Power-sum problem, Bernoulli Numbers and Bernoulli Polynomials.

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Definition 1 ***(Power Sum Problem)*** *Find the sum* $S_p(n) := 1^p + 2^p + ... + n^p$ *where* $p, n \in \mathbb{N}$ *(or, using sum notation,* $S_p(n) = \sum_{k=1}^{n} k^p$*) in closed form.*

Recurrence for $S_p(n)$

Exercise 2 *Using representations* $1 = (k+1) - k, 2k = k(k+1) - (k-1)k, 3k = k(k+1)(k+2) - (k-1)k(k+1)$

find $S_p(n)$ for $p = 1, 2, 3$ and $n \in \mathbb{N}$.

Exercise 3 *By summing differences* $k^2 - (k-1)^2 = 2k - 1, k^3 - (k-1)^3 = 3k^2 - 3k + 1,$

$k^4 - (k-1)^4 = 4k^3 - 6k^2 + 4k - 1$ for k running from 1 to n find $S_p(n)$ for $p = 2, 3, 4$.

General case

Exercise 4 *For any* $p \in \mathbb{N}$ *by summing differences* $(k+1)^{p+1} - k^{p+1} = \sum_{i=1}^{p+1} \binom{p+1}{i} k^{p+1-i}$ *for* k *running from* 1 *to* n *prove that*

$$S_p(n) = \frac{(n+1)^{p+1} - n - 1 - \sum_{i=1}^{p-1} \binom{p+1}{i} S_i(n)}{p+1} \qquad \textbf{(1)}$$

Exercise 5 *For any* $p \in \mathbb{N}$ *by summing differences* $k^{p+1} - k - 1^{p+1} = \sum_{i=1}^{p+1} \binom{p+1}{i} k^{p+1-i}$ *for* k *running from* 1 *to* n *prove that*

$$S_p(n) = \frac{1}{p+1}\left(n^{p+1} + \sum_{i=1}^{p} (-1)^{i+1} \binom{p+1}{i+1} S_{p-i}(n)\right) \qquad \textbf{(2)}$$

Recurrences **(1)** and **(2)** give opportunity, starting from $S_0(n) = \sum_{k=1}^{n} k^0 = n$, constructively find representation of $S_p(n)$ as polynomial of n.

Since any polynomial degree of m uniquely defined by their values in $m+1$ distinct points ((**1**) or (**2**) holds for any natural n) then, by such,

polynomials $S_p(x)$ are defined for any $x \in \mathbb{R}$ and $p \in \mathbb{N}$, more precisely, defined sequence of polynomials $\left(S_p(x)\right)_{p\in\mathbb{N}}$ by recurrence

$$S_p(x) = \frac{(x+1)^{p+1} - 1 - \sum_{i=0}^{p-1} \binom{p+1}{i} S_i(x)}{p+1} \qquad \textbf{(1')}$$

(or by recurrence

$$S_p(x) = \frac{1}{p+1}\left(x^{p+1} + \sum_{i=1}^{p} (-1)^{i+1} \binom{p+1}{i+1} S_{p-i}(x)\right)) \qquad \textbf{(2')}$$

with initial condition $S_0(x) = x$.

1 Bernoulli Numbers and Bernoulli Polynomials

Our goal is to solve this recurrence in closed form, that is to find a regular polynomial representation of $S_p(x)$.

Since $S_p(0) = 0$ for any $p = 0, 1, 2, ...$ then we should find real numbers $s_1, ..., s_{p+1}$ such that $S_p(x) = s_1 x + ... s_{p+1} x^{p+1}$.

Note that the problem would simply be solved if we had known for some polynomial $H(x)$ of degree $p+1$ such that $H(x+1) - H(x) = cx^p$ where c is some constant.

Then $S_p(n) = \sum\limits_{k=1}^{n} k^n = \frac{1}{c} \sum\limits_{k=1}^{n} (H(k+1) - H(k)) = \frac{H(n+1) - H(1)}{c}$.

In a sense, we already have one such polynomial (up to an arbitrary constant c), $H(x) = S_p(x-1) + c$ since $H(x+1) - H(x) = S_p(x) - S_p(x-1) = x^p$

But our problem is that $S_p(x)$ is not yet represented in terms of powers of x.

By differentiation of $S_{p+1}(x) - S_{p+1}(x-1) = x^{p+1}$ we obtain $S'_{p+1}(x) - S'_{p+1}(x-1) = (p+1)x^p$; then $S'_{p+1}(x-1)$ can be co sidered as another candidate for the role of $H(x)$, which does not look better than $S_p(x-1)$ for the same reason.

We know that $S_0(x) = x, S_1(x) = \frac{x(x+1)}{2}, S_2(x) = \frac{x(x+1)(2x+1)}{6}, S_3(x) = \frac{x^2(x+1)^2}{4}$

Applying the recurrences **(1)** or **(2)** we obtain

$S_4(x) = \frac{x(x+1)(2x+1)\left(3x^2+3x-1\right)}{30}$ and $S_5(x) = \frac{x^2(x+1)^2\left(2x^2+2x-1\right)}{12}$.

Accordingly, we also have $S'_0(x) = 1, S'_1(x) = x + \frac{1}{2}, S'_2(x) = x^2 + x + \frac{1}{6}, S'_3(x) = x^3 + \frac{3x^2}{2} + \frac{x}{2}$,

$S'_4(x) = x^4 + 2x^3 + x^2 - \frac{1}{30}, S'_5(x) = x^5 + \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{x}{6}$

$S''_0(x) = 0, S''_1(x) = 1, S''_2(x) = 2x + 1 = 2\left(x + \frac{1}{2}\right) = 2S'_1(x)$,

$S''_3(x) = 3x^2 + 3x + \frac{1}{2} = 3\left(x^2 + x + \frac{1}{6}\right) = 3S'_2(x)$,

$S''_4(x) = 4x^3 + 6x^2 + 2x = 4\left(x^3 + \frac{3x^2}{2} + \frac{x}{2}\right) = 4S'_3(x)$,

$S''_5(x) = 5x^4 + 10x^3 + 5x^2 - \frac{1}{6} = 5\left(x^4 + 2x^3 + x^2 - \frac{1}{30}\right) = 5S'_4(x)$.

The above equations lead to the conclusion that the correlation $S''_p(x) = pS'_{p-1}(x)$ holds for any $p \in \mathbb{N}$.

In fact, assuming $S''_i(x) = pS'_{i-1}(x), i = 1, 2, ..., p-1$,and by differentiating **(1')** twice, we obtain

$$S'_p(x) = \frac{(p+1)(x+1)^p - \sum\limits_{i=0}^{p-1}\binom{p+1}{i}S'_i(x)}{p+1} = \frac{(p+1)(x+1)^p - 1 - \sum\limits_{i=1}^{p-1}\binom{p+1}{i}S'_i(x)}{p+1} \quad \text{and}$$

$$S''_p(x) = \frac{(p+1)p(x+1)^{p-1} - \sum\limits_{i=1}^{p-1}\binom{p+1}{i}S''_i(x)}{p+1} =$$

$$\frac{(p+1)p(x+1)^{p-1} - \sum\limits_{i=1}^{p-1}\binom{p+1}{i}iS'_{i-1}(x)}{p+1} =$$

$$\frac{(p+1)p(x+1)^{p-1} - (p+1)\sum\limits_{i=1}^{p-1}\binom{p}{i-1}S'_{i-1}(x)}{p+1} =$$

$$p(x+1)^{p-1} - \sum\limits_{i=0}^{p-2}\binom{p}{i}S'_{i-1}(x) = p \cdot \frac{(x+1)^{p-1} - \sum\limits_{i=0}^{p-2}\binom{p}{i}S'_{i-1}(x)}{p} = pS'_{p-1}(x).$$

Exercise 6 *Prove that $S''_p(x) = pS'_{p-1}(x)$, for any $p \in \mathbb{N}$ using **(2')**.*

Thus, by induction, $S''_p(x) = pS'_{p-1}(x)$ for any $p \in \mathbb{N}$.

Coming back to the polynomial $S'_p(x-1)$, we denote it by $B_p(x)$, and then by replacing x with $x-1$ in the recurrence

$$S'_p(x) = \frac{(p+1)(x+1)^p - \sum_{i=0}^{p-1}\binom{p+1}{i}S'_i(x)}{p+1}$$

we obtain the following recurrence for polynomials $B_p(x), p \in \mathbb{N}$:

$$B_p(x) = (x-1)^p + \frac{\sum_{i=1}^{p}(-1)^{i+1}\binom{p+1}{i+1}B_{p-i}(x)}{p+1}. \qquad \textbf{(B2)}$$

2 Properties.

P0. $\deg B_p(x) = \deg S'_p(x-1) = p$;

P1. $B_0(x) = S'_1(x-1) = 1$;

P2. $B'_p(x) = \left(S'_p(x-1)\right)' = S''_p(x-1) = pS'_{p-1}(x) = pB_{p-1}(x)$;

P3. $B_p(x+1) - B_p(x) = S'_p(x) - S'_p(x-1) = px^{p-1}, p \in \mathbb{N}$.

We call such polynomials *Bernoulli Polynomials.*

We already have the first few polynomials $B_p(x)$, namely,

$B_1(x) = S'_1(x-1) = x - 1 + \frac{1}{2} = x - \frac{1}{2}, B_2(x) = S'_2(x-1) = (x-1)^2 + (x-1) + \frac{1}{6} = x^2 - x + \frac{1}{6},$

$B_3(x) = S'_3(x-1) = (x-1)^3 + \frac{3(x-1)^2}{2} + \frac{(x-1)}{2} = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x;$

$B_4(x) = S'_4(x-1) = x^4 - 2x^3 + x^2 - \frac{1}{30}, B_5(x) = S'_5(x-1) = x^5 - \frac{5x^4}{2} + \frac{5x^3}{3} - \frac{x}{6}.$

We can see that $B_1(0) = -\frac{1}{2}, B_1(1) = \frac{1}{2}$, but $B_2(0) = B_2(1) = \frac{1}{6}, B_3(0) = B_3(1) = 0, B_4(0) = B_4(1) = -\frac{1}{30}, B_5(0) = B_5(1) = 0$

and in general $B_p(0) = B_p(1)$ for any $p \geq 2$. Furthermore, $B_{2p+1}(0) = B_{2p+1}(1) = 0$.

Since $B_p(x+1) - B_p(x) = px^{p-1}$, then for $x = 0$ we obtain $B_p(1) - B_p(0) = p \cdot 0^{p-1} \iff B_p(1) = B_p(0)$, for all $p \geq 2$.

(Hypothesis $B_{2p+1}(0) = B_{2p+1}(1) = 0, p \in \mathbb{N}$ is equivalent to dividing $B_{2p+1}(x)$ by x which we will prove later).

Note that the recursion $B'_p(x) = pB_{p-1}(x), p \in \mathbb{N}$ with initial condition $B_0(x) = 1$ allows us to obtain polynomials $B_1(x), B_2(x), B_3(x), ...$and thus easier than by recurrence **(B1)** or **(B2).**

Indeed, assume that we already know polynomial $B_{p-1}(x)$,then $B_p(x) - B_p(1) = \int_1^x B'_p(t)\,dt = \int_1^x pB_{p-1}(t)\,dt \iff B_p(x) = B_p(0) + p\int_1^x B_{p-1}(t)\,dt$.

Let $B_p := B_p(0), p \in \mathbb{N} \cup \{0\}$. We call such numbers *Bernoulli Numbers*.

By replacing x in **(B1)** or in **(B2)** with 0 we obtain

$$B_p = \frac{-\sum_{i=0}^{p-1}\binom{p+1}{i}B_i}{p+1} \qquad \textbf{(B3)}$$

or

$$B_p = (-1)^p + \frac{\sum_{i=1}^{p}(-1)^{i+1}\binom{p+1}{i+1}B_{p-i}}{p+1}. \qquad \textbf{(B4)}$$

Any of these recurrences allows to get consistently numbers $B_1, B_2, B_3, ...$

Exercise 7 *Find the first 5 terms of sequence $(B_p)_{p\geq 0}$.*

We show that by $B_k, k = 1, 2, ...$, we can obtain polynomial $B_p(x)$.

Let $B_p(x) = b_px^p + b_{p-1}x^{p-1} + ... + b_1x + b_0$,where b_k should be determined.

Since $B_p(0) = B_p$ then $b_0 = B_p$. Also since $B'_p(x) = pB_{p-1}(x)$ then $B_p^{(k)}(x) = p(p-1)...(p-k+1)B_{p-k}(x)$ and $B_p^{(k)}(x) = \left(b_px^p + b_{p-1}x^{p-1} + ... + b_1x + b_0\right)^{(k)} = \left(b_px^p + b_{p-1}x^{p-1} + ... + b_{k+1}x^k\right)^{(k)} + b_kk!$ yields

$B_p^{(k)}(0) = b_kk! \iff p(p-1)...(p-k+1)B_{p-k}(0) = b_kk! \iff b_k = \frac{p(p-1)...(p-k+1)}{k!}B_{p-k} \iff$

$$b_k = \binom{p}{k} B_{p-k}, k = 1, 2, ..., p.$$

Hence, $B_p(x) = B_p + \binom{p}{1} B_{p-1} x^1 + ... + \binom{p}{p-1} B_1 x^{p-1} + B_0 x^p = \sum_{k=0}^{p} \binom{p}{k} B_{p-k} x^k.$

In particular $B_0(x) = x, B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6},$

$B_3(x) = B_0 x^3 + 3B_1 x^2 + 3B_2 x + B_3 = x^3 + 3\left(-\frac{1}{2}\right) x^2 + 3 \cdot \frac{1}{6} x = x^3 - \frac{3}{2} x^2 + \frac{1}{2} x.$

More properties of Bernoulli polynomials and numbers.

P4. $\int_0^1 B_p(x)\,dx = 0$ for any $p \in \mathbb{N}$.

Proof. Because of **P2**. we have $B'_{p+1}(x) = (p+1) B_p(x)$ then

$(p+1)\int_0^1 B_p(x)\,dx = (p+1)\int_0^1 B'_{p+1}(x)\,dx = (p+1)\left(B_{p+1}(x)\right)_0^1 = (p+1)\left(B_{p+1}(1) - B_{p+1}(0)\right) = (p+1)\cdot 0 = 0 \implies \int_0^1 B_p(x)\,dx = 0.$

We will prove that properties **P1.**,**P2.**,**P3.** determine polynomials $B_p(x)$ uniquely.

Let $\left(Q_p(x)\right)_{p\geq 0}$ be a sequence of polynomials such that $Q_0(x) = 1, Q'_n(x) = nQ_{n-1}(x), n \in \mathbb{N}$ and $Q_p(x+1) - Q_p(x+1) = px^{p-1}, p \in \mathbb{N}.$

First note that $Q_0(x) = 1 = B_0(x)$ .Also, $Q_n(1) = Q_n(0)$ for $n \geq 2$ since $Q_p(1) - Q_p(0) = p \cdot 0^{p-1} = 0, p \geq 2.$

This yields $\int_0^1 Q_p(x)\,dx = 0, p \in \mathbb{N}.$

Indeed, $p\int_0^1 Q_p(x)\,dx = \int_0^1 Q'_{p+1}(x)\,dx = Q_{n+1}(1) - Q_{n+1}(0) = 0.$ Since $Q'_1(x) = 1 \cdot Q_0(x) = 1$ then $Q_1(x) = x + c$ and, therefore, $Q'_2(x) = 2Q_1(x)$

yields $Q_2(x) = x^2 + 2cx + d$. Then $Q_2(x+1) - Q_2(x) = 2x \iff (x+1)^2 + 2c(x+1) - x^2 - 2cx = 2x \iff 2c + 1 = 0 \iff c = -\frac{1}{2}$.Hence, $Q_1(x) = x - \frac{1}{2} = B_1(x)$

Assume that $Q_p(x) = B_p(x)$ then $Q'_{p+1}(x) = (p+1) Q_p(x) = (p+1) B_p(x) = B'_{p+1}(x) \iff$

$Q_{p+1}(x) = B_{p+1}(x) + c$.Therefore $0 = \int_0^1 Q_{p+1}(x)\,dx = \int_0^1 \left(B_{p+1}(x) + c\right) dx = \int_0^1 B_{p+1}(x)\,dx + c = c.$

So, by induction $Q_p(x) = B_p(x)$ for any $p \in \mathbb{N}$. ■

P5. $B_p(x) = (-1)^p B_p(1-x), p \geq 0$.(Complement property)

Proof. Let $Q_p(x) := (-1)^p B_p(1-x), p \in \mathbb{N} \cup \{0\}$ then:

1. By **P1** $Q_0(x) = B_0(1-x) = 1;$

2. By **P2**. $Q'_p(x) = (-1)^p \left(B_p(1-x)\right)' = (-1)^p \left(B_p(1-x)\right)' = -(-1)^p B'_p(1-x) = p(-1)^{p-1} B_{p-1}(1-x) = pQ_{p-1}(x);$

3. By **P3**. $Q_p(x+1) - Q_p(x) = (-1)^p B_p(1-(x+1)) - (-1)^p B_p(1-x) =$

$(-1)^p \left(B_p(-x) - B_p(1 + (-x))\right) = (-1)^{p+1} \left(B_p((-x) + 1) - B_p(-x)\right) = p(-1)^{p+1}(-x)^{p-1} = px^{p-1}.$

Therefore, by property of uniqueness we get $(-1)^p B_p(1-x) = B_p(x)$. ■

Corollary 8 *For $p = 2m + 1, m \in \mathbb{N}$ holds $B_p(0) = 0$.*

Indeed, if $p = 2m+1$ then $B_p(x) = -B_p(1-x)$ and, therefore, for $x = 0$ we have $B_p(0) = -B_p(1) = -B_p(0) \implies 2B_p(0) = 0 \iff B_p(0) = 0.$

Corollary 9 *By replacing x in $B_p(x) = (-1)^p B_p(1-x)$ with $x + 1$ we obtain*

$B_p(x+1) = (-1)^p B_p(1-(x+1)) = (-1)^p B_p(-x) = (-1)^p \sum_{k=0}^{p} \binom{p}{k} B_{p-k}(-x)^k =$

$\sum_{k=0}^{p} (-1)^{n-k} \binom{p}{k} B_{p-k} x^k = \sum_{k=0}^{p} (-1)^k \binom{p}{k} B_{p-k} x^k.$

Now, we write $S_p(n)$ in polynomial form by powers of n.

Since $B_{p+1}(x+1) - B_{p+1}(x) = (p+1)x^p$ then $(p+1) S_p(n) = (p+1)\sum_{k=1}^{n} k^p =$

$\sum_{k=1}^{n} \left(B_{p+1}(k+1) - B_{p+1}(k)\right) = B_{p+1}(n+1) - B_{p+1}(1) = B_{p+1}(n+1) - B_{p+1}(0)$

and, therefore, $(p+1) S_p(n) = B_{p+1}(n+1) - B_{p+1} \iff$

$S_p(n) = \dfrac{B_{p+1}(n+1) - B_{p+1}}{p+1} = \dfrac{1}{p+1}\left(\sum_{k=0}^{p+1}(-1)^k\binom{p+1}{k}B_{p+1-k}n^k - B_{p+1}\right) =$
$\dfrac{1}{p+1}\sum_{k=1}^{p+1}(-1)^k\binom{p+1}{k}B_{p+1-k}n^k.$

(★) $\qquad S_p(n) = \dfrac{1}{p+1}\sum_{k=1}^{p+1}(-1)^k\binom{p+1}{k}B_{p+1-k}n^k$ (Faulhaber's Formula).

Problem 1

Prove that $B_{2m+1}(x)$ is divisible by $S_2(x-1)$ for any $m \in \mathbb{N}$.

Problem 2

Prove that $Sign(B_{2m}) = (-1)^{m+1}$ and $\max\limits_{[0,1]} B_{4m-2}(x) = B_{4m-2}$, $\min\limits_{[0,1]} B_{4m}(x) = B_{4m}, m \in \mathbb{N}$.

Hint (use induction).

1. A. M. Alt-Variations on a theme-The sum of equal powers of natural numbers, part 1_Crux vol.40,n.8;

2. A. M. Alt-Variations on a theme-The sum of equal powers of natural numbers, part 2_Crux vol.40,n.10.

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