Junior Problems

J445. Find all pairs (p, q) of primes such that $p^2 + q^3$ is a perfect cube.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J446. Let a, b, c be positive real numbers such that $ab + bc + ca = 3abc$. Prove that

$$\frac{1}{2a^2+b^2} + \frac{1}{2b^2+c^2} + \frac{1}{2c^2+a^2} \leq 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J447. Let $N = \overline{d_0 d_1 \cdots d_9}$ be a 10−digit number with $d_{k+5} = 9 - d_k$, for $k = 0, 1, 2, 3, 4$. Prove that N is divisible by 41.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J448. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$4 \leq \sqrt{a^4+b^2+c^2+1} + \sqrt{b^4+c^2+a^2+1} + \sqrt{c^4+a^2+b^2+1} \leq 3\sqrt{2}.$$

Proposed by An Zhenping, Xianyang Normal University, China

J449. A square of area 1 is inscribed in a rectangle such that each side of the rectangle contains precisely a vertex of the square. What is the greatest possible area of the rectangle?

Proposed by Mircea Becheanu, Montreal, Canada

J450. Prove that in any triangle ABC

$$\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} \geq \sqrt{\frac{3(4R+r)}{2R}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Senior Problems

S445. Solve in integers the equation:

$$x^3 - y^3 - 1 = (x + y - 1)^2.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

S446. Let a and b be positive real numbers such that $ab = 1$. Prove that

$$\frac{2}{a^2 + b^2 + 1} \le \frac{1}{a^2 + b + 1} + \frac{1}{a + b^2 + 1} \le \frac{2}{a + b + 1}.$$

Proposed by An Zhenping, Xianyang Normal University, China

S447. Let a, b, c, $d \ge -1$ such that $a + b + c + d = 4$. Find the maximum of

$$\left(a^2 + 3\right)\left(b^2 + 3\right)\left(c^2 + 3\right)\left(d^2 + 3\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S448. Let ABC be a triangle with area Δ. Prove that for any point P in the plane of the triangle

$$AP + BP + CP \ge 2\sqrt[4]{3}\sqrt{\Delta}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S449. Find the maximum of

$$\left(\frac{9b + 4c}{a} - 6\right)\left(\frac{9c + 4a}{b} - 6\right)\left(\frac{9a + 4b}{c} - 6\right),$$

over all positive real numbers a, b, c.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S450. Let ABC be a triangle and D the foot of the altitude from B. The tangents in B and C to the circumcircle of ABC meet in S. Let P be the intersection of BD and AS. We know that $BP = PD$. Calculate $\angle ABC$.

Proposed by Mihaela Berindeanu, Bucharest, România

Undergraduate Problems

U445. Let a, b, c be the roots of the equations $x^3 + px + q = 0$, where $q \neq 0$. Evaluate the sum

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$$

in terms of p and q.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U446. Find the minimum of $\max\left\{|1+z|,\, |1+z^2|\right\}$, when z runs over all complex numbers.

Proposed by Robert Bosch, USA

U447. If F_n is the n^{th} Fibonacci number, then for fixed p show that

$$\sum_{k=1}^{n} \binom{n}{k} F_p^k F_{p-1}^{n-k} F_k = F_{pn}.$$

Proposed by Tarit Goswami, West Bengal, India

U448. Let $p \geq 5$ be a prime number. Prove that the polynomial

$$2X^p - p3^pX + p^2$$

is irreducible in $\mathbb{Z}[X]$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U449. Evaluate

$$\int_0^{\frac{\pi}{4}} \ln \frac{\tan \frac{x}{3}}{(\tan x)^2}\, dx.$$

Proposed by Perfetti Paolo, Università degli studi di Tor Vergata Roma, Italy

U450. Let P be a nonconstant polynomial with integer coefficients. Prove that for each positive integer n there are pairwise relatively prime positive integers $k_1, k_2, \ldots, k_n$ such that $k_1k_2\cdots k_n = |P(m)|$ for some positive integer m.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Olympiad Problems

O445. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\sqrt[8]{\frac{a^3+b^3+c^3}{3}} \leq \frac{3}{ab+bc+ca}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O446. Prove that in any triangle ABC the following inequality holds:

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2} \leq \sqrt{2+\frac{r}{2R}}.$$

Proposed by Dragoljub Miloševići, Gornji Milanovac, Serbia

O447. Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2 \geq a^3+b^3+c^3$. Prove that

$$a^3b^3+b^3c^3+c^3a^3 \leq a^2b^2+b^2c^2+c^2a^2.$$

Proposed by An Zhenping, Xianyang Normal University, China

O448. Prove that for any positive integers m and n there are m consecutive positive integer numbers such that each number has at least n divisors.

Proposed by Anton Vassilyev, Astana, Kazakhstan

O449. At the AwesomeMath Summer Camp, a teacher wants to challenge his 102 students. He gives them 19 green t-shirts, 25 red t-shirts, 28 purple t-shirts and 30 blue t-shirts, a t-shirt to each student. Then, he calls three students randomly: if they have a t-shirt with different colors, they must wear a t-shirt of the remaining color and must solve a problem given by the teacher. Is it possible that after some time all the students have all the t-shirts of the same color? (Assume that there are sufficient t-shirts for each color in the store).

Proposed by Alessandro Ventullo, Milan, Italy

O450. A computer had randomly assigned all labels from 1 through 64 to an 8×8 electronic board. Then it did it also randomly for the second time. Let n_k be the label of the square that had been originally assigned k. Knowing that $n_{17}=18$, find the probability that

$$|n_1-1|+|n_2-2|+\cdots+|n_{64}-64|=2018.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA