Junior Problems

**J457.** Let $ABC$ be a triangle and let $D$ be a point on segment $BC$. Denote by $E$ and $F$ the orthogonal projections of $D$ onto $AB$ and $AC$, respectively. Prove that

$$\frac{\sin^2 \angle EDF}{DE^2 + DF^2} \leq \frac{1}{AB^2} + \frac{1}{AC^2}.$$  

*Proposed by Adrian Andreescu, University of Texas at Austin, USA*

**J458.** Let $a$, $b$, $c$ be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{1}{\sqrt{a + 3b}} + \frac{1}{\sqrt{b + 3c}} + \frac{1}{\sqrt{c + 3a}} \geq \frac{3}{2}.$$  

*Proposed by Mircea Becheanu, Montreal, Canada*

**J459.** Let $a$ and $b$ be positive real numbers such that

$$a^4 + 3ab + b^4 = \frac{1}{ab}.$$  

Evaluate

$$\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}} - \sqrt{2 + \frac{1}{ab}}.$$  

*Proposed by Adrian Andreescu, University of Texas at Austin, USA*

**J460.** Prove that for all positive real numbers $x$, $y$, $z$

$$(x^3 + y^3 + z^3)^2 \geq 3 \left(x^2y^4 + y^2z^4 + z^2x^4\right).$$  

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

**J461.** Let $a$, $b$, $c$ be real numbers such that $a + b + c = 3$. Prove that

$$(ab + bc + ca - 3)(4(ab + bc + ca) - 15) + 18(a - 1)(b - 1)(c - 1) \geq 0.$$  

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

**J462.** Let $ABC$ a triangle. Prove that

$$\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \leq \frac{3R}{4r}.$$  

*Proposed by Florin Rotaru, Focșani, România*
Senior Problems

S457. Let $a, b, c$ be real numbers such that $ab + bc + ca = 3$. Prove that
\[
a^2(b - c)^2 + b^2(c - a)^2 + c^2(a - b)^2 \leq \left((a + b + c)^2 - 6\right) \left((a + b + c)^2 - 9\right).
\]

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S458. Let $AD$, $BE$, $CF$ be altitudes of triangle $ABC$, and let $M$ be the midpoint of side $BC$. The line through $C$ and parallel to $AB$ intersects $BE$ at $X$, and the line through $B$ and is parallel to $MX$ intersects $EF$ at $Y$. Prove that $Y$ lies on $AD$.

Proposed by Marius Stănean, Zalău, România

S459. Solve in real numbers the system of equations
\[
\begin{align*}
|x^2 - 2| &= \sqrt{y + 2} \\
|y^2 - 2| &= \sqrt{z + 2} \\
|z^2 - 2| &= \sqrt{x + 2}.
\end{align*}
\]

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S460. Let $x, y, z$ be real numbers. Suppose that $0 < x, y, z < 1$ and $xyz = \frac{1}{4}$. Prove that
\[
\frac{1}{2x^2 + yz} + \frac{1}{2y^2 + zx} + \frac{1}{2z^2 + xy} \leq \frac{x}{1 - x^3} + \frac{y}{1 - y^3} + \frac{z}{1 - z^3}.
\]

Proposed by Luke Robitaille, Euless, Texas, USA

S461. Find all triples $(p, q, r)$ of prime numbers such that
\[
\begin{align*}
p &\mid 7^q - 1 \\
q &\mid 7^r - 1 \\
r &\mid 7^p - 1.
\end{align*}
\]

Proposed by Alessandro Ventullo, Milan, Italy

S462. Let $a, b, c$ be positive real numbers. Prove that
\[
\frac{ab + bc + ca}{a^2 + b^2 + c^2} + \frac{2(a^2 + b^2 + c^2)}{ab + bc + ca} \leq \frac{a + b}{2c} + \frac{b + c}{2a} + \frac{c + a}{2b}
\]

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam
Undergraduate Problems

U457. Evaluate
\[
\sum_{n \geq 2} \frac{(-1)^n (n^2 + n - 1)^3}{(n - 2)! + (n + 2)!}.
\]

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U458. Let \(a, b, c\) be positive real numbers such that \(abc = 1\). Prove that
\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{2}{a^2 + b^2 + c^2} \geq \frac{11}{3}.
\]

Proposed by An Zhenping, Xianyang Normal University, China

U459. Let \(a, b, c\) be positive real numbers such that \(a + b + c = 3\). Prove that
\[
\left(1 + \frac{1}{b}\right)^{ab} \left(1 + \frac{1}{c}\right)^{bc} \left(1 + \frac{1}{a}\right)^{ca} \leq 8.
\]

Proposed by Mihaela Berindeanu, Bucharest, România

U460. Let \(L_k\) denote the \(k\)th Lucas number. Prove that
\[
\sum_{k=1}^{\infty} \tan^{-1} \frac{L_{k+1}}{L_k L_{k+2} + 1} \tan^{-1} \frac{1}{L_{k+1}} = \frac{\pi}{4} \tan^{-1} \frac{1}{3}.
\]

Proposed by Angel Plaza, University of Las Palmas de Gran Canaria, Spain

U461. Find all positive integers \(n > 2\) such that the polynomial
\[
X^n + X^2Y + XY^2 + Y^n
\]
is irreducible in the ring \(Q[X,Y]\).

Proposed by Mircea Becheanu, Montreal, Canada

U462. Let \(f : [0, \infty) \rightarrow [0, \infty)\) be a differentiable function with continuous derivative and such that \(f(f(x)) = x^2\), for all \(x \geq 0\). Prove that
\[
\int_0^1 (f'(x))^2 \, dx \geq \frac{30}{31}.
\]

Proposed by Mihai Piticari, Câmpulung Moldovenesc, România
O457. Let $a, b, c$ be real numbers such that $a + b + c \geq \sqrt{2}$ and

$$8abc = 3 \left( a + b + c - \frac{1}{a + b + c} \right).$$

Prove that

$$2(ab + bc + ca) - (a^2 + b^2 + c^2) \leq 3.$$