Junior Problems

J457. Let ABC be a triangle and let D be a point on segment BC. Denote by E and F the orthogonal projections of D onto AB and AC, respectively. Prove that

$$\frac{\sin^2 \angle EDF}{DE^2 + DF^2} \le \frac{1}{AB^2} + \frac{1}{AC^2}.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J458. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{1}{\sqrt{a+3b}} + \frac{1}{\sqrt{b+3c}} + \frac{1}{\sqrt{c+3a}} \ge \frac{3}{2}.$$

Proposed by Mircea Becheanu, Montreal, Canada

J459. Let a and b be positive real numbers such that

$$a^4 + 3ab + b^4 = \frac{1}{ab}.$$

Evaluate

$$\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}} - \sqrt{2 + \frac{1}{ab}}.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J460. Prove that for all positive real numbers x, y, z

$$\left(x^3 + y^3 + z^3\right)^2 \ge 3\left(x^2y^4 + y^2z^4 + z^2x^4\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J461. Let a, b, c be real numbers such that $a + b + c = 3$. Prove that

$$(ab + bc + ca - 3)\left(4(ab + bc + ca) - 15\right) + 18(a-1)(b-1)(c-1) \ge 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J462. Let ABC a triangle. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \le \frac{3R}{4r}.$$

Proposed by Florin Rotaru, Focşani, România

Senior Problems

S457. Let a, b, c be real numbers such that $ab+bc+ca=3$. Prove that

$$a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2 \le \left((a+b+c)^2-6\right)\left((a+b+c)^2-9\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S458. Let AD, BE, CF be altitudes of triangle ABC, and let M be the midpoint of side BC. The line through C and parallel to AB intersects BE at X, and the line through B and is parallel to MX intersects EF at Y. Prove that Y lies on AD.

Proposed by Marius Stănean, Zalău, România

S459. Solve in real numbers the system of equations

$$\begin{aligned} \left|x^2-2\right| &= \sqrt{y+2} \\ \left|y^2-2\right| &= \sqrt{z+2} \\ \left|z^2-2\right| &= \sqrt{x+2}. \end{aligned}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S460. Let x, y, z be real numbers. Suppose that $0<x$, y, $z<1$ and $xyz=\frac{1}{4}$. Prove that

$$\frac{1}{2x^2+yz}+\frac{1}{2y^2+zx}+\frac{1}{2z^2+xy} \le \frac{x}{1-x^3}+\frac{y}{1-y^3}+\frac{z}{1-z^3}.$$

Proposed by Luke Robitaille, Euless, Texas, USA

S461. Find all triples (p, q, r) of prime numbers such that

$$\begin{aligned} p &\mid 7^q-1 \\ q &\mid 7^r-1 \\ r &\mid 7^p-1. \end{aligned}$$

Proposed by Alessandro Ventullo, Milan, Italy

S462. Let a, b, c be positive real numbers. Prove that

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{2\left(a^2+b^2+c^2\right)}{ab+bc+ca} \le \frac{a+b}{2c}+\frac{b+c}{2a}+\frac{c+a}{2b}$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

Undergraduate Problems

U457. Evaluate

$$\sum_{n\geq 2}\frac{(-1)^n\left(n^2+n-1\right)^3}{(n-2)!+(n+2)!}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U458. Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{2}{a^2+b^2+c^2}\geq\frac{11}{3}.$$

Proposed by An Zhenping, Xianyang Normal University, China

U459. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\left(1+\frac{1}{b}\right)^{ab}\left(1+\frac{1}{c}\right)^{bc}\left(1+\frac{1}{a}\right)^{ca}\leq 8.$$

Proposed by Mihaela Berindeanu, Bucharest, România

U460. Let L_k denote the k^{th} Lucas number. Prove that

$$\sum_{k=1}^{\infty}\tan^{-1}\frac{L_{k+1}}{L_kL_{k+2}+1}.\tan^{-1}\frac{1}{L_{k+1}}=\frac{\pi}{4}.\tan^{-1}\frac{1}{3}.$$

Proposed by Angel Plaza, University of Las Palmas de Gran Canaria, Spain

U461. Find all positive integers $n>2$ such that the polynomial

$$X^n+X^2Y+XY^2+Y^n$$

is irreducible in the ring $\mathbb{Q}[X,Y]$.

Proposed by Mircea Becheanu, Montreal, Canada

U462. Let $f:[0,\infty)\longrightarrow[0,\infty)$ be a differentiable function with continuous derivative and such that $f\left(f(x)\right)=x^2$, for all $x\geq 0$. Prove that

$$\int_0^1\left(f'(x)\right)^2\,dx\geq\frac{30}{31}.$$

Proposed by Mihai Piticari, Câmpulung Moldovenesc, România

Olympiad Problems

O457. Let a, b, c be real numbers such that $a+b+c \geq \sqrt{2}$ and

$$8abc = 3\left(a+b+c-\frac{1}{a+b+c}\right).$$

Prove that

$$2(ab+bc+ca)-\left(a^2+b^2+c^2\right) \leq 3.$$

Proposed by Titu Andreescu, University of Texas at Dallas

O458. Let $F_n = 2^{2^n}+1$ be a Fermat prime, $n \geq 2$. Find the sum of periodical digits of

$$\frac{1}{F_n}.$$

Proposed by Doğukan Namli, Turkey

O459. Let a, b, x be real numbers such that

$$(4a^2b^2+1)x^2+9(a^2+b^2) \leq 2018.$$

Prove that

$$20(4ab+1)x+9(a+b) \leq 2018.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O460. Let a, b, c, d be positive real numbers such that

$$a+b+c+d = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

Prove that

$$a^4+b^4+c^4+d^4+12abcd \geq 16.$$

Proposed by Marius Stănean, Zalău, România

O461. Let n be a positive integer and $C > 0$ a real number. Let x_1, x_2, $\ldots$, x_{2n} be real numbers such that $x_1+\ldots+x_{2n} = C$ and $|x_{k+1}-x_k| < \dfrac{C}{n}$ for all $k = 1, 2, \ldots, 2n$. Prove that among these numbers there are n numbers $x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}$ such that

$$\left|x_{\sigma(1)}+x_{\sigma(2)}+\cdots+x_{\sigma(n)}-\frac{C}{2}\right| < \frac{C}{2n}.$$

Proposed by Alessandro Ventullo, Milan, Italy

O462. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{2a^3+a^2+bc}+\frac{1}{2b^3+b^2+ca}+\frac{1}{2c^3+c^2+ab} \geq \frac{3}{4}abc.$$

Proposed by Bui Xuan Tien,Quang Nam, Vietnam