## Junior problems

J451. Solve in positive integers the equation

$$2(6xy+5)^2 - 15(2x+2y)^2 = 2018.$$

*Proposed by Adrian Andreescu, University of Texas at Austin, USA*

*Solution by Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy*
The given equation is equivalent to:

$$2(6xy+5)^2 - 60(x+y)^2 = 2018$$

or

$$(6xy+5)^2 - 30(x+y)^2 = 1009$$

Expanding the squares yields:

$$1009 = 36x^2y^2 + 25 - 30x^2 - 30y^2 = (6x^2-5)(6y^2-5)$$

Because 1009 is prime and $6x^2-5, 6y^2-5 > 0$ if $x, y \in \mathbb{N}$, we only have two possibilities:

$$\begin{cases} 6x^2-5=1 \\ 6y^2-5=1009 \end{cases} \quad \text{or} \quad \begin{cases} 6x^2-5=1009 \\ 6y^2-5=1 \end{cases}$$

By solving the two systems one gets the solutions

$$(x,y) = (\pm1, \pm13), (\pm13, \pm1)$$

*Also solved by Daniel Lasaosa, Pamplona, Spain; Adarsh Kumar, Ryan International School, Mumbai, India; Anish Ray, Institute of Mathematics and Applications, Bhubaneswar, India; Arkady Alt, San Jose, CA, USA; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Frank Gamboa, Faculty of Mathematics and Computer Sciences, Havana, Cuba; George Theodoropoulos, 2nd High school of Agrinio, Greece; Ioannis D. Sfikas, Athens, Greece; Joel Schlosberg, Bayside, NY, USA; Corneliu Mănescu-Avram, Ploiești, Romania and Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua; Shriom Kumar Singh, Fiitjee, Mumbai, India; Titu Zvonaru, Comănești, Romania; Polyahedra, Polk State College, FL, USA; Nikos Kalapodis, Patras, Greece; Adrienne Ko, Fieldston School, New York, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Daniel Cortild; Dumitru Barac, Sibiu, Romania; Mihai Bogdan, Romania; Chanyeol Paul Kim, Seoul International School, Seoul, South Korea; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Kelvin Kim, Bergen Catholic High School, NJ, USA; Navneel Singhal, Delhi, India.*

J452. Let $a, b, c > 0$ and $x, y, z$ be real numbers. Prove that

$$\frac{a\left(y^2+z^2\right)}{b+c}+\frac{b\left(z^2+x^2\right)}{c+a}+\frac{c\left(x^2+y^2\right)}{a+b} \geq xy+yz+zx.$$

*Proposed by An Zhenping, Xianyang Normal University, China*

*Solution by Polyahedra, Polk State College, USA*

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&2(a+b+c)^2\left[\frac{a(y^2+z^2)}{b+c}+\frac{b(z^2+x^2)}{c+a}+\frac{c(x^2+y^2)}{a+b}+\frac{1}{2}(x^2+y^2+z^2)\right]= \\
&\left[(bc+ab)+(ca+bc)+2a^2+(ca+bc)+(ab+ca)+2b^2+(ab+ca)+(bc+ab)+2c^2\right]\cdot \\
&\left[\frac{b^2x^2}{bc+ab}+\frac{c^2x^2}{ca+bc}+\frac{a^2x^2}{2a^2}+\frac{c^2y^2}{ca+bc}+\frac{a^2y^2}{ab+ca}+\frac{b^2y^2}{2b^2}+\frac{a^2z^2}{ab+ca}+\frac{b^2z^2}{bc+ab}+\frac{c^2z^2}{2c^2}\right] \\
&\geq \left[(b+c+a)|x|+(c+a+b)|y|+(a+b+c)|z|\right]^2 \geq (a+b+c)^2(x+y+z)^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{a(y^2+z^2)}{b+c}+\frac{b(z^2+x^2)}{c+a}+\frac{c(x^2+y^2)}{a+b} \\
\geq\ &\frac{1}{2}(x+y+z)^2-\frac{1}{2}(x^2+y^2+z^2)=xy+yz+zx.
\end{aligned}$$

*Also solved by Daniel Lasaosa, Pamplona, Spain; Nikos Kalapodis, Patras, Greece; Adrienne Ko, Fieldston School, New York, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Daniel Cortild; Dumitru Barac, Sibiu, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Anish Ray, Institute of Mathematics and Applications, Bhubaneswar, India; Arkady Alt, San Jose, CA, USA; Frank Gamboa, Faculty of Mathematics and Computer Sciences, Havana, Cuba; Idamia Abdelhamid, Jaafar El Fassi High School, Casablanca, Morocco; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Titu Zvonaru, Comănești, Romania.*

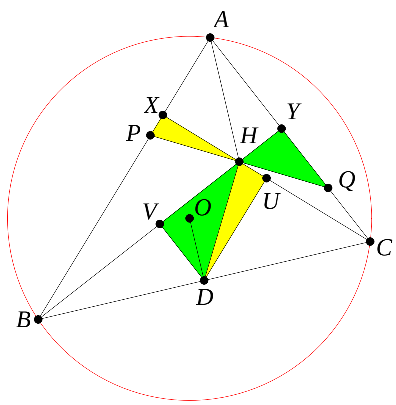
J453. Let $ABC$ be an acute triangle, $O$ its circumcenter and $H$ its orthocenter. Let $D$ be the midpoint of $BC$. The perpendicular in $H$ to $DH$ intersects $AB$ and $AC$ in $P$ and $Q$, respectively. Prove that

$$\overrightarrow{AP} + \overrightarrow{AQ} = 4\overrightarrow{OD}.$$

*Proposed by Mihaela Berindeanu, Bucharest, România*

*Solution by Polyahedra, Polk State College, USA*

Let $X$ and $Y$ be the feet of altitudes on $AB$ and $AC$, and $U$ and $V$ be the feet of perpendiculars from $D$ onto $CX$ and $BY$, respectively.




Then $\triangle HPX \sim \triangle DHU$ and $\triangle HQY \sim \triangle DHV$. Hence,

$$\frac{HP}{DH} = \frac{HX}{DU} = \frac{2HX}{BX} = \frac{2HY}{CY} = \frac{HY}{DV} = \frac{HQ}{DH}.$$

Thus, $H$ is the midpoint of $PQ$. Therefore, $\overrightarrow{AP} + \overrightarrow{AQ} = 2\overrightarrow{AH} = 4\overrightarrow{OD}$.

*Also solved by Daniel Lasaosa, Pamplona, Spain; Nikos Kalapodis, Patras, Greece; Mihai Bogdan, Romania; Frank Gamboa, Faculty of Mathematics and Computer Sciences, Havana, Cuba; Titu Zvonaru, Comănești, Romania; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Jiho Lee, Canterbury School, New Milford, CT, USA; Navneel Singhal, Delhi, India.*

J454. Let $ABCD$ be a square and let $M, N, P, Q$ be arbitrary points on the sides $AB, BC, CD, DA$ respectively. Prove that

$$MN + NP + PQ + QM \geq 2AC.$$

When does the equality hold?

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

*First solution by Daniel Lasaosa, Pamplona, Spain*

Assume wlog since the problem is invariant under scaling, that $AB = BC = CA = AD = 1$, or $AC = \sqrt{2}$. Let $0 \leq t, u, v, w \leq 1$ such that $AM = t$, $BN = u$, $CP = v$ and $DQ = w$. Using the AM-QM inequality, note that

$$MN = \sqrt{(1-t)^2 + u^2} = \sqrt{2}\sqrt{\frac{(1-t)^2 + u^2}{2}} \geq \sqrt{2}\frac{1-t+u}{2},$$

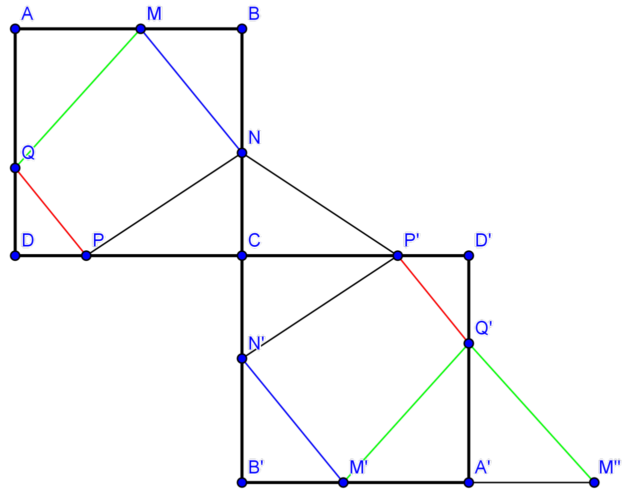
with equality iff $1 - t = u$, and similarly for $NP, PQ, QM$. Then,

$$MN + NP + PQ + QM \geq \sqrt{2}\frac{(1-t+u) + (1-u+v) + (1-v+w) + (1-w+t)}{2} =$$

$$= 2\sqrt{2} = 2AC.$$

The conclusion follows, equality holds iff $t + u = u + v = v + w = w + t = 1$, ie iff $AM = NC = CP = QA$ and $MB = BN = PD = DQ$, or equivalently iff $MN, NP, PQ, QM$ all form angles of $45^\circ$ with the sides of the square.

*Second solution by Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua*




The proof is immediate when you see the figure. The square $A'B'CD'$ is the reflection of the square $ABCD$ with respect to point $C$, and $M''$ is the reflection of the $M'$ with respect to point $A'$.

It is evident that $MM''$ is equal to $AA'$, because $AMM''A'$ is a parallelogram.

Then the shortest distance from $M$ to $M''$ is equals to $2AC$, but the lengths $MN$, $NP' = NP$, $P'Q' = PQ$ and $Q'M'' = QM$ form a polygonal line from $M$ to $M''$. From here the inequality follows.

If $MN + MP + PQ + QM = 2AC$, then in the figure, $M$, $N$, $P'$, $Q'$ and $M''$ are collinear. This implies that $MNPQ$ is a rectangle.

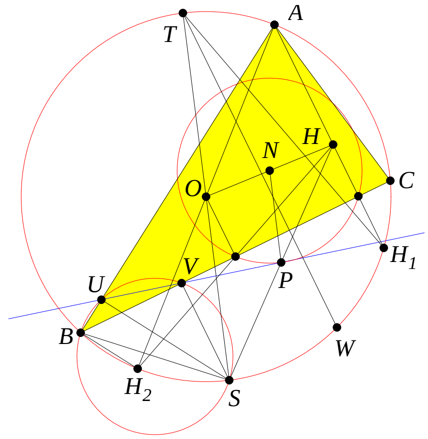
*Also solved by Polyahedra, Polk State College, USA; Nikos Kalapodis, Patras, Greece; Adrienne Ko, Fieldston School, New York, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Daniel Cortild; Mihai Bogdan, Romania; Chanyeol Paul Kim, Seoul International School, Seoul, South Korea; Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Kelvin Kim, Bergen Catholic High School, NJ, USA; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Frank Gamboa, Faculty of Mathematics and Computer Sciences, Havana, Cuba; Ioannis D. Sfikas, Athens, Greece; Joel Schlosberg, Bayside, NY, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Titu Zvonaru, Comănești, Romania; Navneel Singhal, Delhi, India.*

J455. Let $ABC$ be a triangle, $\Gamma$ its circumcircle with center $O$ and $H$ its orthocenter. Let $H_1$ be the reflection of $H$ about the line $BC$ and $H_2$ be the reflection of $H$ through the midpoint of the segment $BC$. Let $S$ be the point on $\Gamma$ such that $\angle SOH_2 = \frac{1}{3}\angle H_1OH_2$. Prove that the Simson line of point $S$ is tangent to the Euler circle of the triangle $ABC$.

*Proposed by Alexandru Gîrban, Constanţa, România*

*Solution by Polyahedra, Polk State College, USA*

Let $N$ and $P$ be the midpoints of $HO$ and $HS$ and $U$ and $V$ the feet of perpendiculars from $S$ onto $AB$ and $BC$, respectively. Also, let $T$ be the reflection of $S$ through $O$ and $W$ the point on $\Gamma$ such that $TW \parallel AH_1$.




Since $\Gamma$ is a dilation of the Euler circle with ratio 2 and center $H$, $H_1$ and $H_2$ are on $\Gamma$ and $P$ is on the Euler circle. It is well known that $P$ is also on the Simson line $UV$.
See, for example, `http://mathworld.wolfram.com/SimsonLine.html`. Since $H_2B \perp AB$,

$$\angle PVC = \angle UVB = \angle USB = \angle H_2BS = \frac{1}{2}\angle STH_1 = \angle STW.$$

Hence, $ST \perp UV$, so $NP \perp UV$. This completes the proof, since $N$ is the center of the Euler circle.

*Also solved by Daniel Lasaosa, Pamplona, Spain.*

J456. Let $a, b, c, d$ be real numbers such that $a+b+c+d=0$ and $a^2+b^2+c^2+d^2=12$. Prove that

$$-3 \le abcd \le 9.$$

*Proposed by Marius Stănean, Zalău, România*

*Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy*

RHS:

$$12 = a^2+b^2+c^2+d^2 \ge 4(a^2b^2c^2d^2)^{\frac{1}{4}} = 4|abcd|^{\frac{1}{2}}$$

thus

$$abcd \le |abcd| \le 9$$

LHS:

If $a, b \le 0$ and $c, d \ge 0$ the inequality holds true so let's suppose $a, b, c \ge 0$ and $d \le 0$.

$$abcd = abc(-a-b-c) \ge -3 \iff abc(a+b+c) \le 3 \tag{1}$$

which evidently holds true if $abc(a+b+c) \le 0$.

$$3abc(a+b+c) \le (ab+bc+ca)^2$$

so it suffices to show $ab+bc+ca \le 3$.

$$a^2+b^2+c^2+d^2 = a^2+b^2+c^2+(-a-b-c)^2 = 12$$

if and only if

$$6 = a^2+b^2+c^2+ab+bc+ca$$

Since

$$(a-b)^2+(b-c)^2+(c-a)^2 \ge 0 \iff a^2+b^2+c^2 \ge ab+bc+ca$$

we get

$$6 = a^2+b^2+c^2+ab+bc+ca \ge 2(ab+bc+ca) \iff ab+bc+ca \le 3$$

and this concludes the proof.

*Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, FL, USA; Adrienne Ko, Fieldston School, New York, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Dumitru Barac, Sibiu, Romania; Mihai Bogdan, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Frank Gamboa, Faculty of Mathematics and Computer Sciences, Havana, Cuba; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania.*

## Senior problems

S451. Find all pairs $(z, w)$ of complex numbers simultaneously satisfying the equations:

$$\frac{2018}{z} - w = 15 + 28i,$$
$$\frac{2018}{w} - z = 15 - 28i.$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*Solution by Takuji Imaiida, Fujisawa, Kanagawa, Japan*

Let

$$\frac{2018}{z} - w = 15 + 28i \quad (1)$$
$$\frac{2018}{w} - z = 15 + 28i. \quad (2)$$

From $(1) \times (2)$, we obtain

$$\begin{aligned} \left(\frac{2018}{z} - w\right)\left(\frac{2018}{w} - z\right) &= (15 + 28i)(15 - 28i) \\ (zw)^2 - 5045zw + 2018^2 &= 0 \\ zw &= 1009, 4036. \end{aligned}$$

If $zw = 1009$, then $w = 1009/z$ and from (1) and (2) we obtain $(z, w) = (15 - 28i, 15 + 28i)$. If $zw = 4036$, then $(z, w) = (-30 + 56i, -30 - 56i)$. Both solutions satisfy equations (1) and (2).

*Also solved by Daniel Lasaosa, Pamplona, Spain; Adrienne Ko, Fieldston School, New York, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Dumitru Barac, Sibiu, Romania; Mihai Bogdan, Romania; Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy; Kelvin Kim, Bergen Catholic High School, NJ, USA; Jiho Lee, Canterbury School, New Milford, CT, USA; Navneel Singhal, Delhi, India; Arkady Alt, San Jose, CA, USA; Frank Gamboa, Faculty of Mathematics and Computer Sciences, Havana, Cuba; Moubinool Omarjee, Lycée Henri IV, Paris, France; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Titu Zvonaru, Comănești, Romania; Suhas Sheikh, Ryan International School, Sanpada, Navi Mumbai, India.*

S452. Let $a, b, c$ be positive real numbers such that $a+b+c=3$. Prove that

$$abc\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)\leq 3.$$

*Proposed by Tran Tien Manh, Vinh City, Vietnam*

*Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy*

$$abc\leq (ab+bc+ca)^{3/2}/(3\sqrt{3}), \qquad \sum a\sqrt{a}\leq (a^2+b^2+c^2)^{3/4}\cdot 3^{1/4}$$

so it suffices to prove

$$\frac{(ab+bc+ca)^{\frac{3}{2}}}{3\sqrt{3}}(a^2+b^2+c^2)^{\frac{3}{4}}3^{\frac{1}{4}}\leq 3 \iff (ab+bc+ca)^{\frac{3}{2}}(a^2+b^2+c^2)^{\frac{3}{4}}\leq 3^{\frac{9}{4}}$$

or

$$(ab+bc+ca)^{\frac{3}{2}}((a+b+c)^2-2(ab+bc+ca))^{\frac{3}{4}}\leq 3^{\frac{9}{4}}$$

if and only if

$$(ab+bc+ca)^{\frac{3}{2}}(9-2(ab+bc+ca))^{\frac{3}{4}}\leq 3^{\frac{9}{4}}$$

Moreover we get upon elevating to the power 4/3

$$(ab+bc+ca)^2(9-2(ab+bc+ca))\leq 27 \iff 2(ab+bc+ca-3)^2(ab+bc+ca+\frac{3}{2})\geq 0$$

and this holds true.

*Also solved by Daniel Lasaosa, Pamplona, Spain; Adrienne Ko, Fieldston School, New York, NY, USA; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Daniel Cortild; Dumitru Barac, Sibiu, Romania; Mihai Bogdan, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Arkady Alt, San Jose, CA, USA; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Paraskevi-Andrianna Maroutsou, Charters Sixth Form, Sunningdale, England, UK; Titu Zvonaru, Comănești, Romania.*

S453. Let $a, b, c \in (-1, 1)$ such that $a^2 + b^2 + c^2 = 2$. Prove that

$$\frac{(a+b)(a+c)}{1-a^2} + \frac{(b+c)(b+a)}{1-b^2} + \frac{(c+a)(c+b)}{1-c^2} \geq 9(ab+bc+ca) + 6.$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*Solution by the author*

$$\frac{2(a+b)(a+c)}{-a^2+b^2+c^2} + \frac{2(b+c)(b+a)}{a^2-b^2+c^2} + \frac{2(c+a)(c+b)}{a^2+b^2-c^2} \geq 9(ab+bc+ca) + 6.$$

By adding $1+1+1=3$ to both sides we obtain the equivalent inequality:

$$\frac{(a+b+c)^2}{-a^2+b^2+c^2} + \frac{(a+b+c)^2}{a^2-b^2+c^2} + \frac{(a+b+c)^2}{a^2+b^2-c^2} \geq \frac{9}{2}(2ab+2bc+2ca+a^2+b^2+c^2)$$

We have equality for $a+b+c=0$ and, for $a+b+c \neq 0$ the inequality reduces to

$$\frac{1}{-a^2+b^2+c^2} + \frac{1}{-a^2+b^2+c^2} + \frac{1}{-a^2+b^2+c^2} \geq \frac{9}{2}.$$

But this is follows from the AM-HM Inequality for the positive numbers $-a^2+b^2+c^2, a^2-b^2+c^2, a^2+b^2-c^2$. Equality holds if and only if

$$a+b+c=0 \text{ or } |a|=|b|=|c|=\sqrt{\frac{2}{3}}$$

*Also solved by Daniel Lasaosa, Pamplona, Spain; Nikos Kalapodis, Patras, Greece; Dumitru Barac, Sibiu, Romania; Mihai Bogdan, Romania; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Adarsh Kumar, Ryan International School, Mumbai, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Nikolaos Evgenidis, M.N.Raptou High School, Larissa, Greece; Frank Gamboa, Faculty of Mathematics and Computer Sciences, Havana, Cuba; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Titu Zvonaru, Comănești, Romania.*

S454. Let $a, b, c, d$ be positive real numbers such that

$$a+b+c+d=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

Prove that

$$a^2+b^2+c^2+d^2+3abcd\geq 7.$$

*Proposed by Marius Stanean, Zalau, România*

*Solution by the author*

Denote $S=a+b+c+d$ and let $a=\frac{xS}{4}$, $b=\frac{yS}{4}$, $c=\frac{zS}{4}$, $d=\frac{tS}{4}$, then $x+y+z+t=4$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{S^2}{4}$.

The inequality becomes

$$\frac{S^2}{16}(x^2+y^2+z^2+t^2)+\frac{3S^4}{4^4}xyzt\geq 7,$$

or

$$4(xyz+yzx+zxt+txy)(x^2+y^2+z^2+t^2)+3(xyz+yzt+ztx+txy)^2\geq 112xyzt,$$

or

$$(xyz+yzx+zxt+txy)\left[4(x^2+y^2+z^2+t^2)+3(xyz+yzx+zxt+txy)\right]\geq 112xyzt. \tag{1}$$

We prove that

$$4(x^2+y^2+z^2+t^2)+3(xyz+yzx+zxt+txy)\geq 28.$$

Indeed, since

$$3(xyz+yzx+zxt+txy)=\sum_{cyc}x^3-6\sum_{cyc}x^2+32$$

we need to prove that

$$4\sum_{cyc}x^2+\sum_{cyc}x^3-6\sum_{cyc}x^2+32\geq 28,$$

or

$$\sum_{cyc}x(x-1)^2\geq 0,$$

true. Returnning to (1), applying AM-GM Inequality we have

$$\begin{aligned} LHS &\geq 28(xyz+yzx+zxt+txy) \\ &= 7(x+y+z+t)(xyz+yzx+zxt+txy) \\ &\geq 112\sqrt[4]{xyzt}\sqrt[4]{x^3y^3z^3t^3}=112xyzt. \end{aligned}$$

Equality holds when $x=y=z=t=1 \Longrightarrow a=b=c=d=1$.

*Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Dumitru Barac, Sibiu, Romania; Mihai Bogdan, Romania; Frank Gamboa, Faculty of Mathematics and Computer Sciences, Havana, Cuba; Ioannis D. Sfikas, Athens, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy.*

S455. Let $a$ and $b$ be real numbers such that all roots of the polynomial $f(X) = X^4 - X^3 + aX + b$ are real numbers. Prove that

$$f\left(-\frac{1}{2}\right) \le \frac{3}{16}.$$

*Proposed by Vladimir Cerbu, România*

*Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia*
Denote roots of given polynomials by $x_1$, $x_2$, $x_3$, $x_4$. By the Viet's theorem we get

$$x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = 0$$
$$-x_1x_2x_3x_4\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right) = a$$
$$x_1x_2x_3x_4 = b.$$

From the first and second equation gives that

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1.$$

Using Cauchy-Schwartz's inequality we have

$$1 = x_1^2 + (x_2^2 + x_3^4 + x_4^2) \ge x_1^2 + \frac{1}{3}(x_2 + x_3 + x_4)^2$$
$$= x_1^2 + \frac{1}{3}(1 - x_1)^2.$$

Hence we have

$$-\frac{1}{2} \le x_1 \le 1.$$

Similarly way

$$-\frac{1}{2} \le x_2, x_3, x_4 \le 1.$$

Hence we get

$$f(1) \ge 0 \iff a + b \ge 0.$$

We sufficient to prove that

$$f\left(-\frac{1}{2}\right) \le \frac{3}{16} \iff a \ge 2b.$$

Let $b \le 0$. From $a + b \ge 0$, we have $a \ge 0$. That case $a \ge 2b$ is true.
Let $b > 0$. Other word $x_1x_2x_3x_4 > 0$. That case we have

$$a \ge 2b \iff \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} \le -2 \qquad (*)$$

That case two roots are positive, two roots are negative. Assume that

$$x_1, x_2 > 0,\ x_3, x_4 < 0$$

$$-\frac{1}{2} \le x_4 \le 1 \Rightarrow 2x_4 + 1,\ 1 - x_4 \ge 0$$

and $x_1x_2x_3 < 0$. Hence we get

$$\begin{aligned} x_4^2(1-x_4) \geq x_1x_2x_3(2x_4+1) &\Leftrightarrow x_4^2(x_1+x_2+x_3) - x_1x_2x_3 \geq 2x_1x_2x_3x_4 \\ &\Leftrightarrow \frac{x_4(x_1+x_2+x_3)}{x_1x_2x_3} - \frac{1}{x_4} \geq 2 \\ &\Leftrightarrow \frac{-x_1x_2 - x_1x_3 - x_2x_3}{x_1x_2x_3} - \frac{1}{x_4} \geq 2 \\ &\Leftrightarrow \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} \leq -2. \end{aligned}$$

$(*)$ is proved.

*Also solved by Dumitru Barac, Sibiu, Romania; Adarsh Kumar, Ryan International School, Mumbai, India; Frank Gamboa, Faculty of Mathematics and Computer Sciences, Havana, Cuba.*

S456. Let $a, b, c$ be the sides of a triangle $ABC$ and $R, r$ its circumradius and inradius, respectively. Prove that

$$\left(\frac{a}{b+c}\right)^2 + \left(\frac{b}{c+a}\right)^2 + \left(\frac{c}{a+b}\right)^2 + \frac{3r}{4R} \geq \frac{9}{8}$$

*Proposed by Titu Zvonaru, Comanesti, România*

*Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy*

We know that

$$\frac{3r}{4R} = \frac{3(s-a)(s-b)(s-c)}{abc}$$

We are reduced to show the sufficient inequality

$$\frac{1}{3}\left(\frac{(a+b+c)^2}{2(ab+bc+ca)}\right)^2 + \frac{3(s-a)(s-b)(s-c)}{abc} \geq \frac{9}{8}$$

Moreover let's change variables $a = y+z$, $b = x+z$, $c = x+y$. The inequality reads as

$$\sum_{\text{cyc}}\left(\frac{y+z}{x+x+y+z}\right)^2 + \frac{3xyz}{(x+y)(y+z)(z+x)} \geq \frac{9}{8}$$

Clearing the denominators we come to

$$\sum_{\text{sym}}(97a^7b^2 + 103a^6b^3 + 25a^5b^4 + 120a^5b^3c + 191a^5b^2c^2 + 553a^6b^2 + 144a^7bc + 28a^8b) \geq$$
$$\sum_{\text{sym}}(823a^4b^3c^2 + 173a^4b^4c + 264(abc)^3)$$

Muirhead's theorem concludes the proof. Indeed $[a,b,c] \succ [a',b',c']$ where $[a,b,c]$ and $[a',b',c']$ appears respectively in the RHS and LHS.

*Also solved by Joshua Siktar, Carnegie Mellon University, PA, USA; Adarsh Kumar, Ryan International School, Mumbai, India; Ioannis D. Sfikas, Athens, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.*

## Undergraduate problems

U451. Let $x_1, x_2, x_3, x_4$ be the roots of the polynomial $2018x^4 + x^3 + 2018x^2 - 1$. Evaluate

$$\left(x_1{}^2 - x_1 + 1\right)\left(x_2{}^2 - x_2 + 1\right)\left(x_3{}^2 - x_3 + 1\right)\left(x_4{}^2 - x_4 + 1\right).$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*First solution by Takuji Imaiida, Fujisawa, Kanagawa, Japan*

From Vieta's formula we obtain

$$\begin{aligned}
&\sum_{cyc} x_1 = -1/2018,\\
&\sum_{cyc} x_1x_2 = 1,\\
&\sum_{cyc} x_1x_2x_3 = 0,\\
&x_1x_2x_3x_4 = -1/2018.
\end{aligned}$$

On the other hand, since $(x^2 - x + 1) = (x-\omega)(x-\overline{\omega})$, where $\omega = (1+\sqrt{-3})/2$,

$$\begin{aligned}
\prod_{cyc}(x_1^2 - x_1 + 1) &= \prod_{cyc}(x_1-\omega)(x_1-\overline{\omega})\\
&= (x_1x_2x_3x_4 - \omega\sum_{cyc} x_1x_2x_3 + \omega^2\sum_{cyc} x_1x_2 - \omega^3\sum_{cyc} x_1 + \omega^4)\\
&\cdot (x_1x_2x_3x_4 - \overline{\omega}\sum_{cyc} x_1x_2x_3 + \overline{\omega}^2\sum_{cyc} x_1x_2 - \overline{\omega}^3\sum_{cyc} x_1 + \overline{\omega}^4)\\
&= (-\frac{1}{2018} + \omega^2 - \frac{1}{2018} - \omega)(-\frac{1}{2018} + \overline{\omega}^2 - \frac{1}{2018} - \overline{\omega})\\
&= \left(1 + \frac{1}{1009}\right)^2.
\end{aligned}$$

*Second solution by Mircea Becheanu, Montreal, Canada*

Let $a \emptyset$ be a real number and $x_1, x_2, x_3, x_4$ be the roots of the polynomial $P(x) = ax^4 + x^3 + ax^2 - 1$. We want to evaluate the expression

$$E = (x_1^2 - x_1 + 1)(x_2^2 - x_2 + 1)(x_3^2 - x_3 + 1)(x_4^2 - x_4 + 1) = \prod_{i=1}^{4}(x_i^2 - x_i + 1).$$

We use complex numbers. Let $w = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ a primitive root of $-1$. Then we have the relations $w^2 - w + 1 = 0, w^3 = -1, w^5 = \overline{w}$ and $w = 1 + \omega$, where $\omega$ is the primitive cubic root of 1. Moreover, we have the splitting

$$x^2 - x + 1 = (x - w)(x - \overline{w}).$$

From this splitting we have

$$E = \prod_{i=1}^{4}(x_i - w)(x_i - \overline{w}).$$

In order to compute this product we consider the splitting

$$P(x) = a(x - x_1)(x - x_2)(x - x_3)(x - x_4).$$

It follows that $P(w)P(\overline{w}) = a^2E$, and therefore

$$E = \frac{P(w)P(\overline{w})}{a^2}$$

We have $P(w) = aw^4 + w^3 + aw^2 - 1 = -aw + aw^2 - 2 = a\omega - a(1 + \omega) - 2 = -(a + 2)$. Taking the conjugation, we also have $P(\overline{w}) = -(a + 2)$. So,

$$E = \frac{(a + 2)^2}{a^2}.$$

In our problem, $a = 2018$ and

$$E = \frac{1010^2}{1009^2}$$

*Third solution by Ricardo Largaespada, Universidad Nacional de Ingeniería, Managua, Nicaragua*
Let $M$ be a matrix that has eigenvalues $x_1, x_2, x_3, x_4$ that are the roots of $p(x) = x^4 + \frac{1}{2018}x^3 + x^2 - \frac{1}{2018}$. It is known that a matrix $M$ it can be equal to:

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2018} & 0 & -1 & -\frac{1}{2018} \end{bmatrix}$$

Now, due to $M$ is diagonalizable, then $M = PDP^{-1}$, where $D$ is a diagonal matrix with elements $[x_1, x_2, x_3, x_4]$.

Now let's consider the matrix:

$$N = M^2 - M + I = (PDP^{-1})(PDP^{-1}) - PDP^{-1} + PIP^{-1} = P(D^2 - D + I)P^{-1}.$$

Then $N$ is a diagonalizable matrix that has eigenvalues: $x_1^2 - x_1 + 1, x_2^2 - x_2 + 1, x_3^2 - x_3 + 1, x_4^2 - x_4 + 1$.

Then the product of those eigenvalues is the determinant of $N$.

$$\det N = \begin{vmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ \frac{1}{2018} & 0 & 0 & -\frac{2019}{2018} \\ -\frac{2019}{2018^2} & \frac{1}{2018} & \frac{2019}{2018} & \frac{2019}{2018^2} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 & 2019 \\ 0 & 1 & -1 & 1 \\ \frac{1}{2018} & 0 & 0 & 0 \\ -\frac{2019}{2018^2} & \frac{1}{2018} & \frac{2019}{2018} & -\frac{2019}{2018} \end{vmatrix}$$
$$= \frac{1}{2018^2}\begin{vmatrix} -1 & 1 & 2019 \\ 1 & -1 & 1 \\ 1 & 2019 & -2019 \end{vmatrix} = \frac{1}{2018^2}\begin{vmatrix} -1 & 1 & 2019 \\ 0 & 0 & 2020 \\ 0 & 2020 & 0 \end{vmatrix} = \left(\frac{2020}{2018}\right)^2.$$

*Also solved by Daniel Lasaosa, Pamplona, Spain; Dumitru Barac, Sibiu, Romania; Mihai Bogdan, Romania; Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy; Navneel Singhal, Delhi, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Frank Gamboa, Faculty of Mathematics and Computer Sciences, Havana, Cuba; Souza Leão, Federal University of Pernambuco, Brazil; Akash Singha Roy, Kolkata, India; Suhas Sheikh, Ryan International School, Sanpada, Navi Mumbai, India.*

U452. Find all finite groups whose all proper subgroups have order 2 or 3.

*Proposed by Mihai Piticari, Câmpulung Moldovenesc, România*

*Solution by Souza Leão, Federal University of Pernambuco, Brazil*
Let us start with stating
*Lagrange's Theorem* Let $G$ be a group, and $F$ one of its subgroup, then $\operatorname{ord} F | \operatorname{ord} G$.
*Sylow's Theorem* Let $G$ be a group and $p$ a prime number. If $\alpha$ is such that $p^\alpha \mid \operatorname{ord} G$, but $p^{\alpha+1} \nmid \operatorname{ord} G$ and $\operatorname{ord} G \neq p^\alpha$, then $\exists F < G$ such that $\operatorname{ord} F = p^\alpha$.

First, let $G$ be a group whose all proper subgroups have order 2, and $n = \operatorname{ord} G$. If $p \neq 2$ divides $n$, then there exists $\alpha$ satisfying the condition of Sylow's Theorem, and $G$ would have a subgroup with order different from 2; therefore $n = 2^k$, $k \geq 1$.

1. If $k = 1$, then $G$ is cyclic, therefore, $G = \mathbb{Z}_2$, except for isomorphism.
2. If $k = 2$, then:
   (a) If $G$ is cyclic, then $G = \mathbb{Z}_4$.
   (b) If $G$ is not cyclic, then $\exists a, b \in G$ such that $a^2 = b^2 = e$, and since $ab \in G$, $G = \{e, a, b, ab\}$. Furthermore, $ba \in G$, but it can't be equal to $e$, $a$, nor $b$. Therefore $ab = ba$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$.
3. If $k \geq 3$, then, since $G$ cannot have subgroups of order bigger than 2, $\forall a \in G, a^2 = e$, otherwise $(a)$ would be such a subgroup. In this case, $a^2 b^2 = e = (ab)^2$, which implies $ab = ba$, therefore $G$ is abelian. But, then $\{e, a, b, ab\}$ would be a subgroup of $G$ $\forall a, b \in G$, therefore such a $G$ cannot exist.

The same argument holds if $G$ is a group whose all proper subgroups have order 3.

Finally, if $G$ has subgroups of order 2 and 3, by Lagrange's Theorem, $6 \mid \operatorname{ord} G$. And if $p \mid \operatorname{ord} G$ for $p \neq 2, 3$ prime or $\exists \alpha > 1$ such that $2^\alpha \mid \operatorname{ord} G$, $3^\alpha \mid \operatorname{ord} G$, then we can use Sylow's Theorem to find a subgroup whose order is neither 2 nor 3. Hence $\operatorname{ord} G = 6$.

It just remains to show if $\operatorname{ord} G = 6$, then

- $G = \mathbb{Z}_6$
- $G = \mathbb{Z}_2 \times \mathbb{Z}_3$
- $G = S_3$

To prove such assertion, suppose $G$ is not cyclic, because if it is, then $G = \mathbb{Z}_6$.

Take $g, h \in G \backslash \{e\}$ such that $g^2 = e$ or $h^3 = e$. There exists such elements because $G$ has subgroups of order 2 and 3.

- If $gh = hg$, then $G = \{e, g, h, h^2, gh, gh^2\}$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_3$, because all this elements belongs to $G$, they are all distincts and $\operatorname{ord} G = 6$. In fact, Suppose $g^{i_1} h^{j_1} = g^{j_2} h^{j_2}$, for $0 \leq i_1, i_2 \leq 1$ and $0 \leq j_1, j_2 \leq 2$. Then $g^{i_1 - i_2} = h^{j_2 - j_1}$, which implies $i_1 = i_2$ and $j_1 = j_2$
- If $gh \neq hg$, then $G = e, g, h, h^2, gh, hg$, which is equal to $S_3$, because we can consider the isomorphism which takes $g$ to a transposition and $h$ to a 3-cycle. And it is also easy to see all of these elements are distinct.

Thus all the groups satisfying the problem's condition are $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_9$, $\mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $S_3$.

*Also solved by Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Akash Singha Roy, Kolkata, India; Frank Gamboa, Faculty of Mathematics and Computer Sciences, Havana, Cuba; Joel Schlosberg, Bayside, NY, USA.*

U453. Let $A$ be a $n \times n$ matrix such that $A^7 = I_n$. Prove that $A^2 - A + I_n$ is invertible and find its inverse.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

*Solution by the author*
We have $A^8 = A$ and so

$$I_n = A^8 - A + I_n = A^2(A^6 - I_n) + A^2 - A + I_n$$
$$= (A^2 - A + I_n)(A^2(A + I_n)(A^3 - I_n) + I_n).$$

Also, clearly,

$$(A^2(A + I_n)(A^3 - I_n) + I_n)(A^2 - A + I_n) = I_n,$$

hence $A^2 - A + I_n$ is invertible and its inverse is $A^6 + A^5 - A^3 - A^2 + I_n$.

*Also solved by Moubinool Omarjee, Lycée Henri IV, Paris, France; Daniel Lasaosa, Pamplona, Spain; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Dumitru Barac, Sibiu, Romania; Mihai Bogdan, Romania; Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Akash Singha Roy, Kolkata, India; Souza Leão, Federal University of Pernambuco, Brazil; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Frank Gamboa, Faculty of Mathematics and Computer Sciences, Havana, Cuba.*

U454. Let $f\colon[0,1]\to[0,1)$ be an integrable function. Prove that

$$\lim_{n\to\infty}\int_0^1 f^n(x)dx=0$$

*Proposed by Mihai Piticari and Sorin Radulescu, România*

*Solution by Takuji Imaiida, Fujisawa, Kanagawa, Japan*

Let $F_n(t)=\int_0^t f^n(x)dx$. By Mean Value Theorem, there exists some $c_n\in(0,1)$ such that

$$\frac{F_n(1)-F_n(0)}{1-0}=f^n(c_n).$$

Since $0\le f(x)<1$ for $x\in[0,1]$, a sequence $(F_n(1))_n$ decreases and bounded from below. Therefore sequence $(F_n(1))_n$ converges to $\alpha$. On the other hand, since $c_n\in(0,1)$, by Bolzano-Weierstrass Theorem sequence $(c_n)_n$ has a subsequence $(c_{i_n})_n$ which converges to $c\in[0,1]$. From $(F_n(1))_n=(f^n(c_n))_n$, $\alpha=\lim_{n\to\infty}f^{i_n}(c)=0$.

*Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Akash Singha Roy, Kolkata, India; Souza Leão, Federal University of Pernambuco, Brazil; Suhas Sheikh, Ryan International School, Sanpada, Navi Mumbai, India.*

U455. For two square matrices $X, Y \in M_n(\mathbb{C})$ we denote by $[X,Y] = XY - YX$ their commutator. Prove that if $A, B, C \in M_n(\mathbb{C})$ satisfy the identity $ABC + A + B + C = AB + BC + AC$ then

$$[A, BC] = [A, B] + [A, C].$$

*Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România*

*Solution by Souza Leão, Federal University of Pernambuco, Brazil*

The condition in the problem is equivalent to $(A - I)(B - I)(C - I) = -I$.
Hence we also have $(B - I)(C - I)(A - I) = -I$, from which we conclude:

$$\begin{aligned}
[A, BC] - [A, B] - [A, C] &= ABC - BCA - AB + BA - AC + CA \\
&= BA + BC + CA - A - B - C - BCA \\
&= BC(I - A) + (B + C - I)A - B - C \\
&= (BC - B - C + I)(I - A) - I \\
&= (B - I)(C - I)(I - A) - I \\
&= 0
\end{aligned}$$

*Also solved by Daniel Lasaosa, Pamplona, Spain; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Dumitru Barac, Sibiu, Romania; Mihai Bogdan, Romania; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Akash Singha Roy, Kolkata, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Moubinool Omarjee, Lycée Henri IV, Paris, France.*

U456. Let $a_1 > \cdots > a_m \geq 1$ be natural numbers and $P_1(x), \ldots, P_m(x)$ be rational functions with rational coefficients such that

$$P_1(n)a_1^n + \cdots + P_m(n)a_m^n,$$

is an integer for all sufficiently large $n$. Prove that $P_1(x), \ldots, P_m(x)$ are polynomials.

*Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran*

*Solution by the author*

Let $H(x)$ be the least common multiplier of the denominators of $P_1(x), \ldots, P_m(x)$, such that $H_i(x) = H(x)P_i(x)$ be a polynomial with integer coefficients. Then,

$$\gcd(H_1(x), \ldots, H_m(x), H(x)) = 1.$$

Assume that $H(x)$ is not constant. Then, there are polynomials $T_1(x), \ldots, T_m(x), T(x)$ with integer coefficients and nonzero integer $A$ such that

$$T_1(x)H_1(x) + \ldots T_m(x)H_m(x) + T(x)H(x) = A.$$

Choose a prime number $p$ large enough such that $H(s) \equiv 0 \pmod{p}$. For some large enough integer $s$

$$P_1(s)a_1^s + \cdots + P_{m-1}(s)a_m^s$$

is integer. Then,

$$H_1(s+rp)a_1^{s+rp} + \cdots + H_{m-1}(s+rp)a_{m-1}^{s+rp} = H(s+rp)(P_1(s)a_1^s + \cdots + P_{m-1}(s)a_{m-1}^s).$$

After taking the above equality *mod p*, we find that:

$$H_1(s+rp)a_1^{s+rp} + \cdots + H_m(s+rp)a_m^{s+rp} \equiv 0 \pmod{p}.$$

Hence, for any positive integer $t = 0, 1, \ldots, m-1$ we find that

$$\sum_{j=1}^{m} H_j(s)a_j^s a_j^{tp} \equiv 0 \pmod{p}$$

Consider the above system of congruencies as a linear system in $\mathbb{Z}_p$ such that $H_j(s)a_j^s$ are unknown. Next, the determinant of the system is (Wandermonde's determinant)

$$\prod_{i<j}(a_i^p - a_j^p) \equiv \prod_{i<j}(a_i - a_j) \not\equiv 0 \text{mod} p.$$

Therefore, the system has a trivial solution, that is

$$H_j(s) \equiv 0 \pmod{p}, j = 1, \ldots, m.$$

Thus, $p$ must divide $A$ which contradicts with the choice of $p$. This shows that $H(x)$ is constant and $P_1(x), \ldots, P_m(x)$ are polynomials.

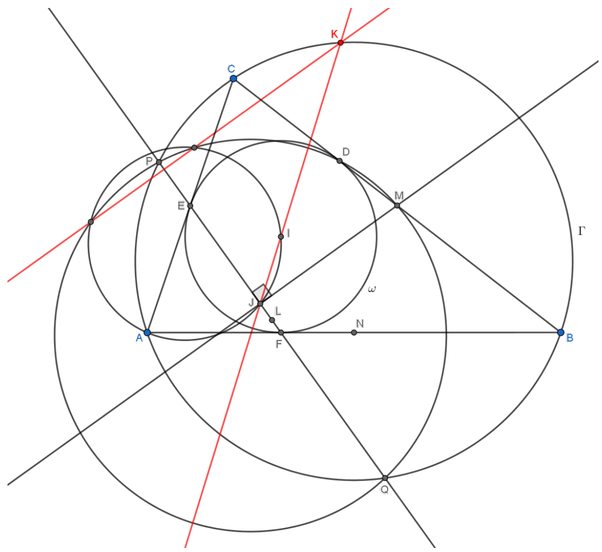
*Also solved by Akash Singha Roy, Kolkata, India.*

## Olympiad problems

O451. Let $ABC$ be a triangle, $\Gamma$ its circumcircle, $\omega$ its incircle and $I$ the incenter. Let $M$ be the midpoint of $BC$. The incircle $\omega$ is tangent to $AB$ and $AC$ at $F$ and $E$, respectively. Suppose $EF$ meets $\Gamma$ at distinct points $P$ and $Q$. Let $J$ denote the point on $EF$ such that $MJ$ is perpendicular on $EF$. Show that $IJ$ and the radical axis of $(MPQ)$ and $(AJI)$ intersect on $\Gamma$.

*Proposed by Toni Wen, USA*

*Solution by Andrea Fanchini, Cantù, Italy*




We use barycentric coordinates with reference to the triangle $ABC$.

We know that

$$E(s-c:0:s-a), \qquad F(s-b:s-a:0), \qquad M(0:1:1)$$

then the line $EF$ is

$$EF:(a-s)x+(s-b)y+(s-c)z=0$$

and the line that passes for $M$ and perpendicular to $EF$ is

$$MEF_{\infty\perp}:(b-c)x+(b+c)y-(b+c)z=0$$

therefore the point $J$ is

$$J=MEF_{\infty\perp}\cap EF=(a(b+c):b(2c-a):c(2b-a))$$

and the line $IJ$ has equation

$$IJ:bc(b-c)x+ac(s-b)y+ab(c-s)z=0$$

the radical axis of $(MPQ)$ and $(AJI)$ has equation

$$bc(b-c)x+c(s-b)(a-2c)y-b(s-c)(a-2b)z=0$$

now the intersection between the line $IJ$ and the radical axis give the point $K$

$$K\left(a(s-b)(s-c):b^2(c-s):c^2(b-s)\right)$$

and it is easy to check that this point is on the circumcircle $\Gamma: a^2yz+b^2zx+c^2xy=0$.

*Also solved by Haosen Chen, Zhejiang, China; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Jafet Baca, Universidad Centroamericana, Nicaragua; Navneel Singhal, Delhi, India.*

O452. Let $a, b, c$ be nonnegative real numbers, at most one being zero. Prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}+\frac{3}{a+b+c}\geq\frac{4}{\sqrt{ab+bc+ca}}.$$

*Proposed by An Zhenping, Xianyang Normal University, China*

*Solution by Daniel Lasaosa, Pamplona, Spain*

Denoting $s = a + b + c$, we have

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}+\frac{3}{a+b+c}=\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}+\frac{3}{s}=$$

$$=\frac{s^3+4(ab+bc+ca)s-3abc}{(ab+bc+ca)s^2-abcs}\geq\frac{4}{\sqrt{ab+bc+ca}}\cdot\frac{s^2\sqrt{ab+bc+ca}-\frac{3abc}{4}}{s^2\sqrt{ab+bc+ca}-\frac{abcs}{\sqrt{ab+bc+ca}}},$$

where we have applied the AM-GM to $s^3$ and $4(ab+bc+ca)s$, and it suffices to show that

$$abc\left(4s-3\sqrt{ab+bc+ca}\right)\geq 0.$$

Now, $s^2 \geq 3(ab+bc+ca)$ by the scalar product inequality, or $4s \geq 4\sqrt{3}\sqrt{ab+bc+ca} > 3\sqrt{ab+bc+ca}$, or this last inequality clearly holds, and equality holds iff $abc = 0$, ie iff exactly one of $a, b, c$ is zero. The conclusion follows, equality holds iff one of $a, b, c$ is zero, and simultaneously $s^2 = 4(ab+bc+ca)$. By symmetry we may assume wlog that $c = 0$, yielding $(a+b)^2 = 4ab$, or $a = b$. Thus, equality holds in the proposed inequality iff $(a, b, c)$ is a permutation of $(k, k, 0)$ for any positive real $k$.

*Also solved by Arkady Alt, San Jose, CA, USA; Mihai Bogdan, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania and Corneliu Mănescu-Avram, Ploiești, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Titu Zvonaru, Comănești, Romania; Shubhajit Roy, Fiitjee Chembur, Mumbai, India.*

O453. Let $a, b, c$ be positive real numbers such that $abc = 1$. Prove that

$$\frac{ab}{a^5 + b^5 + c^2} + \frac{bc}{b^5 + c^5 + a^2} + \frac{ca}{c^5 + a^5 + b^2} \le 1.$$

*Proposed by Florin Rotaru, Focşani, România*

*Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain*
Since $abc = 1$, the proposed inequality may be written as

$$\frac{a^2b^2c}{a^5 + b^5 + abc^3} + \frac{ab^2c^2}{b^5 + c^5 + a^3bc} + \frac{c^2a^2b}{c^5 + a^5 + ab^3c} \le 1.$$

By the AM-GM inequality $a^5 + b^5 + abc^3 \ge 3\sqrt[3]{a^6b^6c^3} = 3a^2b^2c$, and therefore $\frac{a^2b^2c}{a^5 + b^5 + abc^3} \le \frac{1}{3}$.
Applying the same argument to each summand on the left-hand side of the inequality, the result follows.

*Also solved by Daniel Lasaosa, Pamplona, Spain; Nikos Kalapodis, Patras, Greece; Dumitru Barac, Sibiu, Romania; Mihai Bogdan, Romania; Shubhajit Roy, Fiitjee Chembur, Mumbai, India; Adarsh Kumar, Ryan International School, Mumbai, India; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Anish Ray, Institute of Mathematics and Applications, Bhubaneswar, India; Arkady Alt, San Jose, CA, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; Frank Gamboa, Faculty of Mathematics and Computer Sciences, Havana, Cuba; Ioannis D. Sfikas, Athens, Greece; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania and Corneliu Mănescu-Avram, Ploiești, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Titu Zvonaru, Comănești, Romania.*

O454. Let $a, b, c$ be positive real numbers. Prove that

$$\frac{1}{18}\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)+\frac{a}{2a+b+c}+\frac{b}{a+2b+c}+\frac{c}{a+b+2c}\geq\frac{11}{12}$$

*Proposed by Titu Zvonaru, Comanesti, România*

*Solution by the author*

Using the known inequalities $3(x^2+y^2+z^2)=(x+y+z)^2$ and $x^2+y^2+z^2\geq xy+yz+zx$, we obtain

$$3\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)\geq\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2=\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq$$

$$\geq 3\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right),$$

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq\frac{a}{b}\cdot\frac{b}{c}+\frac{b}{c}\cdot\frac{c}{a}+\frac{c}{a}\cdot\frac{a}{b}=\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\Rightarrow$$

$$\Rightarrow\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq\frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}-6+6\right)=3+\sum_{cyc}\frac{(a-b)^2}{2ab}$$

Since

$$\frac{3}{4}-\frac{a}{2a+b+c}-\frac{b}{a+2b+c}-\frac{c}{a+b+2c}=\sum_{cyc}\left(\frac{1}{4}-\frac{a}{2a+b+c}\right)=\sum_{cyc}\frac{b-a+c-a}{4(2a+b+c)}$$

$$=\sum_{cyc}\frac{b-a}{4(2a+b+c)}+\sum_{cyc}\frac{a-b}{4(a+2b+c)}=\sum_{cyc}\frac{(a-b)^2}{4(2a+b+c)(a+2b+c)},$$

it suffices to prove that

$$\frac{1}{18}\sum_{cyc}\frac{(a-b)^2}{2ab}\geq\sum_{cyc}\frac{(a-b)^2}{4(2a+b+c)(a+2b+c)}\Leftrightarrow\sum_{cyc}\frac{(a-b)^2}{9ab}\geq\sum_{cyc}\frac{(a-b)^2}{(2a+b+c)(a+2b+c)}$$

The last inequality is true because

$$(2a+b+c)(a+2b+c)\geq(2a+b)(a+2b)\geq 9ab.$$

*Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania and Corneliu Mănescu-Avram, Ploiești, Romania.*

O455. Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 + a_2 + \ldots + a_n = n$, $n \ge 4$. Prove that

$$\sum_{1\le i<j\le n} 2a_ia_j \ge (n-1)\sqrt{na_1a_2\cdots a_n(a_1^2+a_2^2+\ldots+a_n^2)}$$

*Proposed by Marius Stanean, Zalau, România*

*Solution by the author*

*Lemma:* If $a \le b \le c$ be positive real numbers such that $a+b+c=p$, $ab+bc+ca=q$, where $p$ and $q$ are fixed real numbers satisfying $p^2 \ge 3q$, then $abc$ is maximal when $a=b$.

