Junior problems

J457. Let ABC be a triangle and let D be a point on segment BC. Denote by E and F the orthogonal projections of D onto AB and AC, respectively. Prove that

$$\frac{\sin^2 \angle EDF}{DE^2 + DF^2} \le \frac{1}{AB^2} + \frac{1}{AC^2}.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Polyahedra, Polk State College, USA

Denote by $|XYZ|$ the area of $\triangle XYZ$. By the Cauchy-Schwarz inequality,

$$\begin{aligned}\left(DE^2 + DF^2\right)\left(AB^2 + AC^2\right) &\ge (DE \cdot AB + DF \cdot AC)^2 = 4(|ABD| + |ADC|)^2 \\ &= 4|ABC|^2 = (AB \cdot AC \sin A)^2 = AB^2 AC^2 \sin^2 \angle EDF.\end{aligned}$$

Hence,

$$\frac{1}{AB^2} + \frac{1}{AC^2} \ge \frac{\sin^2 \angle EDF}{DE^2 + DF^2}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Arkady Alt, San Jose, CA, USA; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Albert Stadler, Herrliberg, Switzerland; George Theodoropoulos, 2nd High School of Agrinio, Greece; Dumitru Barac, Sibiu, Romania; Adrienne Ko, Fieldston School, New York, NY, USA; Celine Lee, Chinese International School, Hong Kong; Joehyun Kim, Fort Lee High School, NJ, USA; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Titu Zvonaru, Comănești, Romania; Ioannis D. Sfikas, Athens, Greece.

J458. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{1}{\sqrt{a+3b}} + \frac{1}{\sqrt{b+3c}} + \frac{1}{\sqrt{c+3a}} \geq \frac{3}{2}.$$

Proposed by Mircea Becheanu, Montreal, Canada

Solution by Daniel Lasaosa, Pamplona, Spain

Note that the problem is equivalent to showing that the harmonic mean of $\sqrt{a+3b}, \sqrt{b+3c}, \sqrt{c+3a}$ is at most 2. But their quadratic mean is

$$\sqrt{\frac{4(a+b+c)}{3}} = 2\sqrt{\frac{a+b+c}{3}},$$

or it suffices to show that the arithmetic mean of a, b, c is at most 1. But their quadratic mean is 1. The conclusion follows, equality holds iff $a = b = c$, which is necessary and sufficient for equality to hold in both mean inequalities.

Also solved by Polyahedra, Polk State College, USA; George Theodoropoulos, 2nd High School of Agrinio, Greece; Dumitru Barac, Sibiu, Romania; Adrienne Ko, Fieldston School, New York, NY, USA; Celine Lee, Chinese International School, Hong Kong; Joehyun Kim, Fort Lee High School, NJ, USA; Arkady Alt, San Jose, CA, USA; Daniel Vacaru, Pitești, Romania; Henry Ricardo, Westchester Area Math Circle; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nguyen Minh Khang, VNU-HCM High School for the Gifted, Vietnam; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sarah B. Seales, Prescott, AZ, USA; Albert Stadler, Herrliberg, Switzerland; Bryant Hwang, Korea International School, South Korea; Titu Zvonaru, Comănești, Romania; Ioannis D. Sfikas, Athens, Greece.

J459. Let a and b be positive real numbers such that

$$a^4 + 3ab + b^4 = \frac{1}{ab}.$$

Evaluate

$$\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}} - \sqrt{2 + \frac{1}{ab}}.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Polyahedra, Polk State College, USA

First, the condition on a, b implies that $ab < 1$. Also,

$$\left(\frac{a}{b} + \frac{b}{a}\right)^2 - \left(\frac{1}{ab} - 1\right)^2 \left(2 + \frac{1}{ab}\right) = \frac{1}{a^2b^2}\left(a^4 + 3ab + b^4 - \frac{1}{ab}\right) = 0,$$

thus

$$0 = \frac{a}{b} + \frac{b}{a} - \left(\frac{1}{ab} - 1\right)\sqrt{2 + \frac{1}{ab}} = A^3 - B^3 - 3(A - B) = (A - B)\left(A^2 + AB + B^2 - 3\right),$$

where $A = \sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}}$ and $B = \sqrt{2 + \frac{1}{ab}}$. Since $B^2 > 3$, $A - B = 0$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Dumitru Barac, Sibiu, Romania; Adrienne Ko, Fieldston School, New York, NY, USA; Celine Lee, Chinese International School, Hong Kong; Joehyun Kim, Fort Lee High School, NJ, USA; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Joel Schlosberg, Bayside, NY, USA; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

J460. Prove that for all positive real numbers x, y, z

$$\left(x^3+y^3+z^3\right)^2 \geq 3\left(x^2y^4+y^2z^4+z^2x^4\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, CA, USA

By AM-GM Inequality

$$\left(x^3+y^3+z^3\right)^2 = x^6+y^6+z^6+2x^3y^3+2y^3z^3+2z^3x^3 =$$

$$\sum_{cyc}\left(x^6+2z^3x^3\right) \geq \sum_{cyc} 3\sqrt[3]{x^6\cdot\left(z^3x^3\right)^2} = \sum_{cyc} 3\sqrt[3]{x^{12}z^6} =$$

$$3\sum_{cyc} x^4z^2 = \quad 3\left(x^2y^4+y^2z^4+z^2x^4\right)$$

and the conclusion follows.

Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, USA; Dumitru Barac, Sibiu, Romania; Daniel Vacaru, Pitești, Romania; Idamia Abdelhamid, Jaafar el Fassi High School, Casablanca, Morocco; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sarah B. Seales, Prescott, AZ, USA; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

J461. Let a, b, c be real numbers such that $a+b+c=3$. Prove that

$$(ab+bc+ca-3)(4(ab+bc+ca)-15)+18(a-1)(b-1)(c-1)\geq 0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, USA

Let $x=a-1$, $y=b-1$, and $z=c-1$. Then $x+y+z=0$ and $ab+bc+ca-3=xy+yz+zx=-xy-x^2-y^2$. Thus,

$$\begin{aligned} L &= (ab+bc+ca-3)\left(4(ab+bc+ca)-15\right)+18(a-1)(b-1)(c-1) \\ &= \left(xy+x^2+y^2\right)\Big(4(xy+x^2+y^2)+3\Big)+18xyz. \end{aligned}$$

Clearly, $L\geq 0$ if $xyz\geq 0$. Consider $xyz<0$. We may assume that $x,y>0$ and $z=-x-y<0$. Using $xy+x^2+y^2\geq \frac{3}{4}(x+y)^2$ and $xy\leq \frac{1}{4}(x+y)^2$ we have

$$L\geq \frac{9}{4}(x+y)^2\left((x+y)^2+1\right)-\frac{9}{2}(x+y)^3=\frac{9}{4}(x+y)^2(x+y-1)^2\geq 0,$$

completing the proof.

Also solved by Dumitru Barac, Sibiu, Romania; Bryant Hwang, Korea International School, South Korea; Arkady Alt, San Jose, CA, USA; Nguyen Minh Khang, VNU-HCM High School for the Gifted, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

J462. Let ABC a triangle. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\leq\frac{3R}{4r}.$$

Proposed by Florin Rotaru, Focşani, România

Solution by Daniel Lasaosa, Pamplona, Spain

Denote as usual that $s=\frac{a+b+c}{2}$, and that $rs=S=\frac{abc}{4R}$ is the area of ABC. It is also well known that $9R^2\geq a^2+b^2+c^2$, or $27R^2\geq(a+b+c)^2=4s^2$. It then suffices to prove that

$$76s^3abc+27a^2b^2c^2\leq 8s^4(ab+bc+ca)+36sabc(ab+bc+ca),$$

which rewrites as

$$\sum_{\text{cyc}}a\left(b^3+b^2c+bc^2+c^3+4a^3+abc\right)(b-c)^2+6\left(a^3b^3+b^3c^3+c^3a^3-3a^2b^2c^2\right)\geq 0,$$

where the sum is clearly non negative, being zero iff $a=b=c$, and where the second term in the LHS rewrites as $6\left(u^3+v^3+w^3-3uvw\right)$, where $u=ab$, $v=bc$ and $w=ca$, and this term is non negative, being zero iff $u=v=w$. Note that the necessary condition $a=b=c$ is also clearly sufficient in the proposed inequality. The conclusion follows, equality holds iff ABC is equilateral.

Also solved by Nguyen Viet Hung, Hanoi University of Science, Vietnam; Polyahedra, Polk State College, USA; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nguyen Minh Khang, VNU-HCM High School for the Gifted, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Scott H. Brown, Auburn University Montgomery, Montgomery, AL, USA; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Ioannis D. Sfikas, Athens, Greece.

S457. Let a, b, c be real numbers such that $ab + bc + ca = 3$. Prove that

$$a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2 \leq \left((a+b+c)^2 - 6\right)\left((a+b+c)^2 - 9\right).$$

Proposed by Titu Andreescu, University of Texas at Austin, USA

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy
$9 = \sum_{\text{cyc}} a^2b^2 + 2abc(a+b+c)$ so that the inequality becomes

$$18 - 6abc(a+b+c) - ((a+b+c)^2 - 6)((a+b+c)^2 - 9) \leq 0 \tag{1}$$

This is a linear function of abc and (1) holds true if and only if it holds true for the extreme values of abc. Once fixed the values of $ab + bc + ca$ and $a + b + c$, the extreme values of abc occurs when $a = b$ or cyclic. Thus we set $c = \dfrac{3 - ab}{a + b}$ and $b = a$ getting that (1) becomes

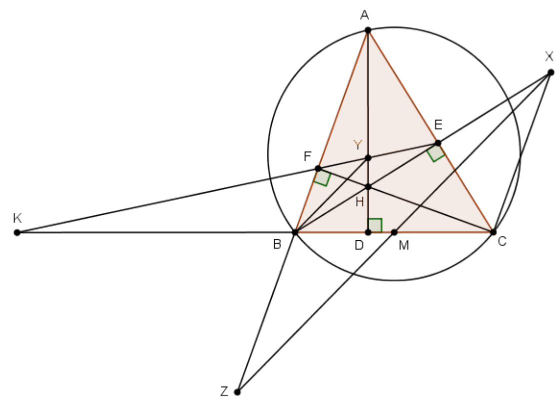
$$-\frac{9}{16a^4}(a-1)^2(a+1)^2(a^2-3)^2 \leq 0$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Dumitru Barac, Sibiu, Romania; Arkady Alt, San Jose, CA, USA; Titu Zvonaru, Comănești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland.

S458. Let AD, BE, CF be altitudes of triangle ABC, and let M be the midpoint of side BC. The line through C and parallel to AB intersects BE at X, and the line through B and is parallel to MX intersects EF at Y. Prove that Y lies on AD.

Proposed by Marius Stănean, Zalău, România

Solution by Andrea Fanchini, Cantù, Italy



We use barycentric coordinates with reference to the triangle ABC.
We know that

$$D(0:S_C:S_B),\qquad E(S_C:0:S_A),\qquad F(S_B:S_A:0),\qquad M(0:1:1)$$

then the line BE and the line through C and parallel to AB are

$$CAB_\infty : x+y=0,\qquad BE: S_Ax-S_Cz=0$$

therefore the point X is

$$X = CAB_\infty \cap BE = (S_C:-S_C:S_A)$$

then the line EF and the line through B and parallel to MX are

$$BMX_\infty : S_Ax-2S_Cz=0,\qquad EF: -S_Ax+S_By+S_Cz=0$$

therefore the point Y is

$$X = BMX_\infty \cap EF = (2S_BS_C : S_AS_C : S_AS_B)$$

that lies on the line $AD: S_By-S_Cz=0$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Titu Zvonaru, Comănești, Romania; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Albert Stadler, Herrliberg, Switzerland.

S459. Solve in real numbers the system of equations

$$\left|x^2-2\right| = \sqrt{y+2}$$
$$\left|y^2-2\right| = \sqrt{z+2}$$
$$\left|z^2-2\right| = \sqrt{x+2}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Arkady Alt, San Jose, CA, USA

First note that

$$\begin{cases} \left|x^2-2\right| = \sqrt{y+2} \\ \left|y^2-2\right| = \sqrt{z+2} \\ \left|z^2-2\right| = \sqrt{x+2} \end{cases} \iff \begin{cases} y = \left(x^2-2\right)^2-2 \\ z = \left(y^2-2\right)^2-2 \\ x = \left(z^2-2\right)^2-2 \end{cases}$$

Noting that $x, y, z \geq -2$ we consider two cases:

First Case: Let $x, y, z \in [-2, 2]$.Then denoting $t := \arccos \frac{x}{2}$ we obtain $x = 2\cos t, t \in [0, \pi]$,

$$y = \left(4\cos^2 t - 2\right)^2 - 2 = 4\cos^2 2t - 2 = 2\cos 4t, z = \left(4\cos^2 4t - 2\right)^2 - 2 = 2\cos 16t$$

and

$$x = \left(4\cos^2 16t - 2\right)^2 - 2 = 2\cos 64t.$$

Hence, for $t \in [0, \pi]$ we have $2\cos t = 2\cos 64t \iff$

$$\cos 64t - \cos t = 0 \iff \left[\begin{array}{l} \sin \frac{65t}{2} = 0 \\ \sin \frac{63t}{2} = 0 \end{array}\right. \iff \left[\begin{array}{l} t = \frac{\pi(2n+1)}{65}, 0 \leq n \leq 32 \\ t = \frac{\pi(2n+1)}{63}, 0 \leq n \leq 31 \end{array}\right. .$$

Thus,

$$(x, y, z) = \left(2\cos\frac{\pi(2n+1)}{65}, 2\cos\frac{4\pi(2n+1)}{65}, 2\cos\frac{16(2n+1)}{65}\right), n = 0, 1, .., 32$$

and

$$(x, y, z) = \left(2\cos\frac{\pi(2n+1)}{63}, 2\cos\frac{4\pi(2n+1)}{63}, 2\cos\frac{16(2n+1)}{63}\right), n = 0, 1, .., 31.$$

Second Case: Let $x, y, z \geq 2$.Then using representation $x = t + \frac{1}{t}, t > 0$ we obtain

$$y = \left(\left(t + \frac{1}{t}\right)^2 - 2\right)^2 - 2 = t^4 + \frac{1}{t^4}, z = \left(\left(t^4 + \frac{1}{t^4}\right)^2 - 2\right)^2 - 2 = t^{16} + \frac{1}{t^{16}},$$

$$x = \left(\left(t^{16} + \frac{1}{t^{16}}\right)^2 - 2\right)^2 - 2 = t^{64} + \frac{1}{t^{64}} = t + \frac{1}{t}.$$

Equation $t^{64} + \frac{1}{t^{64}} = t + \frac{1}{t}$ has only solution $t = 1$,because for any $t > 0$ and any natural $n > 1$ holds inequality $t^n + \frac{1}{t^n} \geq t + \frac{1}{t}$, where equality occurs iff $t = 1$.
Indeed, $t^n + \frac{1}{t^n} \geq t + \frac{1}{t} \iff t^{2n} - t^{n+1} - t^{n-1} + 1 \geq 0 \iff \left(t^{n-1} - 1\right)\left(t^{n+1} - 1\right) \geq 0$

Also solved by Titu Zvonaru, Comănești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Albert Stadler, Herrliberg, Switzerland; Ioannis D. Sfikas, Athens, Greece.

S460. Let x, y, z be real numbers. Suppose that $0 < x, y, z < 1$ and $xyz = \frac{1}{4}$. Prove that

$$\frac{1}{2x^2+yz}+\frac{1}{2y^2+zx}+\frac{1}{2z^2+xy} \le \frac{x}{1-x^3}+\frac{y}{1-y^3}+\frac{z}{1-z^3}$$

Proposed by Luke Robitaille, Euless, TX, USA

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy
The inequality is

$$\sum_{\text{cyc}} \frac{1}{2x^2+\frac{1}{4x}} \le \sum_{\text{cyc}} \frac{x}{1-x^3}$$

that is

$$\sum_{\text{cyc}} \frac{x}{1-x^3} - \frac{4x}{1+8x^3} \ge 0$$

Let $f(x) = \dfrac{x}{1-x^3} - \dfrac{4x}{1+8x^3}$.

$$f''(x) = 18x^2 \frac{(22-133x^3+456x^6+128x^9+256x^12)}{(1-x^3)^3(1+8x^3)^3}$$

Since $22+456x^6 \ge 2\sqrt{22\cdot 456}x^3 \sim 200x^3$, the second derivative is always positive and then $f(x)$ is convex. This implies $(S = x+y+z))$

$$\sum_{\text{cyc}} f(x) \ge 3\left(\frac{\frac{S}{3}}{1-\frac{S^3}{27}} - 4\frac{\frac{S}{3}}{1+8\frac{S^3}{27}}\right) = \frac{81S(-27+4S^3)}{(3-S)(S^2+3S+9)(3+2S)(9-6S+4S^2)}$$

Now $3 \ge x+y+z \ge 3(xyz)^{1/3} = 3/4^{1/3}$ and the inequality holds true.

Also solved by Albert Stadler, Herrliberg, Switzerland.

S461. Find all triples (p, q, r) of prime numbers such that

$$p | 7^q - 1$$

$$q | 7^r - 1$$

$$r | 7^p - 1.$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece
It is $p, q, r > 0$. If one of these primes equals to 7, let $p = 7$, then $7 | 7^q - 1$, which is not true, since $q > 0$. So none of these primes equals to 7.

Without loss of generality $p = max\{p, q, r\}$.

$r | 7^p - 1 \Leftrightarrow 7^p \equiv 1 \pmod r$. Let d be the order of 7. Then $d | p$. Fermat's little theorem (we can use it, since $r \neq 7$) indicates that $7^{r-1} \equiv 1 \pmod r$. As a result $d | r - 1 \Rightarrow d < r$, so $d < p$. Consequently $d \neq p$ and therefore $d = 1$. In other words $7 \equiv 1 \pmod r$, that is to say $r = 2$ or $r = 3$.

1) $r = 2$ $q | 7^2 - 1$, so $q = 2$ or $q = 3$.

i) $q = 2$ $p | 7^2 - 1$, so $p = 2$ or $p = 3$ So we have the triples $(p, q, r) = (2, 2, 2)$ or $(3, 2, 2)$

ii) $q = 3$ $p | 7^3 - 1$, so $p = 3$ or $p = 19$ ($p \neq 2$, since $p \geq q$) So we have the triples $(p, q, r) = (3, 3, 2)$ or $(19, 3, 2)$

2) $r = 3$ $q | 7^3 - 1 \Leftrightarrow q = 2$ or $q = 3$ or $q = 19$

i) $q = 2$ $p | 7^2 - 1$, so $p = 3$ ($p \neq 2$, since $p \geq r$) So we have the triple $(p, q, r) = (3, 2, 3)$

ii) $q = 3$ $p | 7^3 - 1$, so $p = 3$ or $p = 19$ ($p \neq 2$) So we have the triple $(p, q, r) = (3, 3, 3)$ or $(19, 3, 3)$

iii) $q = 19$ $p | 7^{19} - 1$ so $p = 419$ or $p = 4534166740403$ since ($p \geq q$) So we have the triple $(p, q, r) = (419, 19, 3)$ or $(4534166740403, 19, 3)$

To sum up the triples are $(2, 2, 2), (3, 3, 3), (3, 2, 2), (2, 3, 2), (2, 2, 3), (3, 3, 2), (3, 2, 3), (2, 3, 3)(19, 3, 2), (2, 19, 3),$
$(3, 2, 19), (19, 3, 3), (3, 19, 3), (3, 3, 19), (419, 19, 3), (3, 419, 19), (19, 3, 419),$
$(4534166740403, 19, 3), (3, 4534166740403, 19), (19, 3, 4534166740403)$

Also solved by Daniel Lasaosa, Pamplona, Spain; Haosen Chen, Zhejiang, China; Albert Stadler, Herrliberg, Switzerland.

S462. Let a, b, c be positive real numbers. Prove that

$$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{2(a^2+b^2+c^2)}{ab+bc+ca} \le \frac{a+b}{2c} + \frac{b+c}{2a} + \frac{c+a}{2b}$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

Solution by Albert Stadler, Herrliberg, Switzerland

We note that

$$\frac{a+b}{2c} + \frac{b+c}{2a} + \frac{c+a}{2b} - \frac{2(a^2+b^2+c^2)}{ab+bc+ca} = \frac{p(a,b,c)}{2abc(a^2+b^2+c^2)(ab+bc+ca)},$$

where

$$p(a,b,c) = \sum_{sym} a^5b^2 + \sum_{sym} a^4b^3 + 2\sum_{sym} a^4b^2c - \sum_{sym} a^5bc - 3\sum_{sym} a^3b^3c.$$

By Muirhead's inequality,

$$\sum_{sym} a^5b^2 \ge \sum_{sym} a^5bc,$$

$$\sum_{sym} a^4b^3 \ge \sum_{sym} a^3b^3c,$$

$$2\sum_{sym} a^4b^2c \ge 2\sum_{sym} a^3b^3c.$$

Therefore, $p(a,b,c) \ge 0$.

Also solved by Arkady Alt, San Jose, CA, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Ioannis D. Sfikas, Athens, Greece.

Undergraduate problems

U457. Evaluate

$$\sum_{n\geq 2} \frac{(-1)^n \left(n^2+n-1\right)^3}{(n-2)!+(n+2)!}$$

Proposed by Titu Andreescu, University of Texas at Austin, USA

Solution by Albert Stadler, Herrliberg, Switzerland

We have

$$\sum_{n\geq 2} \frac{(-1)^n \left(n^2+n-1\right)^3}{(n-2)!+(n+2)!} = \sum_{n\geq 2} \frac{(-1)^n \left(n^2+n-1\right)^3}{(n-2)!(1+(n-1)n(n+1)(n+2))} =$$

$$\sum_{n\geq 2} \frac{(-1)^n \left(n^2+n-1\right)^3}{(n-2)!(n^2+n-1)^2} = \sum_{n\geq 2} \frac{(-1)^n \left(n^2+n-1\right)}{(n-2)!} =$$

$$\sum_{n\geq 4} \frac{(-1)^n (n-2)(n-3)}{(n-2)!} + 6\sum_{n\geq 3} \frac{(-1)^n (n-2)}{(n-2)!} + 5\sum_{n\geq 2} \frac{(-1)^n}{(n-2)!} =$$

$$\frac{1}{e} - \frac{6}{e} + \frac{5}{e} = 0.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Arkady Alt, San Jose, CA, USA.

U458. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{2}{a^2 + b^2 + c^2} \geq \frac{11}{3}$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Thiago Landim de Souza Leão, Federal University of Pernambuco, Brazil

If one of the variables a, for example, tends to ∞, then at least one other, say, b tends to 0, and $\frac{1}{b}$ tends to ∞, therefore the minimum of the function

$$f(a,b,c) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{2}{a^2 + b^2 + c^2}$$

in the surface $g(a,b,c) = abc = 1$ does not occur in the "border" of the set, but in the interior. Hence we can use Lagrange Multipliers to find it. Let λ be such that $\nabla f = \lambda \nabla g$ in the minimum. Hence

$$-\frac{1}{a^2} - \frac{4a}{(a^2 + b^2 + c^2)^2} = \lambda bc$$
$$-\frac{1}{b^2} - \frac{4b}{(a^2 + b^2 + c^2)^2} = \lambda ca$$
$$-\frac{1}{c^2} - \frac{4c}{(a^2 + b^2 + c^2)^2} = \lambda ab$$

Dividing the first equation by bc, the second by ca and subtracting one from another we get

$$\frac{1}{abc}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{4(b-a)}{(a^2 + b^2 + c^2)^2}.$$

If $a \neq b$, then (supposing WLOG $c \geq 1$)

$$4 = c(a^2 + b^2 + c^2)^2 \geq (3\sqrt[3]{a^2b^2c^2})^2 = 9.$$

Hence $a = b$. Suppose now $a \neq c$. Then by the same argument as above we get the equation (and using $a^2c = 1$)

$$4 = a(a^2 + b^2 + c^2)^2 = a\left(2a^2 + \frac{1}{a^4}\right)$$

implying $4a^7 = 1 + 4a^6 + 4a^{12}$. But since $c > 1$, $a < 1$, therefore $4a^7 < 4a^6$, and we can't have the equality. Hence the minimum occurs when $a = b = c = 1$, and it is $\frac{11}{3}$.

Also solved by Marin Chirciu and Stroe Octavian, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sarah B. Seales, Prescott, AZ, USA; Albert Stadler, Herrliberg, Switzerland.

U459. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\left(1+\frac{1}{b}\right)^{ab}\left(1+\frac{1}{c}\right)^{bc}\left(1+\frac{1}{a}\right)^{ca} \le 8$$

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

The inequality is

$$3\sum_{\text{cyc}} \frac{a}{3} b \ln\left(1+\frac{1}{b}\right) \le 3\ln 2$$

By $\left[x\ln(1+\frac{1}{x})\right]'' = \frac{-1}{x(x+1)^2} < 0$ thus

$$3\sum_{\text{cyc}} \frac{a}{3} b \ln\left(1+\frac{1}{b}\right) \le 3\frac{a+b+c}{3}\ln\left(1+\frac{1}{\frac{a+b+c}{3}}\right) = 3\ln 2$$

and the conclusion follows.

Also solved by Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Dumitru Barac, Sibiu, Romania; Adrienne Ko, Fieldston School, New York, NY, USA; Celine Lee, Chinese International School, Hong Kong; Joehyun Kim, Fort Lee High School, NJ, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland; Thiago Landim de Souza Leão, Federal University of Pernambuco, Brazil; Akash Singha Roy, Chennai Mathematical Institute, India.

U460. Let L_k denote the k^{th} Lucas number. Prove that

$$\sum_{k=1}^{\infty} \tan^{-1}\left(\frac{L_{k+1}}{L_k L_{k+2}+1}\right) \tan^{-1}\left(\frac{1}{L_{k+1}}\right) = \frac{\pi}{4} \tan^{-1}\left(\frac{1}{3}\right).$$

Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Solution by G. C. Greubel, Newport News, VA, USA

By making use of

$$\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}(x) - \tan^{-1}(y)$$

and, for $x > 0$,

$$\tan^{-1}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{x}\right)$$

then it can be determined that

$$\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}\left(\frac{1}{y}\right) - \tan^{-1}\left(\frac{1}{x}\right).$$

Now, by using $L_{k+2} = L_{k+1} + L_k$ it is seen that

$$\tan^{-1}\left(\frac{L_{k+1}}{1+L_k L_{k+2}}\right) = \tan^{-1}\left(\frac{1}{L_{k+2}}\right) - \tan^{-1}\left(\frac{1}{L_k}\right).$$

The summation in question can now be seen in a telescopic form given as

$$\begin{aligned} S &= \sum_{k=1}^{\infty} \tan^{-1}\left(\frac{L_{k+1}}{L_k L_{k+2}+1}\right) \tan^{-1}\left(\frac{1}{L_{k+1}}\right) \\ &= \sum_{k=1}^{\infty} \left(\tan^{-1}\left(\frac{1}{L_k}\right) \tan^{-1}\left(\frac{1}{L_{k+1}}\right) - \tan^{-1}\left(\frac{1}{L_{k+1}}\right) \tan^{-1}\left(\frac{1}{L_{k+2}}\right)\right) \\ &= \tan^{-1}\left(\frac{1}{L_1}\right) \tan^{-1}\left(\frac{1}{L_2}\right) \\ &= \frac{\pi}{4} \tan^{-1}\left(\frac{1}{3}\right). \end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Chennai Mathematical Institute, India; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Albert Stadler, Herrliberg, Switzerland; Thiago Landim de Souza Leão, Federal University of Pernambuco, Brazil.

U461. Find all positive integers $n > 2$ such that the polynomial

$$X^n + X^2Y + XY^2 + Y^n$$

is irreducible in the ring $\mathbb{Q}[X, Y]$.

Proposed by Mircea Becheanu, Montreal, Canada

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that if n is odd, both $X^2Y + XY^2 = XY(X + Y)$ and $X^n + Y^n$ are divisible by $X + Y$, or $n \geq 4$ must be odd for the proposed property to hold.

Assume now that for $n \geq 4$, the polynomial $X^n + X^2Y + XY^2 + Y^n$ may be written as the product of two polynomials $P(X, Y)$ and $Q(X, Y)$ in $\mathbb{Q}[X, Y]$. Let u be the degree of P and v the degree of Q.

Then, $X^n + X^2Y + XY^2 + Y^n = P(X, Y)Q(X, Y)$ has degree $u + v = n$. Denoting by $p(X, Y)$ the sum of the terms of $P(X, Y)$ of degree u and by $q(X, Y)$ the sum of the terms of $Q(X, Y)$ of degree v, clearly $X^n + Y^n = p(X, Y)q(X, Y)$, since every other term in the product has degree other than n. It follows that $X^n + Y^n$ is not irreducible over $\mathbb{Q}[X, Y]$. But this is known to be false for even n, contradiction.

We conclude that $X^n + X^2Y + XY^2 + Y^n$ with $n > 2$ is irreducible in $\mathbb{Q}[X, Y]$ iff n is even.

U462. Let $f:[0,\infty)\to[0,\infty)$be a differentiable function with continuous derivative and such that $f(f(x))=x^2$, for all $x\geq 0$. Prove that

$$\int_0^1 (f'(x))^2dx \geq \frac{30}{31}.$$

Proposed by Mihai Piticari, Câmpulung Moldovenesc, România

Solution by Albert Stadler, Herrliberg, Switzerland

We will prove the stronger inequality $\int_0^1 (f'(x))^2dx \geq 1$. If we replace x by $f(x)$ in $f(f(x))=x^2$ we get $f(f(f(x)))=f(x)^2$. If we apply f to both sides of $f(f(x))=x^2$, we get $f(f(f(x)))=f(x^2)$. So $f(x)^2=f(x^2)$ for all x, which implies that $f(0)$ and $f(1)$ are both either equal to 0 or to 1. Suppose if possible that $f(0)=f(1)$. Then, $0=f(f(0))=f(f(1))=1$ which is a contradiction. Hence, $f(0)\neq f(1)$. We conclude that

$$1=|f(1)-f(0)|=\left|\int_0^1 f'(x)dx\right|\leq\sqrt{\int_0^1 (f'(x))^2dx}\sqrt{\int_0^1 dx}.$$

Also solved by Dumitru Barac, Sibiu, Romania; Akash Singha Roy, Chennai Mathematical Institute, India; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Sarah B. Seales, Prescott, AZ, USA.

O457. Let a, b, c be real numbers such that $a+b+c \geq \sqrt{2}$ and

$$8abc = 3\left(a+b+c-\frac{1}{a+b+c}\right)$$

. Prove that

$$2(ab+bc+ca)-(a^2+b^2+c^2) \leq 3$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

Let $s = a+b+c, q = ab+bc+ca$, and $p = abc$. From Schur's Inequality,

$$s^4 - 5s^2q + 4q^2 + 6sp \geq 0,$$

implying

$$4q^2 - 5s^2q + s^4 + \left(\frac{9}{4}\right)(s^2-1) \geq 0.$$

The left hand side is a quadratic in q and has discriminant

$$(5s^2)^2 - 16s^4 - 16\left(\frac{9}{4}\right)(s^2-1) = (3s^2-6)^2.$$

Because $s^2 \geq 2$, the roots are $q_1 \geq q_2$, where

$$q_1 = \frac{5s^2+(3s^2-6)}{8} = \frac{4s^2-3}{4}$$

and

$$q_2 = \frac{5s^2+(3s^2-6)}{8} = \frac{2s^2+3}{4}$$

So either $q \leq q_2$or $q \geq q_1$. Taking into account that $3q \geq s^2$, we cannot have $q \geq q_1$, as this will imply

$$\frac{12s^2-9}{4} \leq s^2$$

leading to $8s^2 \leq 9$, in contradiction with $s^2 \geq 2$. It follows that $q \leq q_2$, yielding $4q - s^2 \leq 3$. Hence the conclusion.

Also solved by Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

O458. Let $F_n = 2^{2^n} + 1$ be a Fermat prime, $n \geq 2$. Find the sum of periodical digits of

$$\frac{1}{F_n}.$$

Proposed by Doğukan Namli, Turkey

Solution by Daniel Lasaosa, Pamplona, Spain

Claim 1: 5 is a primitive root modulo F_n for any Fermat prime $F_n = 2^{2^n} + 1$ with $n \geq 2$.

Proof 1: The number of primitive roots of any prime p is $\varphi(\varphi(p))$, or the number of primitive roots of F_n is $\varphi(F_n - 1) = \frac{F_n - 1}{2}$, ie, modulo F_n there are exactly as many primitive roots as quadratic non-residues. But every primitive root must be a quadratic non-residue, since quadratic residues, when multiplied, generate only quadratic residues. Or the primitive roots of F_n are exactly its quadratic non-residues. Since both F_n and 5 are congruent to 1 modulo 4, by the Quadratic Reciprocity Law it suffices to show that F_n is a quadratic non-residue modulo 5. But $2^{2^n} = 4^{2^{n-1}} \equiv 1 \pmod 5$, or $F_n \equiv 2 \pmod 5$. The Claim 1 follows from the fact that the quadratic non-residues modulo 5 are $2, 3$.

Claim 2: 10 is a primite root modulo F_n for any Fermat prime $F_n = 2^{2^n} + 1$ with $n \geq 2$.

Proof 2: Assume that it is not so. Then, the sequence $10^0, 10^1, 10^2, \ldots$ is periodic with a period which is a proper divisor of $F_n - 1 = 2^{2^n}$. Or it is periodic with period $\frac{F_n - 1}{2}$. Now, $\frac{F_n - 1}{2} = 2^{2^{n-1}}$ is a multiple of 2^{n+1} for all $n \geq 2$ because $2^{n-1} \geq n + 1$ for all $n \geq 2$, and since $2^{2^n} \equiv -1 \pmod{F_n}$, then $2^{2^{n+1}} \equiv 1 \pmod{F_n}$, ie 2^k is periodic with period $2^{n+1} \leq 2^{2^{n-1}} = \frac{F_n - 1}{2}$. Therefore, $2^{\frac{F_n - 1}{2}} \equiv 2^{\frac{F_n - 1}{2}} \equiv 1 \pmod{F_n}$, and since $10^{\frac{F_n - 1}{2}} \equiv 10^{F_n - 1} \equiv 1 \pmod{F_n}$, we conclude that $5^{\frac{F_n - 1}{2}} \equiv 5^{F_n - 1} \pmod{F_n}$, in contradiction with the Claim 1. The Claim 2 follows.

Returning to the proposed problem, consider what happens when we perform the division $\frac{1}{F_n}$, where F_n is a Fermat prime, which results clearly in a number of the form $0.a_1 a_2 \ldots a_N a_1 a_2 \ldots$, where N is the length of the period and $a_1 a_2 \ldots a_N$ is the period, $a_1, a_2, \ldots, a_N$ being digits. Note that a_1 is the integer quotient of $\frac{10}{F_n}$, a_2 is the integer quotient of $\frac{100}{F_n}$ minus $10a_1$, a_3 is the integer quotient of $\frac{10^3}{F_n}$ minus $100a_1 + 10a_2$, and so on. In other words, a_k is the quotient of of $10r_k$ modulo p, where r_k is the remainder of 10^k when divided by p. Note then that

- The period of $\frac{1}{F_n}$ has length $N = F_n - 1$, since 10 is a primitive root modulo F_n, or r_k takes all possible $F_n - 1$ non-zero values, and the period only begins when the first remainder a_1 repeats itself, after all $F_n - 1$ remainders have appeared exactly once.

- The digits in the period are exactly and in some order, the integer quotients of the $10r$'s modulo F_n, where r takes the value of all possible nonzero remainders modulo F_n.

Now, partition the $F_n - 1 = 2^{2^n}$ nonzero remainders modulo F_n in pairs of the form $(r, F_n - r)$. Clearly, both elements in any such pair appear exactly once when calculating the digits in the period of $\frac{1}{F_n}$, and since $10r + 10(F_n - r) = 10F_n$, their corresponding integer quotients when divided by F_n add up to 9; clearly they add up to less than 10 because neither of them yields an exact division, and they add up to more than 8 since otherwise $10r, 10(F_n - r)$ would add up to less than $10F_n$, absurd. It follows that the period of $\frac{1}{F_n}$ has length $F_n - 1$, and can be partitioned in $\frac{F_n - 1}{2}$ pairs of digits, each pair having sum 9. The total sum of the digits in the period of $\frac{1}{F_n}$ is therefore

$$9 \cdot \frac{F_n - 1}{2} = 9 \cdot 2^{2^n - 1}.$$

Also solved by Albert Stadler, Herrliberg, Switzerland; Ioannis D. Sfikas, Athens, Greece.

O459. Let a, b, x be real numbers such that

$$\left(4a^2b^2+1\right)x^2+9\left(a^2+b^2\right)\le 2018.$$

Prove that

$$20(4ab+1)x+9(a+b)\le 2018.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that the first expression is unchanged under sign inversion of a, b, x, whereas for equal values of $|a|, |b|, |x|$, the second expression is maximum when a, b, x are all non-negative. It thus suffices to consider a, b, x as non-negative reals. Note then that, by the AM-GM,

$$4a^2b^2x^2+1600\ge 160abx, \qquad x^2+400\ge 40x,$$

$$9a^2+9\ge 18a, \qquad 9b^2+9\ge 18b,$$

where equality respectively holds iff $abx=20$, $x=20$, $a=1$ and $b=1$, ie equality holds in all inequalities iff $(a,b,x)=(1,1,20)$. Note now that

$$20(4ab+1)x+9(a+b)\le \frac{4a^2b^2x^2+x^2+9a^2+9b^2}{2}+1009\le 2018.$$

The conclusion follows, equality holds iff $(a,b,x)=(1,1,20)$, since this also produces equality in the condition.

Also solved by Dumitru Barac, Sibiu, Romania; Albert Stadler, Herrliberg, Switzerland.

O460. Let a, b, c, d be positive real numbers such that

$$a+b+c+d=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

Prove that

$$a^4+b^4+c^4+d^4+12abcd \geq 16.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

Denote $S=a+b+c+d$ and let $a=\frac{xS}{4}$, $b=\frac{yS}{4}$, $c=\frac{zS}{4}$, $d=\frac{tS}{4}$, then $x+y+z+t=4$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}=\frac{S^2}{4}$.

The inequality becomes

$$\frac{S^4}{4^4}(x^4+y^4+z^4+t^4+12xyzt) \geq 16,$$

or

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)^2(x^4+y^4+z^4+t^4+12xyzt) \geq 256, \tag{1}$$

But, from well known inequality (Tran Le Bach, Vasile Cîrtoaje)

$$x^4+y^4+z^4+t^4+12xyzt \geq (x+y+z+t)(xyz+yzt+ztx+txy),$$

(which follows, with easy computations, assuming that $x \geq y \geq z \geq t$ and replacing them with $x=t+u$, $y=t+v$, $z=t+w$, $u, v, w \geq 0$) it remains to prove that

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)^3 \geq \frac{4^3}{xyzt},$$

or

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}\right)^3 \geq 4^2\left(\frac{1}{xyz}+\frac{1}{yzt}+\frac{1}{ztx}+\frac{1}{txy}\right),$$

or

$$\frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{t}}{4} \geq \sqrt[3]{\frac{\frac{1}{xyz}+\frac{1}{yzt}+\frac{1}{ztx}+\frac{1}{txy}}{4}},$$

which follows from Maclaurin's Inequality.

Equality holds when $x=y=z=t=1 \Longrightarrow a=b=c=d=1$.

Also solved by Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Albert Stadler, Herrliberg, Switzerland.

O461. Let n be a positive integer and $C > 0$ a real number. Let $x_1, x_2, \dots, x_{2n}$ be real numbers such that $x_1 + \dots + x_{2n} = C$ and $|x_{k+1} - x_k| < \frac{C}{n}$ for all $k = 1, 2, \dots, 2n$. Prove that among these numbers there are n numbers $x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}$ such that

$$\left| x_{\sigma(1)} + x_{\sigma(2)} + \dots + x_{\sigma(n)} - \frac{C}{2} \right| < \frac{C}{2n}.$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Daniel Lasaosa, Pamplona, Spain

Denote $y_i = \max\{x_{2i-1}, x_{2i}\}$, and $z_i = \min\{x_{2i-1}, x_{2i}\}$, and denote

$$s_k = z_1 + z_2 + \dots + z_k + y_{k+1} + y_{k+2} + \dots + y_n,$$

where $s_0 = y_1 + y_2 + \dots + y_n$ and $s_n = z_1 + z_2 + \dots + z_n$.

Note that $s_0 + s_n = C$ with $s_0 \geq s_n$, or $s_0 \geq \frac{C}{2} \geq s_n$. If $s_0 - \frac{C}{2} = \frac{C}{2} - s_n < \frac{C}{2n}$, we are done.

Otherwise, note that for all $k = 1, 2, \dots, n$, we have $0 \leq s_{k-1} - s_k = |x_{2k-1} - x_{2k}| < \frac{C}{n}$, or the sequence $s_0, s_1, \dots, s_n$ decreases monotonically from a value not smaller than $\frac{C}{2} + \frac{C}{2n}$ to a value not larger than $\frac{C}{2} - \frac{C}{2n}$. Therefore, there exists $u \in \{1, 2, \dots, n\}$ such that $s_{u-1} \geq \frac{C}{2} \geq s_u$.

If the proposed result does not hold, then $s_{u-1} \geq \frac{C}{2} - \frac{C}{2n}$ and $s_u \leq \frac{C}{2} - \frac{C}{2n}$, or $\frac{C}{n} > |x_{2u-1} - x_{2u}| = s_{u-1} - s_u \geq \frac{C}{n}$, contradiction. The conclusion follows.

O462. Let a, b, c are positive real numbers such as $a+b+c=3$. Prove that

$$\frac{1}{2a^3+a^2+bc}+\frac{1}{2b^3+b^2+ac}+\frac{1}{2c^3+c^2+ab}\geq\frac{3abc}{4}.$$

Proposed by Bui Xuan Tien,Quang Nam, Vietnam

Solution by the author

By the Cauchy-Schwarz's inequality, we have

$$\sum\frac{1}{2a^3+a^2+bc}=\sum\frac{b^2c^2}{(2a^3+a^2+bc)b^2c^2}\geq\sum\frac{9abc}{b^3c^3+9a^2b^2c^2}$$

It suffices to prove that

$$\sum b^3c^3+9a^2b^2c^2=q^3-9qr+12r^2=f(q)\leq 12$$

with $p=a+b+c; q=ab+bc+ca; r=abc$ We have $f'(q)=3q^2-9r\geq 0$.

From Schur's inequality

$$(a+b+c)^3+9abc\geq 4(a+b+c)(ab+bc+ca)$$

we get $9-4q+3r\geq 0$

Therefore,

$$f(q)\leq\left(\frac{9+3r}{4}\right)^3-9\left(\frac{9+3r}{4}\right)r+12r^2\leq 12\Leftrightarrow(r-1)(9r^2+346r+445)\leq 0$$

The equality holds for $a=b=c=1$.

Also solved by Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland.