Junior Problems

**J445.** Find all pairs \((p, q)\) of primes such that \(p^2 + q^3\) is a perfect cube.

*Proposed by Adrian Andreescu, University of Texas at Austin, USA*

**J446.** Let \(a, b, c\) be positive real numbers such that \(ab + bc + ca = 3abc\). Prove that

\[
\frac{1}{2a^2 + b^2} + \frac{1}{2b^2 + c^2} + \frac{1}{2c^2 + a^2} \leq 1.
\]

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

**J447.** Let \(N = d_0d_1 \cdots d_9\) be a 10-digit number with \(d_{k+5} = 9 - d_k\), for \(k = 0, 1, 2, 3, 4\). Prove that \(N\) is divisible by 41.

*Proposed by Adrian Andreescu, University of Texas at Austin, USA*

**J448.** Let \(a, b, c\) be real numbers such that \(a^2 + b^2 + c^2 = 1\). Prove that

\[
4 \leq \sqrt{a^4 + b^2 + c^2 + 1} + \sqrt{b^4 + c^2 + a^2 + 1} + \sqrt{c^4 + a^2 + b^2 + 1} \leq 3\sqrt{2}.
\]

*Proposed by An Zhenping, Xianyang Normal University, China*

**J449.** A square of area 1 is inscribed in a rectangle such that each side of the rectangle contains precisely a vertex of the square. What is the greatest possible area of the rectangle?

*Proposed by Mircea Becheanu, Montreal, Canada*

**J450.** Prove that in any triangle \(ABC\)

\[
\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} \geq \sqrt{\frac{3(4R + r)}{2R}}.
\]

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*
Senior Problems

S445. Solve in integers the equation:

\[ x^3 - y^3 - 1 = (x + y - 1)^2. \]

Proposed by Adrian Andreescu, University of Texas at Austin, USA

S446. Let \( a \) and \( b \) be positive real numbers such that \( ab = 1 \). Prove that

\[ \frac{2}{a^2 + b^2 + 1} \leq \frac{1}{a^2 + b + 1} + \frac{1}{a + b^2 + 1} \leq \frac{2}{a + b + 1}. \]

Proposed by An Zhenping, Xianyang Normal University, China

S447. Let \( a, b, c, d \geq -1 \) such that \( a + b + c + d = 4 \). Find the maximum of

\[ (a^2 + 3) (b^2 + 3) (c^2 + 3) (d^2 + 3). \]

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S448. Let \( ABC \) be a triangle with area \( \Delta \). Prove that for any point \( P \) in the plane of the triangle

\[ AP + BP + CP \geq 2 \sqrt{3} \sqrt{\Delta}. \]

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S449. Find the maximum of

\[ \left( \frac{9b + 4c}{a} - 6 \right) \left( \frac{9c + 4a}{b} - 6 \right) \left( \frac{9a + 4b}{c} - 6 \right), \]

over all positive real numbers \( a, b, c \).

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S450. Let \( ABC \) be a triangle and \( D \) the foot of the altitude from \( B \). The tangents in \( B \) and \( C \) to the circumcircle of \( ABC \) meet in \( S \). Let \( P \) be the intersection of \( BD \) and \( AS \). We know that \( BP = PD \). Calculate \( \angle ABC \).

Proposed by Mihaela Berindeanu, Bucharest, România
Undergraduate Problems

U445. Let $a$, $b$, $c$ be the roots of the equations $x^3 + px + q = 0$, where $q \neq 0$. Evaluate the sum

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$$

in terms of $p$ and $q$.

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

U446. Find the minimum of $\max \{|1 + z|, |1 + z^2|\}$, when $z$ runs over all complex numbers.

*Proposed by Robert Bosch, USA*

U447. If $F_n$ is the $n^{th}$ Fibonacci number, then for fixed $p$ show that

$$\sum_{k=1}^{n} \binom{n}{k} F_p^k F_{p-1}^{n-k} F_k = F_{pn}.$$  

*Proposed by Tarit Goswami, West Bengal, India*

U448. Let $p \geq 5$ be a prime number. Prove that the polynomial

$$2X^p - p^3X + p^2$$

is irreducible in $\mathbb{Z}[X]$.

*Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran*

U449. Evaluate

$$\int_{0}^{\frac{\pi}{4}} \ln \frac{\tan \frac{x}{2}}{(\tan x)^2} dx.$$  

*Proposed by Perfetti Paolo, Università degli studi di Tor Vergata Roma, Italy*

U450. Let $P$ be a nonconstant polynomial with integer coefficients. Prove that for each positive integer $n$ there are pairwise relatively prime positive integers $k_1, k_2, \ldots, k_n$ such that $k_1 k_2 \cdots k_n = |P(m)|$ for some positive integer $m$.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*
Olympiad Problems

O445. Let $a$, $b$, $c$ be positive real numbers such that $a + b + c = 3$. Prove that
\[ \sqrt[3]{a^3 + b^3 + c^3} \leq \frac{3}{ab + bc + ca}. \]

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O446. Prove that in any triangle $ABC$ the following inequality holds:
\[ \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \leq \sqrt{2 + \frac{r}{2R}}. \]

Proposed by Dragoljub Miloševići, Gornji Milanovac, Serbia

O447. Let $a$, $b$, $c$ be nonnegative real numbers such that $a^2 + b^2 + c^2 \geq a^3 + b^3 + c^3$. Prove that
\[ a^3b^3 + b^3c^3 + c^3a^3 \leq a^2b^2 + b^2c^2 + c^2a^2. \]

Proposed by An Zhenping, Xianyang Normal University, China

O448. Prove that for any positive integers $m$ and $n$ there are $m$ consecutive positive integer numbers such that each number has at least $n$ divisors.

Proposed by Anton Vassilyev, Astana, Kazakhstan

O449. At the AwesomeMath Summer Camp, a teacher wants to challenge his 102 students. He gives them 19 green t-shirts, 25 red t-shirts, 28 purple t-shirts and 30 blue t-shirts, a t-shirt to each student. Then, he calls three students randomly: if they have a t-shirt with different colors, they must wear a t-shirt of the remaining color and must solve a problem given by the teacher. Is it possible that after some time all the students have all the t-shirts of the same color? (Assume that there are sufficient t-shirts for each color in the store).

Proposed by Alessandro Ventullo, Milan, Italy

O450. A computer had randomly assigned all labels from 1 through 64 to an $8 \times 8$ electronic board. Then it did it also randomly for the second time. Let $n_k$ be the label of the square that had been originally assigned $k$. Knowing that $n_{17} = 18$, find the probability that
\[ |n_1 - 1| + |n_2 - 2| + \cdots + |n_{64} - 64| = 2018. \]

Proposed by Titu Andreescu, University of Texas at Dallas, USA