Junior problems

J445. Find all pairs (p, q) of primes such that $p^2 + q^3$ is a perfect cube.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Let r be the integer such that $p^2 + q^3 = r^3$. Since $p^2 > 0$, $2 \le q < r$ are positive integers, and clearly $r^2 + rq + q^2 > r - q$. Now, $p^2 = (r-q)\left(r^2 + rq + q^2\right)$, or $r - q$ and $r^2 + rq + q^2$ must be either both equal to p (absurd since they are different) or one equal to 1 and the other one equal to p^2. Hence $r = q + 1$, and $p^2 = r^2 + rq + q^2 = (2q+1)^2 - q(q+1)$, or

$$(2q + 1 - p)(2q + 1 + p) = q(q + 1).$$

Now, q divides either $2q + 1 - p$ or $2q + 1 + p$, hence q divides either $p - 1$ or $p + 1$. On the other hand, $p^2 = 3q^2 + 3q + 1 < 4q^2 + 4q + 1 = (2q+1)^2$, or $p - 1 < 2q$. It follows that either $q = p - 1$, or $q = p + 1$, or $2q = p + 1$. In the first case, $q = 2$ and $p = 3$ are the only two consecutive primes, for $p^2 + q^3 = 17$, which is not a cube. In the second case, $q = 3$ and $p = 2$ are again the only two consecutive primes, for $p^2 + q^3 = 31$, again not a cube. Therefore, $p = 2q - 1$, for $3q^2 + 3q + 1 = 4q^2 - 4q + 1$, and $q = 7$, for $p = 13$, yielding $p^2 + q^3 = 169 + 343 = 512 = 8^3$. The only possible pair is therefore $(p, q) = (13, 7)$.

Also solved by Albert Stadler, Herrliberg, Switzerland; Alessandro Ventullo, Ithaca, NY, USA; Anish Ray, Institute of Mathematics and Applications, Bhubaneswar, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Ioannis D. Sfikas, Athens, Greece; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paul Revenant, Lycée du Parc, Lyon, France; Paul Vanborre-Jamin, Lycée Henri IV, Paris, France; Raja Damanik, University of Amsterdam, Netherlands; Polyahedra, Polk State College, FL, USA; Vincelot Ravoson and Terence Ngo, Paris, France; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Athanasios Peppas, Evangelliki Model School of Smyrna, Athens, Greece; Titu Zvonaru, Comănești, Romania.

J446. Let a, b, c be positive real numbers such that $ab + bc + ca = 3abc$. Prove that

$$\frac{1}{2a^2+b^2}+\frac{1}{2b^2+c^2}+\frac{1}{2c^2+a^2}\leq 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Polyahedra, Polk State College, USA

Let $x = bc$, $y = ca$, and $z = ab$. Then $\frac{x+y+z}{3x} = \frac{3abc}{3bc} = a$, $\frac{x+y+z}{3y} = b$, $\frac{x+y+z}{3z} = c$, and the inequality becomes

$$\frac{9x^2y^2}{x^2+2y^2}+\frac{9y^2z^2}{y^2+2z^2}+\frac{9z^2x^2}{z^2+2x^2}\leq (x+y+z)^2.$$

By the AM-GM inequality, $x^2 + 2y^2 \geq 3\sqrt[3]{x^2y^4}$, so $\frac{9x^2y^2}{x^2+2y^2} \leq 3\sqrt[3]{x^4y^2} \leq x^2 + 2xy$. Adding to this the other two analogous inequalities completes the proof.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Vincelot Ravoson and Terence Ngo, Paris, France; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Albert Stadler, Herrliberg, Switzerland; Arkady Alt, San Jose, CA, USA; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu and Octavian Stroe, Colegiul National Zinca Golescu, Pitesti, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Raja Damanik, University of Amsterdam, Netherlands; Guadalupe Russelle, University of the Philippines, Diliman, Quezon City, Philippines; Nikos Kalapodis, Patras, Greece; Titu Zvonaru, Comănești, Romania.

J447. Let $N = \overline{d_0 d_1 \cdots d_9}$ be a 10-digit number with $d_{k+5} = 9 - d_k$, for $k = 0, 1, 2, 3, 4$. Prove that N is divisible by 41.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Albert Stadler, Herrliberg, Switzerland

N is of the form

$$N = \sum_{j=0}^{4} (a_j 10^5 + 9 - a_j) 10^j = \left(\sum_{j=0}^{4} 99999 a_j 10^j \right) + 99999,$$

where a_0, a_1, a_2, a_3, a_4 are digits from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

N is divisible by 41, since 99999 is.

Also solved by Konstantinos Kritharidis, Evangelliki Model School of Smyrna, Athens, Greece; Polyahedra, Polk State College, FL, USA; Vincelot Ravoson and Terence Ngo, Paris, France; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Athanasios Peppas, Evangelliki Model School of Smyrna, Athens, Greece; ANanduud Problem Solving Group, Ulaanbaatar, Mongolia; Francisco Javier Martínez Aguinaga, Universidad Complutense de Madrid, Spain; Henry Ricardo, Westchester Area Math Circle, NY, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Ioannis D. Sfikas, Athens, Greece; Joel Schlosberg, Bayside, NY, USA; Daniel Lasaosa, Pamplona, Spain; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paul Revenant, Lycée du Parc, Lyon, France; Paul Vanborre-Jamin, Lycée Henri IV, Paris, France; Raja Damanik, University of Amsterdam, Netherlands; Timothy Miller, Buffalo, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Anderson Torres, Sao Paulo, Brazil; Titu Zvonaru, Comănești, Romania.

J448. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$4 \le \sqrt{a^4 + b^2 + c^2 + 1} + \sqrt{b^4 + c^2 + a^2 + 1} + \sqrt{c^4 + a^2 + b^2 + 1} \le 3\sqrt{2}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

From the given condition $b^2 + c^2 = 1 - a^2$ and $a^2 - 1 \le 0$. Hence we get

$$\sqrt{a^4 + b^2 + c^2 + 1} = \sqrt{a^4 - a^2 + 2} = \sqrt{a^2(a^2 - 1) + 2} \le \sqrt{2}.$$

Similarly, we have

$$\sqrt{b^4 + c^2 + a^2 + 1} \le \sqrt{2}, \quad \sqrt{c^4 + a^2 + b^2} \le \sqrt{2}$$

Adding above three inequalities, we have

$$\sqrt{a^4 + b^2 + c^2 + 1} + \sqrt{b^4 + c^2 + a^2 + 1} + \sqrt{c^4 + a^2 + b^2 + 1} \le 3\sqrt{2}.$$

RHS is proved. Equality holds only when $\{a, b, c\} = \{\pm 1, 0, 0\}$.

LHS of given problem is equivalent to (1). That is

$$\sqrt{x^2 - x + 2} + \sqrt{y^2 - y + 2} + \sqrt{z^2 - z + 2} \ge 4 \tag{1}$$

for x, y, z is nonnegative numbers such that $x + y + z = 1$.

$f(t) = \sqrt{t^2 - t + 2}$ function is strictly convex on $[0, +\infty)$. Applying Jensen's inequality we get

$$\begin{aligned} f(x) + f(y) + f(z) &\ge 3f\left(\frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z\right) \\ \Leftrightarrow \sqrt{x^2 - x + 2} &+ \sqrt{y^2 - y + 2} + \sqrt{z^2 - z + 2} \\ &\ge 3\sqrt{\left(\frac{1}{3}(x + y + z)\right)^2 - \frac{1}{3}(x + y + z) + 2} = 4. \end{aligned}$$

Hence (1) is proved.

Also solved by Daniel Lasaosa, Pamplona, Spain; Nikos Kalapodis, Patras, Greece; Konstantinos Kritharidis, Evangelliki Model School of Smyrna, Athens, Greece; Polyahedra, Polk State College, FL, USA; Vincelot Ravoson and Terence Ngo, Paris, France; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Athanasios Peppas, Evangelliki Model School of Smyrna, Athens, Greece; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Albert Stadler, Herrliberg, Switzerland; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Arkady Alt, San Jose, CA, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Ioannis D. Sfikas, Athens, Greece; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paul Revenant, Lycée du Parc, Lyon, France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Raja Damanik, University of Amsterdam, Netherlands; Titu Zvonaru, Comănești, Romania.

J449. A square of area 1 is inscribed in a rectangle such that each side of the rectangle contains precisely a vertex of the square. What is the greatest possible area of the rectangle?

Proposed by Mircea Becheanu, Montreal, Canada

First solution by Polyahedra, Polk State College, USA

Since the length and width of the rectangle are both less than or equal to the diagonal of the square, the area of the rectangle is no greater than $(\sqrt{2})(\sqrt{2}) = 2$. This maximum is attained when the rectangle is a square with the midpoints of its sides as the vertices of the inscribed square.

Second solution by Solution by Daniel Lasaosa, Pamplona, Spain

Let $ABCD$ be the square, and $TUVW$ the rectangle, such that $A \in TU$, $B \in UV$, $C \in VW$ and $D \in WT$. Denote $\angle BAU = \alpha$, or clearly $\angle CBV = \angle DCW = \angle ADT = \alpha$ and $\angle TAD = \angle UBA = \angle VCB = \angle WDC = 90^\circ - \alpha$, or $AU = BV = CW = DT = \cos\alpha$, $TA = UB = VC = WD = \sin\alpha$. The rectangle $TUVW$ is then a square with sidelength $\cos\alpha + \sin\alpha$, and by the AM-QM inequality, its area is

$$(\cos\alpha + \sin\alpha)^2 \le 4\left(\frac{\cos^2\alpha + \sin^2\alpha}{2}\right) = 2.$$

The maximum area is obtained when the rectangle is a square, whose sides have for midpoints the vertices of the square of area 1.

Also solved by Vincelot Ravoson and Terence Ngo, Paris, France; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Ioannis D. Sfikas, Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Timothy Miller, Buffalo, NY, USA; George Theodoropoulos, 2nd High School of Agrinio, Greece; Titu Zvonaru, Comănești, Romania.

J450. Prove that in any triangle ABC

$$\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c}\geq\sqrt{\frac{3\,(4R+r)}{2R}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy
If r and F denote respectively the inradius and the area of the triangle we have

$$r_a=\frac{sr}{a(s-a)}=\frac{F}{a(s-a)}$$

Thus, the inequality becomes $(R=abc/(4sr))$

$$F\sum_{\text{cyc}}\frac{1}{a(s-a)}\geq\sqrt{\frac{3(4R+r)}{2R}}=\sqrt{\frac{3}{2}\frac{4sr^2}{abc}+6}$$

Now, let's change variables $a=y+z$, $b=x+z$, $c=x+y$ and get

$$\sqrt{(x+y+z)xyz}\sum_{\text{cyc}}\frac{1}{(y+z)x}\geq\sqrt{6+6\frac{xyz}{(x+y)(y+z)(z+x)}}$$

Upon squaring the inequality becomes equivalent to

$$\sum_{\text{cyc}}x^4y^4\geq\sum_{\text{cyc}}x^4y^2z^2$$

and the conclusion follows from the AGM inequality.

Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, FL, USA; Arkady Alt, San Jose, CA, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland.

