Junior problems

J463. Let a, b, c be non-negative real numbers such that

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}=1.$$

Prove that

$$\frac{1}{6} \leq a+b+c \leq \frac{1}{4}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

First solution by Henry Ricardo, Westchester Area Math Circle, NY, USA
The Cauchy-Schwarz inequality gives us

$$1=\left(\sqrt{a+b} \cdot 1+\sqrt{b+c} \cdot 1+\sqrt{c+a} \cdot 1\right) \leq (2(a+b+c))^{1/2} \cdot \sqrt{3}.$$

This is equivalent to $a+b+c \geq 1/6$. Equality holds if and only if $a=b=c=1/18$.

To prove the right-hand inequality, we observe that

$$\begin{aligned} 1 &= 1^2=(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a})^2 \\ &= 2(a+b+c)+2\left(\sqrt{a+b}\sqrt{b+c}+\sqrt{b+c}\sqrt{c+a}+\sqrt{c+a}\sqrt{a+b}\right) \\ &\geq 2(a+b+c)+2\left(\sqrt{b}\sqrt{b}+\sqrt{c}\sqrt{c}+\sqrt{a}\sqrt{a}\right) \\ &= 4(a+b+c), \end{aligned}$$

which implies that $a+b+c \leq 1/4$. Equality holds if and only if $(a, b, c) \in \{(1/4,0,0),(0,1/4,0),(0,0,1/4)\}$.

Second solution by Polyahedra, Polk State College, USA
Applying Jensen's inequality to the concave function $\sqrt{x}$, we have

$$1 = \sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} \le 3\sqrt{\frac{(a+b)+(b+c)+(c+a)}{3}} = \sqrt{6(a+b+c)},$$

establishing the left inequality. On the other hand, $\sqrt{(a+b)(b+c)} \ge b$, etc, thus

$$\begin{aligned} 1 &= \left(\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a}\right)^2 \\ &= 2(a+b+c) + 2\left(\sqrt{(a+b)(b+c)} + \sqrt{(b+c)(c+a)} + \sqrt{(c+a)(a+b)}\right) \ge 4(a+b+c), \end{aligned}$$

establishing the right inequality.

Also solved by Ioannis D. Sfikas, Athens, Greece; Daniel Lasaosa, Pamplona, Spain; Arkady Alt, San Jose, CA, USA; Daniel Vacaru, Pitești, Romania; Lukas Seier, Charters Sixth Form, Sunningdale, England, UK; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Joehyun Kim, Fort Lee High School, NJ, USA; Andrew Lee, Choate Rosemary Hall, Wallingford, CT, USA; Bryant Hwang, Korea International School, South Korea; Nikos Kalapodis, Patras, Greece; Othmane Hadi, Casablanca, Morocco.

J464. Let p and q be real numbers such that one of the roots of the quadratic equation $x^2 + px + q = 0$ is the square of the other. Prove that $p \le \frac{1}{4}$ and

$$p^3 - 3pq + q^2 + q = 0.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Arkady Alt, San Jose, CA, USA

Let a and a^2 be the roots of the quadratic equation $x^2 + px + q = 0$.

Then

$$\begin{cases} a + a^2 = -p \\ a \cdot a^2 = q \end{cases} \iff \begin{cases} p = -\left(a + a^2\right) \\ q = a^3 \end{cases}$$

and, therefore,

$$p \le \frac{1}{4} \iff -\left(a + a^2\right) \iff \left(a + \frac{1}{2}\right)^2 \ge 0$$

and

$$\begin{aligned} p^3 - 3pq + q^2 + q &= -\left(a + a^2\right)^3 + 3\left(a + a^2\right)a^3 + a^6 + a^3 = \\ &= -a^6 - 3a^5 - 3a^4 - a^3 + 3a^5 + 3a^4 + a^6 + a^3 = 0. \end{aligned}$$

Also solved by Ioannis D. Sfikas, Athens, Greece; Daniel Lasaosa, Pamplona, Spain; Young Joon Kim, Horace Mann School, Bronx, NY, USA; Polyahedra, Polk State College, USA; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Joehyun Kim, Fort Lee High School, NJ, USA; Andrew Lee, Choate Rosemary Hall, Wallingford, CT, USA; Bryant Hwang, Korea International School, South Korea; Nikos Kalapodis, Patras, Greece; Yi Won Kim, Taft School, Watertown, CT, USA; Aenakshee Roy, Mumbai, India; Daniel Vacaru, Pitești, Romania; Jamal Gadirov, Istanbul, Turkey; Konstantinos Kritharidis, American College of Greece, Pierce, Athens, Greece; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Nick Iliopoulos, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Saloni Gole, Mumbai, India; Sarah B. Seales, Prescott, AZ, USA; Soumyadeep Paul, Haldia, India; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Bazar Tumurkhan, National University of Mongolia, Mongolia; Anderson Torres, Sao Paulo, Brazil; Taeyang David Park, Peddie School, Hightstown, NJ, USA; C.R.Aditya, Bangalore, India; Evangelos Panagiotakis, Evangeliki Model High School of Smyrna, Athens, Greece; George Theodoropoulos, 2nd High School of Agrinio, Greece; Sebastian Foulger, Charters Sixth Form, Sunningdale, England, UK; Stanislav Chobanov, USA and Ravi Kumar B, Hyderabad, India; