The sequence $\left(\{n^d\alpha\}\right)$ is dense in $[0,1]$. An elementary proof:

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Kronecker's density theorem (in some sources: Weyl's density theorem) is so well-known that it might seem to many of us to be folkloric (which is not). It states that, if α is an irrational real number, the set of fractional parts of the numbers $n\alpha$, $n = 1, 2, \ldots$ is dense in $[0,1]$. The most common proof proceeds by showing first that, for a given $\epsilon > 0$, one can find a positive integer n such that $\{n\alpha\} < \epsilon$, then this fact leads easily to the desired conclusion that, for any $0 \le u < v \le 1$ there exists a positive integer n such that $u < \{n\alpha\} < v$. In what follows we intend to show that the same proof works for the following more general result:

Proposition. Let $d \ge 1$ be a positive integer, let α be an irrational number, and let P be the polynomial αX^d. Then the sequence $(\{P(n)\})_{n\ge1} = (\{n^d\alpha\})_{n\ge1}$ of the fractional parts of the values of P for the positive integers is dense in $[0,1]$.

The proof (which we never found in the literature; but, of course, this doesn't mean that it doesn't exist somewhere) will use Van der Waerden's theorem on arithmetic progressions (see [3]), namely its weak form stating that however we partition the set of positive integers into a finite number of sets, one of these sets will contain an arithmetic progression with as many terms as we want. We will also use the well-known identity

$$\sum_{j=0}^{d}(-1)^{d-j}\binom{d}{j}(x+jy)^d = d!y^d,$$

for any (real, complex) numbers x and y, and positive integer d.

Perhaps the simplest way to get this equation is to use Lagrange's interpolation formula

$$f(T) = \sum_{j=0}^{d} f(a_j) \prod_{0\le i\le d,\, i\ne j} \frac{T-a_i}{a_j-a_i}$$

for the polynomial (of degree d) $f(T) = (x+Ty)^d$ and for $a_i = i$ ($0 \le i \le d$); comparing the coefficients of T^d in both sides yields the desired identity.

Proof of the proposition. First we prove that, given a positive real number ϵ there exists a positive integer n such that either $\{n^d\alpha\} < \epsilon$ or $\{n^d\alpha\} > 1-\epsilon$.

Consider $M \in \mathbb{N}^*$ such that $\dfrac{1}{M} < \dfrac{\epsilon}{2^{d-1}(d!)^{d-1}}$. We partition the positive integers into M sets $S_0, S_1, \ldots, S_{M-1}$, where

$$S_i = \left\{n \in \mathbb{N}^* \mid \{n^d\alpha\} \in \left[\frac{i}{M}, \frac{i+1}{M}\right)\right\},\ \ 0 \le i \le M-1.$$

By Van der Waerden's theorem, one of these sets contains an arithmetic progression of length $d+1$ which we denote by $x, x+y, \ldots, x+dy$ (with positive integers x and y).

If follows that all the fractional parts $\{(x+jy)^d\alpha\}$, $j = 0, 1, \ldots, d$ are in one and the same interval $\left[\dfrac{i}{M}, \dfrac{i+1}{M}\right)$ for some $0 \le i \le M-1$. Thus, the absolute value of the difference of any two of these fractional parts is less than $\dfrac{1}{M}$. Therefore

$$\left|\sum_{j=0}^{d}(-1)^{d-j}\binom{d}{j}\{(x+jy)^d\alpha\}\right| =$$

$$= \left|\sum_{j=0}^{d-1}(-1)^{d-1-j}\binom{d-1}{j}\left(\{(x+(j+1)y)^d\alpha\}-\{(x+jy)^d\alpha\}\right)\right| \leq$$

$$\leq \sum_{j=0}^{d-1}\binom{d-1}{j}\left|\{(x+(j+1)y)^d\alpha\}-\{(x+jy)^d\alpha\}\right| <$$

$$< \frac{1}{M}\sum_{j=0}^{d-1}\binom{d-1}{j} = \frac{2^{d-1}}{M} < \frac{\epsilon}{(d!)^{d-1}}$$

Now we have

$$(d!)^{d-1}\left(\sum_{j=0}^{d}(-1)^{d-j}\binom{d}{j}\{(x+jy)^d\alpha\}\right) =$$

$$= (d!)^{d-1}\left(\sum_{j=0}^{d}(-1)^{d-j}\binom{d}{j}(x+jy)^d\alpha\right) - (d!)^{d-1}m =$$

$$= (d!)^d y^d\alpha - (d!)^{d-1}m$$

where m is the integer

$$m = \sum_{j=0}^{d}(-1)^{d-j}\binom{d}{j}\left\lfloor (x+jy)^d\alpha\right\rfloor,$$

consequently the above inequality for the sum with fractional parts becomes

$$\left|(d!)^d y^d\alpha - (d!)^{d-1}m\right| < \epsilon.$$

Thus we obtained a positive integer y such that $(d!y)^d\alpha$ is at distance at most ϵ from the integer $(d!)^{d-1}m$, thus such that either

$$\{(d!y)^d\alpha\} < \epsilon$$

or

$$\{(d!y)^d\alpha\} > 1-\epsilon,$$

as we wanted to get. (Of course, this is interesting only for $\epsilon < 1$, otherwise it says nothing; but we *want* ϵ to be small.)

Further we prove that, given a positive $\epsilon < 1$, we can find a positive integer n such that $\{n^d\alpha\} < \epsilon$. Indeed, if this doesn't happen, we can find (according to what we just proved), for any $\epsilon' \leq \epsilon$, some positive integer k (depending on ϵ') such that

$$\{k^d\alpha\} > 1-\epsilon'.$$

Since

$$[0,1) = \bigcup_{j=1}^{\infty}\left[1-\frac{1}{j^d}, 1-\frac{1}{(j+1)^d}\right)$$

this means that, for any big enough positive integer j, we can find k (depending on j) such that

$$1-\frac{1}{j^d} \leq \{k^d\alpha\} < 1-\frac{1}{(j+1)^d}.$$

We thus choose a big enough positive integer N and a positive integer k such that

$$1 - \frac{1}{N^d} \le \{k^d\alpha\} < 1 - \frac{1}{(N+1)^d},$$

and also such that

$$1 - \left(\frac{N}{N+1}\right)^d < \epsilon;$$

this is, of course, possible because

$$\lim_{N\to\infty}\left(1 - \left(\frac{N}{N+1}\right)^d\right) = 0.$$

Since

$$N^d - 1 \le N^d\{k^d\alpha\} < N^d$$

we have

$$\begin{aligned}\{N^d\{k^d\alpha\}\} &= N^d\{k^d\alpha\} - (N^d - 1)\\ &< N^d\left(1 - \frac{1}{(N+1)^d}\right) - N^d + 1\\ &= 1 - \left(\frac{N}{N+1}\right)^d\\ &< \epsilon.\end{aligned}$$

But, for integer s and real z we have

$$\{s\{z\}\} = \{sz - s\lfloor z\rfloor\} = \{sz\},$$

hence, in fact, we have $\{N^d\{k^d\alpha\}\} = \{N^d k^d\alpha\}$, meaning that we actually got

$$\{(Nk)^d\alpha\} < \epsilon.$$

Thus we see that the supposition that it does not exist n with $\{n^d\alpha\} < \epsilon$ leads to a contradiction with the very same supposition; so, anyway, for every $\epsilon > 0$, we can find a positive integer n such that $\{n^d\alpha\} < \epsilon$.

Now take any interval (u, v) with $0 \le u < v \le 1$. Choose a positive integer p such that

$$\{p^d\alpha\} < \left(\sqrt[d]{v} - \sqrt[d]{u}\right)^d \Leftrightarrow \sqrt[d]{\frac{v}{\{p^d\alpha\}}} - \sqrt[d]{\frac{u}{\{p^d\alpha\}}} > 1.$$

Note that this is the only part of the proof where we use the irrationality of α. Namely, we need to know that $\{p^d\alpha\}$ is nonzero, in order to put the inequality in the second form. This happens precisely because α is irrational, hence $p^d\alpha$ cannot be an integer for integer p.

The above inequalities show that there exists an integer q such that

$$\sqrt[d]{\frac{u}{\{p^d\alpha\}}} < q < \sqrt[d]{\frac{v}{\{p^d\alpha\}}} \Leftrightarrow u < q^d\{p^d\alpha\} < v,$$

that is, finally, we have $u < \{n^d\alpha\} < v$, for $n = pq$, and we are done.

Remarks. 1) One sees immediately that, for rational α, the fractional parts $\{n^d\alpha\}$ assume only finitely many values. Thus, in this case, the set of these fractional parts is not dense in $[0,1]$ (as it is well-known from the case $d=1$ of Kronecker's density theorem).

2) One can easily infer from the above proposition that, if P is a nonconstant polynomial with real coefficients of which precisely one (but not the constant term) is an irrational number, then the set of fractional parts of the numbers $P(n)$, $n=1,2,\ldots$ is dense in $[0,1]$. We invite the reader to deduce this fact from the proposition.

3) Actually the following much more general result holds:

Theorem. *(Weyl)* Let P be a nonconstant polynomial with real coefficients. If P has at least one irrational coefficient, other than the constant term, then the sequence $(P(n))_{n\geq 1}$ is equidistributed modulo 1.

In general, one says that the sequence $(x_n)_{n\geq 1}$ of real numbers is equidistributed modulo 1 if

$$\lim_{N\to\infty}\frac{|\{n\in\{1,\ldots,N\}\mid x_n\in[a,b]\}|}{N}=b-a$$

for any $0\leq a<b\leq 1$. It is easy to see that the set of the terms of an equidistributed sequence is dense in $[0,1]$. Thus, in particular, Weyl's theorem says that the set of fractional parts of the numbers $P(n)$, $n=1,2,\ldots$ is dense in $[0,1]$ (for a polynomial P with at least one irrational coefficient, other than the constant term). Clearly, even this statement is more powerful than that from the previous remark. (Could it be deduced from the proposition?) It is not, however, our purpose here to enter in more details about this topic. For the notion of equidistribution and Weyl's equidistribution theorem we refer the interested reader to the books [1,2].

References

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