Junior Problems

J463. Let a, b, c be non-negative real numbers such that

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}=1.$$

Prove that

$$\frac{1}{6} \leq a+b+c \leq \frac{1}{4}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J464. Let p and q be real numbers such that one of the roots of the quadratic equation $x^2+px+q=0$ is the square of the other. Prove that $p \leq \frac{1}{4}$ and

$$p^3-3pq+q^2+q=0.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J465. Let x, y be real numbers such that $xy \geq 1$. Prove that

$$\frac{1}{1+x^2}+\frac{1}{1+xy}+\frac{1}{1+y^2} \geq \frac{3}{1+\left(\frac{x+y}{2}\right)^2}.$$

Proposed by Anish Ray, Institute of Mathematics, Bhubaneswar, India

J466. Let ABC be a triangle and P a point on segment AB. Prove that

$$\frac{PA}{BC^2}+\frac{PB}{AC^2} \geq \frac{AB}{PA \cdot PB+PC^2}.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J467. Find all pairs (x, y) of positive real numbers such that

$$\frac{\sqrt{x}}{3x+y}+\frac{\sqrt{y}}{x+3y}=\sqrt{x}+\sqrt{y}=1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J468. Let a, b, c positive numbers. Prove that

$$\sqrt{\frac{a}{b}}+\sqrt[3]{\frac{b}{c}}+\sqrt[5]{\frac{c}{a}}>2.$$

Proposed by Florin Rotaru, Focşani, România

Senior Problems

S463. Solve in real numbers the equation:

$$\sqrt[3]{x^3+3x^2-4}-x=\sqrt[3]{x^3-3x+2}-1.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

S464. Prove that in any regular 31-gon, $A_0A_1\ldots A_{30}$ the following inequality holds:

$$\frac{1}{A_0A_1}<\frac{1}{A_0A_2}+\frac{1}{A_0A_3}+\cdots+\frac{1}{A_0A_{15}}.$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

S465. Let $ABCD$ be a quadrilateral which has no parallel sides. The sides AB and CD meet in the point E, the sides BC and AD meet in the point F and the diagonals AC and BD meet in the point O. The line l which passes through O and is parallel to EF intersects the lines AB, BC, CD and AD in the points M, P, N and Q, respectively. Prove that $OM=ON$ and $OP=OQ$.

Proposed by Mihai Miculiţa, Oradea, România

S466. Let a,b,c be real numbers such that $a^2+b^2+c^2=6$. Find all possible values of the expression

$$\left(\frac{a+b+c}{3}-a\right)^5+\left(\frac{a+b+c}{3}-b\right)^5+\left(\frac{a+b+c}{3}-c\right)^5.$$

Proposed by Marius Stănean, Zalău, România

S467. Let a,b,c be real numbers, such that $a,b,c\geq\frac{1}{3}$ and $a+b+c=2$. Prove that

$$\left(a^3-2ab+b^3+\frac{8}{27}\right)\left(b^3-2bc+c^3+\frac{8}{27}\right)\left(c^3-2ca+a^3+\frac{8}{27}\right)\leq\left[\frac{10}{3}\left(\frac{4}{3}-ab-bc-ca\right)\right]^3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S468. Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a}{a^2+bc+1}+\frac{b}{b^2+ca+1}+\frac{c}{c^2+ab+1}\leq 1.$$

Proposed by An Zhenping, Xianyang Normal University, China

Undergraduate Problems

U463. Let x_1, x_2, x_3, x_4 be the roots of the polynomial $P(X) = 2X^4 - 5X + 1$. Find the sum

$$\frac{1}{(1-x_1)^3} + \frac{1}{(1-x_2)^3} + \frac{1}{(1-x_3)^3} + \frac{1}{(1-x_4)^3}.$$

Proposed by Mircea Becheanu, Montreal, Canada

U464. Evaluate

$$\sum_{k=1}^{n} \cot^{-1}\left(\frac{k^2+k}{2} + \frac{1}{k}\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U465. Let n be an odd positive integer. Prove that

$$\int_1^n (x-1)(x-2)\cdots(x-n)\,dx = 0.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U466. Let a, b, c positive real numbers. Prove that

$$\left(1+\frac{b}{a}\right)^{a^2/b}\left(1+\frac{c}{b}\right)^{b^2/c}\left(1+\frac{a}{c}\right)^{c^2/a} \geq 2^{a+b+c}$$

Proposed by Mihaela Berindeanu, Bucharest, România

U467. Let A and B be square matrices of dimension 2018×2018 with real entries such that

$$A^2 + B^2 = AB.$$

Prove that the matrix $AB - BA$ is singular.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

U468. Let $a < b$ be real numbers and $f : [a,b] \longrightarrow [a,b]$ be a function with the following properties:

(a) f has left and right limits at any point $x \in (a,b)$ and $f(x-0) \leq f(x+0)$;

(b) there exist limits $f(a+0)$ and $f(b-0)$.

Prove that there exists a point $x_0 \in [a,b]$ such that

$$\lim_{x \to x_0} f(x) = x_0.$$

Proposed by Mihai Piticari and Dan Stefan Marinescu, România

Olympiad Problems

O463. Let ABC ($AB \neq AC$) be an acute triangle with circumcircle $\Gamma(O)$ and let M be the midpoint of the side BC. The circle with diameter AM intersect Γ in a second point A'. Let D and E be the feet of the perpendiculars from A' to AB and AC, respectively. Prove that the line through M and parallel to AO bisect the segment DE.

Proposed by Marius Stănean, Zalău, România

O464. Let a, b, c be nonnegative real numbers such that $\dfrac{a}{b+c} \geq 2$. Prove that

$$5\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \geq \frac{a^2+b^2+c^2}{ab+bc+ca}+10.$$

Proposed by Marius Stănean, Zalău, România

O465. Let $C_0 = \{i_1, i_2, \ldots, i_n\}$ be an ordered set of n positive integers. A transformation of C_0 is the sequence of positive integers

$$C_1 = \{1, 2, \ldots, i_1 - 1, 1, 2, \ldots, i_2 - 1, \ldots, 1, 2, \ldots, i_n - 1\},$$

i.e, each $i_k > 1$ is replaced by the sequence $1, 2, \ldots, i_k - 1$. Similarly, the sequence C_i is obtained by a transformation from C_{i-1}. (For example, if $C_0 = \{1, 2, 6, 3\}$, then $C_1 = \{1, 1, 1, 2, 3, 4, 5, 1, 2\}$)
a) Assuming that $C_0 = \{1, 2, \ldots, n\}$, find the number of occurrences of i in C_j.
b) Let $C_F = \{1, 1, 1, 1, \ldots, 1\}$ be the final sequence obtained after performing maximum possible number of transformations to $C_0 = \{1, 2, \ldots, n\}$. Find the number of occurrences of 1 in C_F.

Proposed by Anish Ray, Institute of Mathematics, Bhubaneswar, India

O466. Let $n \geq 2$ be an integer. Prove that there exists a set S of $n-1$ real numbers such that whenever $a_1, \ldots, a_n$ are mutually different real numbers satisfying

$$a_1 + \frac{1}{a_2} = a_2 + \frac{1}{a_3} = \cdots = a_{n-1} + \frac{1}{a_n} = a_n + \frac{1}{a_1},$$

then the common value of all these sums is a number from S.

Proposed by Josef Tkadlec, Vienna, Austria

O467. Let ABC be a triangle with $\angle A > \angle B$. Prove that $\angle A = 3\angle B$ if and only if

$$\frac{AB}{BC - CA} = \sqrt{1 + \frac{BC}{CA}}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O468. Let A_n be the number of entries in the $n-$th row of Pascal's triangle that are 1 modulo 3. Let B_n be the number of entries in the $n-$th row which are 2 modulo 3. Prove that $A_n - B_n$ is a power of 2 for all positive integers n.

Proposed by Enrique Trevino, Lake Forest College, USA