

minimum is attained when $(x,y,z)=\left(\frac{1}{4},2,2\right) \Rightarrow c=2a, b=4a$ and cyclic permutations

maximum is attained when $(x,y,z)=\left(\frac{1}{2},\frac{1}{2},4\right) \Rightarrow b=2a, c=4a$ and cyclic permutations

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy.