Junior Problems

J439. Solve in real numbers the system of equations:

$$\begin{cases} 2x^2 - 3xy + 2y^2 = 1 \\ y^2 - 3yz + 4z^2 = 2 \\ z^2 + 3zx - x^2 = 3 \end{cases}$$

Proposed by Adrian Andreescu, University of Texas at Austin

J440. Let a, b, c, d be distinct nonnegative real numbers. Prove that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-d)^2} + \frac{c^2}{(d-a)^2} + \frac{d^2}{(a-b)^2} > 2.$$

Proposed by An Zhenping, Xianyang Normal University, China

J441. Prove that for any positive real numbers a, b, c the following inequality holds

$$\frac{(a+b+c)^3}{3abc} + 1 \geq \left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2 + (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J442. Let ABC be an equilateral triangle with center O. A line passing through O intersects sides AB and AC at M and N, respectively. Segments BN and CM intersect at K and segments AK and BO intersect at P. Prove that $MB = MP$.

Proposed by Anton Vassilyev, Kazakhstan

J443. Find all pairs (m, n) of integers such that both equations

$$x^2 + mx - n = 0,$$
$$x^2 + nx - m = 0$$

have integers roots.

Proposed by Alessandro Ventullo, Milan, Italy

J444. Let a, b, c, d be nonnegative real numbers such that $a + b + c + d = 4$. Prove that

$$a^3b + b^3c + c^3d + d^3a + 5abcd \leq 27.$$

Proposed by Marius Stănean, Zalău, România

Senior Problems

S439. Let ABC be a triangle. Let points D and E be on segment BC and line AC, respectively, such that $\triangle ABC \sim \triangle DEC$. Let M be the midpoint of BC. Let P be a point such that $\angle BPM = \angle CBE$ and $\angle MPC = \angle BED$ and A, P lie on the same side of BC. Let Q be the intersection of lines AB and PC. Prove that the lines AC, BP, QD are either concurrent or all parallel.

Proposed by Grant Yu, East Setauket NY, USA

S440. Prove that for any positive real numbers a, b, c the following inequality holds:

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq \frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S441. Let a, b, c be positive numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{ab}{4-a^2} + \frac{bc}{4-b^2} + \frac{ca}{4-c^2} \leq 1.$$

Proposed by An Zhenping, Xianyang Normal University, China

S442. Solve in integers the system of equations:

$$\begin{cases} x^3 - y^2 - 7z^2 = 2018 \\ 7x^2 + y^2 + z^3 = 1312. \end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S443. Let ABC be a triangle, and let r_a, r_b, r_c be its exradii. Prove that

$$r_a \cos\frac{A}{2} + r_b \cos\frac{B}{2} + r_c \cos\frac{C}{2} \leq \frac{3}{2}s.$$

Proposed by Dragoljub Miloševič, Gornji Milanovac, Serbia

S444. Let $x_1, \ldots, x_n$ be positive real numbers. Prove that

$$\sum_{k=1}^{n} \frac{x_k}{x_k + \sqrt{x_1^2 + \cdots + x_n^2}} \leq \frac{n}{1+\sqrt{n}}.$$

Proposed by Angel Plaza, University of Las Palmas de Gran Canaria, Spain

Undergraduate Problems

U439. Evaluate
$$\int_{\frac{1}{2}}^{2} \frac{x^2+2x+3}{x^4+x^2+1}\,dx.$$

Proposed by Alessandro Ventullo, Milan, Italy

U440. Let a, b, c, $t \geq 1$. Prove that
$$\frac{1}{ta^3+1}+\frac{1}{tb^3+1}+\frac{1}{tc^3+1} \geq \frac{3}{tabc+1}.$$

Proposed by An Zhenping, Xianyang Normal University, China

U441. Let x, y, z be nonnegative real numbers such that $x+y+z=1$, and let $1 \leq \lambda \leq \sqrt{3}$. Determine the minimum and maximum of
$$f(x,\, y,\, z) = \lambda(xy+yz+zx)+\sqrt{x^2+y^2+z^2}$$
in terms of λ.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U442. Let $(p_k)_{k\geq 1}$ be the sequence of primes and $q_n = \prod_{k\leq n} p_k$. For every positive integer n, $\omega(n)$ denotes the number of prime divisors of n. Evaluate
$$\lim_{n\to\infty} \frac{\sum_{p\mid q_n} (\log p)^{\alpha}}{\omega(q_n)^{1-\alpha}(\log q_n)^{\alpha}},$$
where $\alpha \in (0,\, 1)$ is a real number.

Proposed by Alessandro Ventullo, Milan, Italy

U443. Find
$$\lim_{n\to\infty} \int_0^{\pi} \frac{\sin x}{1+\cos^2 nx} dx.$$

Proposed by Robert Bosch, USA

U444. Let $p > 2$ be a prime and let $f(x) \in \mathbb{Q}[x]$ be a polynomial such that $\deg(f) < p-1$ and $x^{p-1}+x^{p-2}+\cdots+1$ divides $f(x)f(x^2)\cdots f(x^{p-1})-1$. Prove that there exists a polynomial $g(x) \in \mathbb{Q}[x]$ and a positive integer i such that $i < p$, $\deg(g) < p-1$, and $x^{p-1}+x^{p-2}+\cdots+1 \mid g(x^i)f(x)-g(x)$.

Proposed by Sreejata Kishor Bhattacharya, Chennai Mathematical Institute, India

Olympiad Problems

O439. Find all triples (x, y, z) of integers such that

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = 2018.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O440. Prove that in any triangle ABC the following inequality holds

$$\left(\frac{a}{b+c}\right)^2 + \left(\frac{b}{c+a}\right)^2 + \left(\frac{c}{a+b}\right)^2 + \frac{r}{2R} \geq 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O441. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{\sqrt{2(a^4+b^4)}+4ab} + \frac{1}{\sqrt{2(b^4+c^4)}+4bc} + \frac{1}{\sqrt{2(c^4+a^4)}+4ca} + \frac{a+b+c}{3} \geq \frac{3}{2}.$$

When does equality hold?

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

O442. Let a, b, c be real numbers such that $a+b+c=3$. Prove that

$$7(a^4+b^4+c^4) + 27 \geq (a+b)^4 + (b+c)^4 + (c+a)^4.$$

Proposed by Marius Stănean, Zalău, România

O443. Let $f(n)$ be the number of permutations of the set $\{1, 2, \ldots, n\}$ such that no pair of consecutive integers appears in that order; that is, 2 does not follow 1, 3 does not follow 2, and so on.

(i) Prove that $f(n) = (n-1)f(n-1) + (n-2)f(n-2)$.

(ii) For any real number α, denote by $[\alpha]$ the nearest integer to α. Prove that

$$f(n) = \frac{1}{n}\left[\frac{(n+1)!}{e}\right].$$

Proposed by Rishub Thaper, Hunterdon Central Regional High School, Flemington, NJ, USA

O444. Let T be Toricelli point of a triangle ABC. Prove that

$$\frac{1}{BC^2} + \frac{1}{CA^2} + \frac{1}{AB^2} \geq \frac{9}{(AT+BT+CT)^2}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam