Junior Problems

J451. Solve in positive integers the equation

$$2(6xy+5)^2 - 15(2x+2y)^2 = 2018.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J452. Let $a, b, c > 0$ and x, y, z be real numbers. Prove that

$$\frac{a\left(y^2+z^2\right)}{b+c} + \frac{b\left(z^2+x^2\right)}{c+a} + \frac{c\left(x^2+y^2\right)}{a+b} \geq xy + yz + zx.$$

Proposed by An Zhenping, Xianyang Normal University, China

J453. Let ABC be an acute triangle, O its circumcenter and H its orthocenter. Let D be the midpoint of BC. The perpendicular in H to DH intersects AB and AC in P and Q, respectively. Prove that

$$\overrightarrow{AP} + \overrightarrow{AQ} = 4\overrightarrow{OD}.$$

Proposed by Mihaela Berindeanu, Bucharest, România

J454. Let $ABCD$ be a square and let M, N, P, Q be arbitrary points on the sides AB, BC, CD, DA, respectively. Prove that

$$MN + NP + PQ + QM \geq 2AC.$$

When does the equality hold?

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J455. Let ABC be a triangle, Γ its circumcircle with center O and H its orthocenter. Let H_1 be the reflection of H about the line BC and H_2 be the reflection of H through the midpoint of the segment BC. Let S be the point on Γ such that $\angle SOH_2 = \frac{1}{3}\angle H_1OH_2$. Prove that the Simson line of point S is tangent to the Euler circle of the triangle ABC.

Proposed by Alexandru Gîrban, Constanţa, România

J456. Let a, b, c, d be real numbers such that $a+b+c+d=0$ and $a^2+b^2+c^2+d^2=12$. Prove that

$$-3 \leq abcd \leq 9.$$

Proposed by Marius Stănean, Zalău, România

Senior Problems

S451. Find all pairs (z, w) of complex numbers simultaneously satisfying the equations:

$$\frac{2018}{z} - w = 15 + 28i$$

$$\frac{2018}{w} - z = 15 - 28i.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S452. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$abc\left(a\sqrt{a} + b\sqrt{b} + c\sqrt{c}\right) \leq 3.$$

Proposed by Tran Tien Manh, Vinh City, Vietnam

S453. Let $a, b, c \in (-1, 1)$ such that $a^2 + b^2 + c^2 = 2$. Prove that

$$\frac{(a+b)(a+c)}{1-a^2} + \frac{(b+c)(b+a)}{1-b^2} + \frac{(c+a)(c+b)}{1-c^2} \geq 9(ab + bc + ca) + 6.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S454. Let a, b, c, d be positive real numbers such that

$$a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

Prove that

$$a^2 + b^2 + c^2 + d^2 + 3abcd \geq 7.$$

Proposed by Marius Stănean, Zalău, România

S455. Let a and b be real numbers such that all roots of the polynomial $f(X) = X^4 - X^3 + aX + b$ are real numbers. Prove that

$$f\left(-\frac{1}{2}\right) \leq \frac{3}{16}.$$

Proposed by Vladimir Cerbu, România

S456. Let a, b, c be the sides of a triangle ABC and R, r its circumradius and inradius, respectively. Prove that

$$\left(\frac{a}{b+c}\right)^2 + \left(\frac{b}{c+a}\right)^2 + \left(\frac{c}{a+b}\right)^2 + \frac{3r}{4R} \geq \frac{9}{8}$$

Proposed by Titu Zvonaru, Comăneşti, România

Undergraduate Problems

U451. Let x_1, x_2, x_3, x_4 be the roots of the polynomial $2018x^4 + x^3 + 2018x^2 - 1$. Evaluate

$$\left(x_1^2 - x_1 + 1\right)\left(x_2^2 - x_2 + 1\right)\left(x_3^2 - x_3 + 1\right)\left(x_4^2 - x_4 + 1\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U452. Find all finite groups whose all proper subgroups have order 2 or 3.

Proposed by Mihai Piticari, Câmpulung Moldovenesc, România

U453. Let A be a $n \times n$ matrix such that $A^7 = I_n$. Prove that $A^2 - A + I_n$ is invertible and find its inverse.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U454. Let $f : [0, 1] \longrightarrow [0, 1)$ be an integrable function. Prove that

$$\lim_{n\to\infty} \int_0^1 f^n(x)dx = 0.$$

Proposed by Mihai Piticari and Sorin Rădulescu, România

U455. For two square matrices $X, Y \in \mathrm{M}_n(\mathbb{C})$ we denote by $[X, Y] = XY - YX$ their commutator. Prove that if $A, B, C \in \mathrm{M}_n(\mathbb{C})$ satisfy the identity $ABC + A + B + C = AB + BC + AC$ then

$$[A, BC] = [A, B] + [A, C].$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

U456. Let $a_1 > \cdots > a_m$ be positive integers and $P_1(x), \ldots, P_m(x)$ be rational functions with rational coefficients. Assume that

$$P_1(n)a_1^n + \cdots + P_m(n)a_m^n$$

is an integer for all sufficiently large n. Prove that $P_1(x), \ldots, P_m(x)$ are polynomials.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Olympiad Problems

O451. Let ABC be a triangle, Γ its circumcircle, ω its incircle and I the incenter. Let M be the midpoint of BC. The incircle ω is tangent to AB and AC at F and E, respectively. Suppose EF meets Γ at distinct points P and Q. Let J denote the point on EF such that MJ is perpendicular on EF. Show that IJ and the radical axis of (MPQ) and (AJI) intersect on Γ.

Proposed by Toni Wen, USA

O452. Let a, b, c be nonnegative real numbers, at most one being zero. Prove that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{3}{a+b+c} \geq \frac{4}{\sqrt{ab+bc+ca}}$$

Proposed by An Zhenping, Xianyang Normal University, China

O453. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{ab}{a^5+b^5+c^2} + \frac{bc}{b^5+c^5+a^2} + \frac{ca}{c^5+a^5+b^2} \leq 1.$$

Proposed by Florin Rotaru, Focşani, România

O454. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{18}\left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right) + \frac{a}{2a+b+c} + \frac{b}{a+2b+c} + \frac{c}{a+b+2c} \geq \frac{11}{12}$$

Proposed by Titu Zvonaru, Comăneşti, România

O455. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \ldots + a_n = n$, $n \geq 4$. Prove that

$$\sum_{1 \leq i < j \leq n} 2\, a_i a_j \geq (n-1)\sqrt{n a_1 a_2 \cdots a_n (a_1^2 + a_2^2 + \cdots + a_n^2)}.$$

Proposed by Marius Stănean, Zalău, România

O456. Find all positive integers n for which the equation

$$x^2 + [x]^2 + \{x\}^2 = n$$

has solutions $x \geq 0$. (Here, $[x]$ and $\{x\}$ denotes the integer part and the fractional part of the real number x, respectively.)

Proposed by Dorin Andrica and Dan-Stefan Marinescu, România.