*Generalization of Lemma:* Let $x_1 \le x_2 \le \ldots \le x_n$ be positive real numbers such that $x_1+x_2+\ldots+x_n = p$, $x_1^2+x_2^2+\ldots+x_n^2 = q$, where $p$ and $q$ are fixed real numbers satisfying $p^2 \le nq$. Then, the product $x_1x_2\ldots x_n$ is maximal when $x_1 = x_2 = \ldots = x_{n-1}$.

Now, let us prove our inequality. We have $a_1+a_2+\ldots+a_n = n$ and there is a real number $t \in [0,1)$ such that $\sum_{1\le i<j\le n} a_ia_j = \binom{n}{2}(1-t^2)$ (because $\sum_{1\le i<j\le n}(a_i-a_j)^2 \ge 0 \Longleftrightarrow \sum_{1\le i<j\le n} a_ia_j \le \binom{n}{2}$).

According to *Generalization of Lemma*, without losing the generality assuming that $a_1 \le a_2 \le \ldots \le a_n$, the product $a_1a_2\ldots a_n$ is maximal when $a_1 = a_2 = \ldots = a_{n-1} = 1-t$, $a_n = 1+(n-1)t$, so $a_1a_2\ldots a_n \le (1-t)^{n-1}[1+(n-1)t]$.

Hence, we will prove that

$$\begin{aligned}(1-t^2) &\ge \sqrt{[1+(n-1)t^2](1-t)^{n-1}[1+(n-1)t]}\\ \Longleftrightarrow (1-t^2)^2 &\ge (1-t)^{n-1}[1+(n-1)t^2][1+(n-1)t]. \qquad (1)\end{aligned}$$

We will focus on the right hand side of the inequality and I will show that it is decreasing in relation to $n$, i.e. for $m > 4$:

$$\begin{aligned}&(1-t)^{m-1}[1+(m-1)t^2][1+(m-1)t] \ge (1-t)^m[1+mt^2][1+mt]\\ \Longleftrightarrow &(1-t)^{m-1}\left[1+(m-1)t+(m-1)t^2+(m-1)^2t^3-1-mt-m^2t^3+t+mt^3+m^2t^4\right] \ge 0\\ \Longleftrightarrow &(1-t)^{m-1}t^2[m^2t^2-(m-1)t+m-1] \ge 0,\end{aligned}$$

which is obviously true. Equality holds when $t=0$.

Therefore it is sufficient to prove the inequality (1) for $n=4$, i.e.

$$(1-t^2)^2 \ge (1-t)^3[(1+3t^2)(1+3t) \Longleftrightarrow (1-t)^2t^2(1-3t)^2 \ge 0,$$

which is obviously true. Equality holds when $t = 0 \Longrightarrow a_1 = a_2 = a_3 = a_4 = 1$ or $t = \frac{1}{3} \Longrightarrow a_1 = a_2 = a_3 = \frac{2}{3}$, $a_4 = 2$. For $n > 4$ the equality holds when $a_1 = a_2 = \ldots = a_n = 1$.

*Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Shubhajit Roy, Fiitjee Chembur, Mumbai, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.*

O456. Find all positive integers $n$ for which the equation

$$x^2 + [x]^2 + \{x\}^2 = n$$

has solutions $x \geq 0$. (Here, $[x]$ and $\{x\}$ denote the integer part and the fractional part of the real number $x$, respectively.)

*Proposed by Dorin Andrica and Dan-Stefan Marinescu, România*

*Solution by the author*

Let us note that if the equation is solvable, then it has unique solution. Indeed, if it has two solutions $x$ and $y$, then we obtain

$$x^2 - y^2 + [x]^2 - [y]^2 = \{y\}^2 - \{x\}^2.$$

Assume $x > y$. If $[x] \neq [y]$, then $[x] \geq [y] + 1$, therefore in the left side of the above relation we have a real number $> 1$, not possible because $\{y\}^2 - \{x\}^2 \in (-1, 1)$. It follows $[x] = [y]$. Let $x = k + \alpha$ and $y = k + \beta$, where $[x] = [y] = k$. From the relation above we obtain

$$(k + \alpha)^2 - (k + \beta)^2 = \beta^2 - \alpha^2,$$

hence $(\alpha - \beta)(k + \alpha + \beta) = 0$. We get $\alpha = \beta$, therefore $x = y$.

The equation is equivalent to

$$x^2 - x[x] + [x]^2 - \frac{n}{2} = 0. \qquad (1)$$

If $n = 0$, the equation has the unique solution $x = 0$.

If $n = 1$, the equation has the unique solution $x = \frac{1}{\sqrt{2}}$.

Suppose $n \geq 2$. Consider the set of positive integers

$$J_s = \{2s^2, 2s^2 + 1, \ldots, 2s^2 + 2s + 1\}, s = 0, 1, \ldots$$

We have $J_0 = \{0, 1\}, J_1 = \{2, 3, 4, 5\}, J_2 = \{8, 9, 10, 11, 12, 13\}, \ldots$.

We will prove that the equation (1) has a solution in real numbers if and only if there is a positive integer $s$ such that $n \in J_s$. If $x_n$ is the solution to our equation, then consider $s = [x_n]$. The equation (1) becomes

$$x^2 - sx + s^2 - \frac{n}{2} = 0,$$

with the root $x_n$ in the interval $[s, s + 1)$. It follows

$$s \leq \frac{s + \sqrt{2n - 3s^2}}{2} < s + 1,$$

that is equivalent to $n \in J_s$. Conversely, with the argument above we obtain that if we have $n \in J_s$, then the interval $[s, s + 1)$ contains a unique solution to the considered equation.

Finally, all positive integers $n$ satisfying the desired property are the elements of the set $\cup_{s \geq 0} J_s$.

*Remarks:*

1. We can show that if $n \in J_s$, then we have $s = [\frac{1}{2}\sqrt{2n}]$ and the solution $x_n$ is given by the formula

$$x_n = \frac{[\frac{1}{2}\sqrt{2n}] + \sqrt{2n - 3[\frac{1}{2}\sqrt{2n}]^2}}{2}.$$

2. Because $x_n = [\frac{1}{2}\sqrt{2n}] + \{x_n\}$, we easily obtain the asymptotic formula for the sequence $(x_n)$

$$\lim_{n \to \infty} \frac{x_n}{\sqrt{n}} = \frac{\sqrt{2}}{2}.$$

*Also solved by Anish Ray, Institute of Mathematics and Applications, Bhubaneswar, India; Jae Yong Park, The Lawrenceville School, Lawrenceville, NJ, USA; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Daniel Cortild; Luca Ferrigno, Università degli studi di Tor Vergata, Roma, Italy; Daniel Lasaosa, Pamplona, Spain; Navneel Singhal, Delhi, India; Shubhajit Roy, Fiitjee Chembur, Mumbai, India; Adarsh Kumar, Ryan International School, Mumbai, India; Anish Ray, Institute of Mathematics and Applications, Bhubaneswar, India; Joel Schlosberg, Bayside, NY, USA; Mohammed Kharbach, ADNOC AGP, UAE.*