Senior problems

S445. Solve in integers the equation:

$$x^3 - y^3 - 1 = (x + y - 1)^2$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note that the RHS is non negative, or $x^3 > y^3$, hence $d = x - y$ is a positive integer, whereas $s = x + y$ is an integer. After some algebra, the proposed equation rewrites as

$$4s^2 - 8s + 8 = 3ds^2 + d^3.$$

Note now that if $d \geq 3$, and since s^2 is a nonnegative integer, we have

$$4s^2 - 8s + 8 = 3ds^2 + d^3 \geq 9s^2 + 27 = 4s^2 - 8s + 8 + 4(s + 1)^2 + s^2 + 15 > 4s^2 - 8s + 8,$$

absurd, or $d \in \{1, 2\}$. If $d = 1$, then $0 = s^2 - 8s + 7 = (s - 1)(s - 7)$, with solutions $(x, y) = (1, 0)$ and $(x, y) = (4, 3)$, which are indeed found to satisfy the proposed equation. If $d = 2$, then $0 = s^2 + 4s = s(s + 4)$, with solutions $(x, y) = (1, -1)$ and $(x, y) = (-1, -3)$, which also satisfy the proposed equation. There can be no more integer solutions.

Also solved by Takuji Imaiida, Fujisawa, Kanagawa, Japan; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Albert Stadler, Herrliberg, Switzerland; G. C. Greubel, Newport News, VA, USA; Ioannis D. Sfikas, Athens, Greece.

S446. Let a and b be positive real numbers such that $ab = 1$. Prove that

$$\frac{2}{a^2+b^2+1} \le \frac{1}{a^2+b+1} + \frac{1}{a+b^2+1} \le \frac{2}{a+b+1}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy

Applying Cauchy–Schwarz to the RHS yields

$$(a^2+b+1)(1+b+1) \ge (a+b+1)^2, \quad (a+b^2+1)(a+1+1) \ge (a+b+1)^2$$

Hence, the RHS is implied by

$$\frac{2+b}{(a+b+1)^2} + \frac{2+a}{(a+b+1)^2} \le \frac{2}{a+b+1} \iff \frac{4+a+b}{(a+b+1)^2} \le \frac{2}{a+b+1}$$

We get

$$a+b+1 \ge 2 + \frac{a+b}{2} \iff a+b \ge 2$$

and this follows by the AGM inequality

$$a+b \ge 2\sqrt{ab} = 2$$

Now, applying Cauchy–Schwarz to the LHS gives

$$\frac{1}{a^2+b+1} + \frac{1}{b^2+a+1} \ge \frac{(1+1)^2}{a^2+b^2+a+b+2}$$

$$\frac{2}{a^2+b^2+1} \le \frac{4}{a^2+b^2+a+b+2}$$

which simplifies to

$$a^2+b^2 \ge a+b$$

and the conclusion follows.

Also solved by Daniel Lasaosa, Pamplona, Spain; Vincelot Ravoson and Terence Ngo, Paris, France; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Nikos Kalapodis, Patras, Greece; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Anderson Torres, Sao Paulo, Brazil; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Albert Stadler, Herrliberg, Switzerland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Arkady Alt, San Jose, CA, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu and Octavian Stroe, Colegiul National Zinca Golescu, Pitesti, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Titu Zvonaru, Comănești, Romania.

S447. Let $a,b,c,d \geq -1$ such that $a+b+c+d=4$. Find the maximum of

$$\left(a^2+3\right)\left(b^2+3\right)\left(c^2+3\right)\left(d^2+3\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Vincelot Ravoson and Terence Ngo, Paris, France

Let $F = \prod_{cyc}(a^2+3)$.

Suppose that a,b,c,d are positive. Hence, by AM-GM inequality we have :

$$F \leq \left(\frac{a^2+b^2+c^2+d^2}{4}+3\right)^4,$$

and : $a^2+b^2+c^2+d^2 \leq (a+b+c+d)^2 = 4^2$, so :

$$F \leq (4+3)^4 = 2401.$$

Suppose WLOG that $d \geq 0$ and a,b,c are positive. Then : $d^2 \leq 1$, with equality iff $d=-1$, and : $-1 \leq d = 4-(a+b+c)$, so : $a+b+c \leq 5$ and : $a^2+b^2+c^2 \leq (a+b+c)^2 \leq 5^2$, so by AM-GM :

$$F \leq \left(\frac{a^2+b^2+c^2}{3}+3\right)^2 \cdot 4 \leq \left(\frac{5^2}{3}+3\right) \cdot 4 = \frac{136}{3}.$$

Suppose WLOG that : $c,d \leq 0$ and $a,b \geq 0$. Then: $c^2+3 \leq 4$, $d^2+4 \leq 4$, and : $-2 \leq c+d = 4-(a+b)$, so : $a^2+b^2 \leq (a+b)^2 \leq 6^2$, and by AM-GM we get that :

$$F \leq \left(\frac{6^2}{2}+3\right) \cdot 4^2 = 336.$$

Finally, suppose WLOG that $a \geq 0$ and $b,c,d \geq 0$. Then for all $t \in \{b,c,d\}$, $t^2+3 \leq 4$ and : $-3 \leq b+c+d = 4-a$, so $a \leq 7$, and :

$$F \leq (7^2+3) \cdot 4^3 = 3328,$$

with equality if and only if $a=7$ and $b=c=d=-1$ (with permutations).

In conclusion :

$$\max_{a,b,c,d \geq -1}(a^2+3)(b^2+3)(c^2+3)(d^2+3) = 3328.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Ioannis D. Sfikas, Athens, Greece; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Albert Stadler, Herrliberg, Switzerland.

S448. Let ABC be a triangle with area Δ. Prove that for any point P in the plane of the triangle

$$AP + BP + CP \geq 2\sqrt[4]{3}\sqrt{\Delta}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain

If one of A, B, C is at least $120°$, wlog $A \geq 120°$, then the point P in the plane of ABC such that $AP+BP+CP$ is maximum is $P = A$, or $AP + BP + CP = b + c$. Moreover, $\Delta = \frac{bc\sin A}{2} \leq \frac{bc\sin 120°}{2} = \frac{bc\sqrt{3}}{4}$, and it suffices to show that

$$b + c \geq \sqrt{3bc},$$

always true and holding strictly since by the AM-GM inequality we have $b + c \geq 2\sqrt{bc} > \sqrt{3bc}$.

If $A, B, C \leq 120°$, then the point P in the plane of ABC such that $AP + BP + CP$ is the Torricelli point, and it is well known that $AP + BP + CP = AA' = BB' = CC'$, where BCA', CAB', ABC' are equilateral triangles constructed outside of ABC. Therefore, using the Cosine Law, we have

$$(AP + BP + CP)^2 \geq AA'^2 = AB^2 + BA'^2 - 2AB \cdot BA' \cos(B + 60°) =$$

$$= c^2 + a^2 - ca\cos B + \sqrt{3}ca\sin B = \frac{a^2 + b^2 + c^2}{2} + 2\sqrt{3}\Delta.$$

Squaring both sides of the proposed inequality, and using that $\Delta = \frac{abc}{4R}$ where R is the circumradius of ABC, it suffices to show that

$$R\left(a^2 + b^2 + c^2\right) \geq \sqrt{3}abc.$$

Now, it is well known (or it can be proved using the Sine Law and trigonometric relations) that $a^2 + b^2 + c^2 \leq 9R^2$, or it suffices to show that

$$\left(a^2 + b^2 + c^2\right)^3 \geq 27\sqrt[3]{a^2b^2c^2},$$

which is true by the AM-GM inequality, and with equality iff $a = b = c$. The conclusion follows, and since this last condition is also sufficient for all other equalities, equality holds in the proposed relation iff ABC is equilateral and P is its center.

Also solved by Vincelot Ravoson and Terence Ngo, Paris, France; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Albert Stadler, Herrliberg, Switzerland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioannis D. Sfikas, Athens, Greece; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Titu Zvonaru, Comănești, Romania.

S449. Find the maximum of

$$\left(\frac{9b+4c}{a}-6\right)\left(\frac{9c+4a}{b}-6\right)\left(\frac{9a+4b}{c}-6\right)$$

over all positive real numbers a, b, c.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

The maximum is 7^3, achieved for $a = b = c$. It suffices to show that

$$(9b+4c-6a)(9c+4a-6b)(9a+4b-6c) \le 7^3abc.$$

Let $x = 9b+4c-6a, y = 9c+4a-6b, z = 9a+4b-6c$. We have

$$2x+3y = 35c,\ 2y+3z = 35a,\ 2z+3x = 35b.$$

This shows that maximum one of x, y, z can be nonpositive. The desired inequality is reduced to

$$xyz \le \frac{1}{5^3}(2x+3y)(2y+3z)(2z+3x).$$

If precisely one of x, y, z is nonpositive, then this is obvious. If x, y, z are all positive, then, by the AM-GM Inequality,

$$\sqrt[5]{x^2y^3} \le \frac{1}{5}(x+x+y+y+y) = \frac{1}{5}(2x+3y),$$
$$\sqrt[5]{y^2z^3} \le \frac{1}{5}(y+y+z+z+z) = \frac{1}{5}(2y+3z),$$
$$\sqrt[5]{z^2x^3} \le \frac{1}{5}(z+z+x+x+x) = \frac{1}{5}(2z+3x),$$

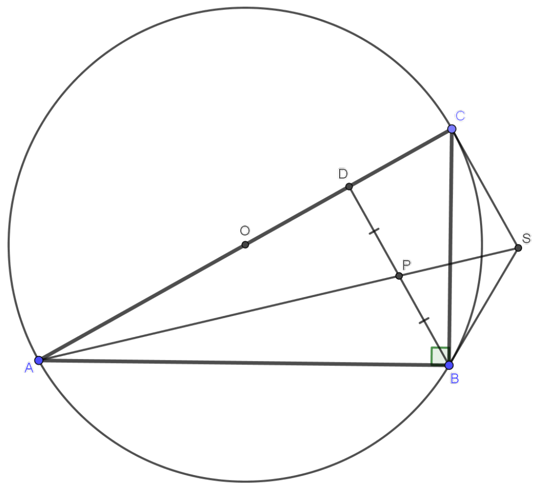
and, by multiplication, the conclusion follows.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Daniel Lasaosa, Pamplona, Spain; Guadalupe Russelle, University of the Philippines, Diliman, Quezon City, Philippines; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Albert Stadler, Herrliberg, Switzerland; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu and Octavian Stroe, Colegiul National Zinca Golescu, Pitesti, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

S450. Let ABC be a triangle and D the foot of the altitude from B. The tangents in B and C to the circumcircle of ABC meet in S. Let P be the intersection of BD and AS. We know that $BP = PD$. Calculate $\angle ABC$.

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Andrea Fanchini, Cantù, Italy



We use barycentric coordinates with reference to the triangle ABC.

- *Coordinates of points S, P.*

The tangents in B and C to the circumcircle of ABC are

$$BBO_{\infty\perp} : c^2x + a^2z = 0, \qquad CCO_{\infty\perp} : b^2x + a^2y = 0$$

then point S is

$$S = BBO_{\infty\perp} \cap CCO_{\infty\perp} = (-a^2 : b^2 : c^2)$$

The foot of the altitude from B is $D(S_C : 0 : S_A)$, so we have

$$BD : S_Ax - S_Cz = 0, \qquad AS : c^2y - b^2z = 0$$

then point P is

$$P = BD \cap AS = (c^2S_C : b^2S_A : c^2S_A)$$

- *Distances BP, PD.*

$$BP = \frac{c^2S}{b(2S_A + S_B)}, \qquad PD = \frac{S_AS}{b(2S_A + S_B)}$$

now if $BP = PD$ then $c^2 = S_A$, therefore $S_B = 0$ that is $\angle ABC = 90°$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Mihai Miculita, Oradea, Romania; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Daniel Ghenghea; Ioannis D. Sfikas, Athens, Greece; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Albert Stadler, Herrliberg, Switzerland; Telemachus Baltsavias, Kerameies Junior High School, Kefalonia, Greece; Titu Zvonaru, Comănești, Romania.

U445. Let a, b, c be the roots of the equations $x^3 + px + q = 0$, where $q \neq 0$. Evaluate the sum

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$$

in terms of p and q.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Oana Prajitura, University of Pittsburgh, PA, USA

By Vieta's relations,

$$a + b + c = 0 \qquad ab + bc + ca = p \qquad abc = -q.$$

Also, since a, b, c are roots of the equation,

$$a^3 = -pa - q \qquad b^3 = -pb - q \qquad c^3 = -pc - q.$$

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} = \frac{a^3c + b^3a + c^3b}{abc} = \frac{(-pa - q)c + (-pb - q)a + (-pc - q)b}{-q}$$

$$= \frac{-p(ab + bc + ca) - q(a + b + c)}{-q} = \frac{p^2}{q}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Vincelot Ravoson and Terence Ngo, Paris, France; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Joehyun Kim, Cresskill, NJ, USA; Anderson Torres, Sao Paulo, Brazil; Albert Stadler, Herrliberg, Switzerland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; G. C. Greubel, Newport News, VA, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma, Rome, Italy; George Theodoropoulos, 2nd High School of Agrinio, Greece; Titu Zvonaru, Comănești, Romania; Juan Jose Granier, Universidad de Chile, Santiago, Chile.

U446. Find the minimum of $\max\left\{|1+z|,\left|1+z^2\right|\right\}$, when z runs over all complex numbers.

Proposed by Robert Bosch, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Let $z = x + iy$, and denote

$$f(x,y) = |1+z|^2 = (1+x)^2 + y^2,$$

$$g(x,y) = \left|1+z^2\right|^2 = \left(1+x^2-y^2\right)^2 + 4x^2y^2,$$

and $h(x,y) = \max\{f(x,y), g(x,y)\}$. Note first that f, g are both differentiable, and both grow without bounds for any arbitrarily large x or y, hence the absolute minimum of h is necessarily also a local minimum. At this minimum, we may have either $f > g$ and $h = f$, or $f < g$ and $h = g$, or $f = g = h$.

If $h = f > g$, then $f = h$ at a neighborhood of the minimum, or f has a local minimum at the absolute minimum of h. Now,

$$\frac{\partial f(x,y)}{\partial x} = 2(1+x), \qquad \frac{\partial f(x,y)}{\partial y} = 2y,$$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = 0, \qquad \frac{\partial^2 f(x,y)}{\partial x^2} = 2, \qquad \frac{\partial^2 f(x,y)}{\partial y^2} = 2,$$

or a local minimum of f occurs iff $x = -1$ and $y = 0$, for $h = f = 0$. But then $g > f$, contradiction. Hence the minimum of h does not occur in this case.

If $f < g = h$, then g has a local minimum at the minimum of h. Considering g as a function of $x^2 = u$ and $y^2 = v$, we have

$$\frac{\partial g(u,v)}{\partial u} = 2\left(1+u+v\right), \qquad \frac{\partial g(u,v)}{\partial v} = 2\left(u+v-1\right),$$

$$\frac{\partial^2 g(u,v)}{\partial u \partial v} = \frac{\partial^2 g(u,v)}{\partial u^2} = \frac{\partial^2 g(u,v)}{\partial v^2} = 2,$$

or a local minimum of g occurs iff $u = 0$ for $x = 0$, and $v = 1$ for $y = \pm 1$, yielding $g = 0$. But then $f > g$, contradiction. Hence the minimum of h does not occur in this case.

If $f = g = h$, let us find when $f(x,y) = g(x,y)$, considering it as a quadratic equation in $v = y^2$:

$$v^2 + v(2x^2 - 3) + x^4 + x^2 - 2x = 0, \qquad v = \frac{3 - 2x^2 \pm \sqrt{9 + 8x - 16x^2}}{2}.$$

Note however that not all values of x are possible for $f = g$, indeed x needs to satisfy $(4x-1)^2 \le 10$, and the resulting value of v needs to be nonnegative. Inserting these expressions for $v = y^2$ in both f, g, both collapse into

$$h_+(x) = 2x + \frac{5}{2} + \frac{\sqrt{9+8x-16x^2}}{2}, \qquad h_-(x) = 2x + \frac{5}{2} - \frac{\sqrt{9+8x-16x^2}}{2}.$$

The derivative of the first expression with respect to x yields

$$h'_+(x) = 2 + \frac{2-8x}{\sqrt{9+8x-16x^2}},$$

which becomes zero when $4x^2 - 2x - 1 = 0$ and $x > \frac{1}{4}$, ie when $x = \frac{1+\sqrt{5}}{4}$, or after calculating the second derivative of h_+ with respect to x, evaluating it at $x = \frac{1+\sqrt{5}}{4}$, and finding it to be positive, the minimum of h in this case is found to be

$$\min\{h_+\} = 3,$$

occurring for $x = \frac{1+\sqrt{5}}{4}$, which yields indeed a positive value of v. The derivative of the second expression with respect to x yields

$$h'_-(x) = 2 - \frac{2-8x}{\sqrt{9+8x-16x^2}},$$

which becomes zero when $4x^2 - 2x - 1 = 0$ and $x < \frac{1}{4}$, ie when $x = \frac{1-\sqrt{5}}{4}$, or the minimum of h in this case is found to be

$$\min\{h_-\} = 3 - \sqrt{5},$$

occurring when $x = \frac{1-\sqrt{5}}{4}$, which yields indeed a positive value of v. It follows that the minimum of $h(x,y)$ is $3-\sqrt{5}$, and occurs when $x = \frac{1-\sqrt{5}}{4}$ and $y = \pm\frac{\sqrt{3}(1-\sqrt{5})}{4}$, or the minimum of the proposed expression is

$$\sqrt{3-\sqrt{5}} = \frac{\sqrt{5}-1}{\sqrt{2}},$$

which occurs for

$$z = \left(1 \pm i\sqrt{3}\right)\frac{1-\sqrt{5}}{4}.$$

Also solved by Juan Jose Granier, Universidad de Chile, Santiago, Chile; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland.

U447. If F_n is the n^{th} Fibonacci number, then for fixed p show that

$$\sum_{k=1}^{n} \binom{n}{k} F_p^k F_{p-1}^{n-k} F_k = F_{pn}.$$

Tarit Goswami, West Bengal, India

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Let $\varphi = \dfrac{1+\sqrt{5}}{2}$. Using that

$$\varphi^m = F_{m-1} + F_m \varphi$$

for positive integer m. Then we have

$$\begin{aligned} F_{pn} \cdot \varphi + F_{pn-1} &= \varphi^{pn} = (\varphi^p)^n \\ &= (F_p \cdot \varphi + F_{p-1})^n \\ &= \sum_{k=1}^{n} \binom{n}{k} (F_p \varphi)^k F_{p-1}^{n-k} \\ &= \sum_{k=1}^{n} \binom{n}{k} F_p^k F_{p-1}^{n-k} (F_k \cdot \varphi + F_{k-1}) \\ &= \left(\sum_{k=1}^{n} \binom{n}{k} F_p^k F_{p-1}^{n-k} F_k \right) \varphi + \sum_{k=1}^{n} \binom{n}{k} F_p^k F_{p-1}^{n-k} F_{k-1}. \end{aligned}$$

Hence we get

$$F_{pn} = \sum_{k=1}^{n} \binom{n}{k} F_p^k F_{p-1}^{n-k} F_k.$$

Also solved by Vincelot Ravoson and Terence Ngo, Paris, France; Henry Ricardo, Westchester Area Math Circle, NY, USA; Albert Stadler, Herrliberg, Switzerland; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; G. C. Greubel, Newport News, VA, USA; Ioannis D. Sfikas, Athens, Greece; Guadalupe Russelle, University of the Philippines, Diliman, Quezon City, Philippines.

U448. Let $p \geq 5$ be a prime number. Prove that the polynomial

$$2X^p - p3^pX + p^2$$

is irreducible in $\mathbb{Z}[X]$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Assume the contrary: $2x^p - p \cdot 3^p x + p^2 = f(x) \cdot g(x)$, where $f(x), g(x)$ are polynomials with integer coefficients. Denote $\deg f(x) = d, \deg g(x) = e$. Because not all the coefficients of $x^p - p \cdot 3^p x + p^2$ are divisible by p, we find that the same statement holds true for the polynomials $f(x), g(x)$. That is, one can write

$$f(x) = x^s f_1(x) + p f_2(x)$$

and

$$g(x) = x^c g_1(x) + p g_2(x)$$

where e, c are the least monomials in $f(x), g(x)$ such that their coefficients are not divisible by p. Therefore, the constant terms of $f_1(x), g_1(x)$ are not divisible by p. Hence,

$$f(x)g(x) = x^{c+s} f_1(x)g_1(x) + p\left(x^s f_1(x)g_2(x) + x^c g_1(x)f_2(x)\right) + p^2 f_2(x)g_2(x).$$

It is easy to find that, $c + s = p$. Thus, $c = e, s = d$. Hence,

$$f(x) = a_d x^d + p f_2(x), g(x) = b_e x^e + p g_2(x)$$

This implies that,

$$2x^p - p \cdot 3^p x + p^2 = a_d b_e x^p + p\left(a_d x^d g_2(x) + b_e x^e f_2(x)\right) + p^2 f_2(x)g_2(x)$$

Therefore, comparing the coefficients of x, one can find that $\min(d, e) \leq 1$. Therefore, one of $f(x), g(x)$ must be linear. In this case, the polynomial $2x^p - p \cdot 3^p x + p^2$ must have a rational root. Using the Rational Root Theorem, one can find that, this root must be of the form of $\pm p, \pm p^2$ or $\pm \frac{p}{2}, \pm \frac{p^2}{2}$. Now, we consider four cases:

Case 1: $2(\pm p)^p \mp p^2 3^p + p^2 = 0$, then $\pm 2p^{p-2} \mp 3^p + 1 = 0$. Then, by Fermat's little theorem, we find that p must divide 2. Absurd.

Case 2: $2(\pm p)^{2p} \mp p^3 3^p + p^2 = 0$, then $\pm 2p^{2p-2} \mp p3^p + 1 = 0$, which is clearly wrong.

Case 3: $2\left(\pm \frac{p}{2}\right)^p \mp \frac{p^2}{2} \cdot 3^p + p^2 = 0$, then $\pm \frac{p^{p-2}}{2^{p-1}} \mp \frac{3^p}{2} + 1 = 0$. Therefore,

$$\pm p^{p-2} = \pm 2^{p-2} 3^p - 2^{p-1}$$

Then, by Fermat's little theorem, we find that p must divide either 1 or 5. If $p = 5$, then

$$-5^3 = -2^3 \cdot 3^5 - 2^4.$$

Absurd.

Case 4: $2\left(\pm \frac{p}{2}\right)^{2p} \mp \frac{p^3}{2} \cdot 3^p + p^2 = 0$, then $\pm \frac{p^{2p-2}}{2^{2p-1}} \mp \frac{p3^p}{2} + 1 = 0$. Contradiction.

Also solved by Vincelot Ravoson and Terence Ngo, Paris, France; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

U449. Evaluate

$$\int_0^{\pi/4} \ln\left(\frac{\tan\left(\frac{x}{3}\right)}{\tan^2 x}\right) dx.$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy

Solution by G. C. Greubel, Newport News, VA, USA
First note that

$$\int \ln(\tan x)\, dx = \frac{i}{2}\,\left(Li_2(i\tan x) - Li_2(-i\tan x)\right) + \ln\left(\frac{1 - i\tan x}{1 + i\tan x}\right)\ln(\tan x) \tag{1}$$

and that

$$\int_0^{\pi/4} \ln(\tan x)\, dx = -G \tag{2}$$

$$\int_0^{\pi/12} \ln(\tan x)\, dx = -\frac{2\,G}{3}, \tag{3}$$

where $Li_2(x)$ is the diligarithm function and G is Catalan's constant. Now, the integral in question can be seen as:

$$\begin{aligned} I &= \int_0^{\pi/4} \ln\left(\frac{\tan\left(\frac{x}{3}\right)}{\tan^2 x}\right) dx \\ &= \int_0^{\pi/4} \ln\left(\tan\left(\frac{x}{3}\right)\right) dx - 2\,\int_0^{\pi/4} \ln(\tan x)\, dx \\ &= 3\,\int_0^{\pi/12} \ln(\tan x)\, dx - 2\,\int_0^{\pi/4} \ln(\tan x)\, dx \\ &= 3\left(\frac{-2G}{3}\right) - 2(-G) = 0. \end{aligned}$$

This leads to

$$\int_0^{\pi/4} \ln\left(\frac{\tan\left(\frac{x}{3}\right)}{\tan^2 x}\right) dx = 0.$$

Also solved by Albert Stadler, Herrliberg, Switzerland; Ioannis D. Sfikas, Athens, Greece; Juan Jose Granier, Universidad de Chile, Santiago, Chile.

U450. Let P be a nonconstant polynomial with integer coefficients. Prove that for each positive integer n there are pairwise relatively prime positive integers $k_1, k_2, \ldots, k_n$ such that $k_1 k_2 \cdots k_n = |P(m)|$ for some positive integer m.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

By Schur's theorem exists distinct prime numbers $p_1, p_2, \ldots, p_n$ and positive integer $m_1, m_2, \ldots, m_n$ such that

$$\begin{aligned} P(m_1) &\equiv 0 \pmod{p_1} \\ P(m_2) &\equiv 0 \pmod{p_2} \\ \vdots \quad & \quad \vdots \quad \quad \vdots \\ P(m_n) &\equiv 0 \pmod{p_n}. \end{aligned}$$

By the Chinese Remainder theorem, exist positive integer m such that

$$\begin{aligned} m &\equiv m_1 \pmod{p_1} \\ m &\equiv m_2 \pmod{p_2} \\ \vdots \quad & \quad \vdots \quad \quad \vdots \\ m &\equiv m_n \pmod{p_n}. \end{aligned}$$

We have $\forall i \in \{1, 2, \ldots, n\} : m \equiv m_i \pmod{p_i}$ hence we get

$$P(m) \equiv P(m_i) \equiv 0 \pmod{p_i}.$$

Thus $p_1 p_2 \ldots p_n$ is divide $P(m)$. Hence we get

$$|P(m)| = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n} \cdot A,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$ and $A \in \mathbb{N}$. Choosing

$$k_1 = p_1^{\alpha_1},\ k_2 = p_2^{\alpha_2}, \ldots, k_{n-1} = p_{n-1}^{\alpha_{n-1}}, \quad k_n = p_n^{\alpha_n} \cdot A$$

then we have $i \neq j$ that gives $(k_i, k_j) = 1$ and

$$k_1 k_2 \cdot \ldots \cdot k_n = |P(m)|.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Anderson Torres, Sao Paulo, Brazil; Juan Jose Granier, Universidad de Chile, Santiago, Chile; Albert Stadler, Herrliberg, Switzerland.

O445. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\sqrt[8]{\frac{a^3 + b^3 + c^3}{3}} \leq \frac{3}{ab + bc + ca}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain

Using the well known relation $a^3+b^3+c^3-3abc = (a+b+c)^3-3(ab+bc+ca)(a+b+c)$, the proposed inequality rewrites as

$$\left(\frac{ab + bc + ca}{3}\right)^8 (9 - 3(ab + bc + ca) + abc) \leq 1.$$

Using the weighted AM-GM inequality, it then suffices to show that

$$8\frac{ab + bc + ca}{3} + 9 - 3(ab + bc + ca) + abc \leq 9, \qquad ab + bc + ca \geq 3abc.$$

Multiplying both sides by $a + b + c = 3$, it suffices to show that

$$(a + b + c)(ab + bc + ca) \geq 9abc,$$

which clearly holds by the AM-GM inequality applied to a, b, c and to ab, bc, ca, and equality holds iff $a = b = c$. The conclusion follows, the necessary condition $a = b = c = 1$ for equality in the last step also being clearly sufficient for equality in the proposed inequality.

Also solved by Vincelot Ravoson and Terence Ngo, Paris, France; Albert Stadler, Herrliberg, Switzerland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu and Octavian Stroe, Colegiul National Zinca Golescu, Pitesti, Romania; Nguyen Ngoc Tu, Ha Giang, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy.

O446. Prove that in any triangle ABC the following inequality holds:

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{B}{2}\le\sqrt{2+\frac{r}{2R}}$$

Proposed by Dragoljub Miloševići, Gornji Milanovac, Serbia

Solution by Nikos Kalapodis, Patras, Greece

Let a,b,c be the sides of triangle ABC. Using the well-known substitution $a=y+z, b=z+x, c=x+y$ we have that

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}=\sqrt{\frac{yz}{(x+y)(x+z)}}$$

$$r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}=\sqrt{\frac{xyz}{x+y+z}}$$

and

$$R=\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}=\frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}.$$

Therefore the given inequality becomes

$$\sqrt{\frac{yz}{(x+y)(x+z)}}+\sqrt{\frac{zx}{(y+z)(y+x)}}+\sqrt{\frac{xy}{(z+x)(z+y)}}\le\sqrt{2+\frac{2xyz}{(x+y)(y+z)(z+x)}}$$

or

$$\sqrt{yz(y+z)}+\sqrt{zx(z+x)}+\sqrt{xy(x+y)}\le\sqrt{2[(x+y)(y+z)(z+x)+xyz]}$$

$$\sqrt{yz(y+z)}+\sqrt{zx(z+x)}+\sqrt{xy(x+y)}\le\sqrt{2(x+y+z)(xy+yz+zx)},$$

which follows by the Cauchy-Schwarz inequality. Equality holds iff $x=y=z$, i.e. $a=b=c$.

Also solved by Arkady Alt, San Jose, CA, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Scott H. Brown, Auburn University Montgomery, Montgomery, AL, USA; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul National Zinca Golescu, Pitesti, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

O447. Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2 \geq a^3+b^3+c^3$. Prove that

$$a^3b^3+b^3c^3+c^3a^3 \leq a^2b^2+b^2c^2+c^2a^2$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Albert Stadler, Herrliberg, Switzerland

It is sufficient to prove that

$$a^3b^3+b^3c^3+c^3a^3 \leq \left(\frac{a^3+b^3+c^3}{a^2+b^2+c^2}\right)^2 (a^2b^2+b^2c^2+c^2a^2),$$

which is equivalent to (after clearing denominators)

$$\sum_{sym} a^7b^3 + \frac{1}{2}\sum_{sym} a^4b^3c^3 \leq \sum_{sym} a^8b^2 + \frac{1}{2}\sum_{sym} a^6b^2c^2.$$

However this inequality is true, since by Muirhead's inequality,

$$\sum_{sym} a^7b^3 \leq \sum_{sym} a^8b^2, \text{ and } \sum_{sym} a^4b^3c^3 \leq \sum_{sym} a^6b^2c^2.$$

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ioannis D. Sfikas, Athens, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Titu Zvonaru, Comănești, Romania.

O448. Prove that for any positive integers m and n there are m consecutive positive integer numbers such that each number has at least n divisors.

Proposed by Anton Vassilyev, Astana, Kazakhstan

Solution by Joel Schlosberg, Bayside, NY, USA

Let p_k be the kth prime, and for any positive integer j let

$$P_j = \prod_{k=(j-1)n+1}^{jn} p_k.$$

For $j_1 \neq j_2$, P_{j_1} and P_{j_2} are relatively prime. Therefore, by the Chinese Remainder Theorem there exists a positive integer M such that for all $j \in \{1, \ldots, m\}$,

$$M \equiv -j \pmod{P_j}.$$

Then $M+1, \ldots, M+m$ is a sequence of m consecutive integers such that $M+j$ is divisible by P_j and thus has at least the n divisors $p_{(j-1)n+1}, \ldots, p_{jn}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Anderson Torres, Sao Paulo, Brazil; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia, Albert Stadler, Herrliberg, Switzerland.

O449. At the AwesomeMath Summer Camp, a teacher wants to challenge his 102 students. He gives them 19 green t-shirts, 25 red t-shirts, 28 purple t-shirts and 30 blue t-shirts, a t-shirt to each student. Then, he calls three students randomly: if they have a t-shirt with different colors, they must wear a t-shirt of the remaining color and must solve a problem given by the teacher. Is it possible that after some time all the students have all the t-shirts of the same color? (Assume that there are sufficient t-shirts for each color in the store).

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Daniel Lasaosa, Pamplona, Spain

Note that at each step, the parity of the number of t-shirts of each color changes, because either exactly one t-shirt of that color is removed, or exactly three t-shirts of that color are added. Therefore, since at the beginning there are two colors with an odd number of t-shirts, and two colors with an even number of t-shirts, at each step in the process there will be two colors with an odd number of t-shirts, and two colors with an even number of t-shirts. The desired final situation has all four numbers of t-shirts of each colors even (three 0, one 102), and cannot thus be obtained.

Also solved by Takuji Imaiida, Fujisawa, Kanagawa, Japan; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Joel Schlosberg, Bayside, NY, USA; Paraskevi-Andrianna Maroutsou, Charters Sixth Form, Sunningdale, England, UK; Albert Stadler, Herrliberg, Switzerland.

O450. A computer had randomly assigned all labels from 1 through 64 to an 8×8 electronic board. Then it did it also randomly for the second time. Let n_k be the label of the square that had been originally assigned k. Knowing that $n_{17} = 18$, find the probability that

$$|n_1 - 1| + |n_2 - 2| + \ldots + |n_{64} - 64| = 2018.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

Without the restriction $n_{17} = 18$ the maximum of $|n_1 - 1| + |n_2 - 2| + \ldots + |n_{64} - 64|$ is

$$(-1 - 1 - 2 - 2 - \ldots - 32 - 32) + (33 + 33 + 34 + 34 + \ldots + 64 + 64) = 2048,$$

achieved if and only if $\{n_1, n_2, \ldots, n_{32}\} = \{33, 34, \ldots, 64\}$ and $\{n_{33}, n_{34}, \ldots, n_{64}\} = \{1, 2, \ldots, 32\}$. The swap between -18 and 33 with -33 and 18 lowers the sum to exactly $2048 - 2 \times 15 = 2018$. The value 33 must be assumed by some n_k, with $k \in \{33, 34, \ldots, 64\}$, implying

$$\{n_{33}, n_{34}, \ldots, n_{64}\} \setminus \{k\} = \{1, 2, \ldots, 32\} \setminus \{18\}.$$

This could be done in $32 \times (31!)$ ways. Independently,

$$\{n_1, n_2, \ldots, n_{32}\} \setminus \{17\} = \{34, 35, \ldots, 64\}.$$

Hence the desired probability is $32 \times (31!)(31!)/64! = \frac{1}{32 \times \binom{64}{32}}$.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.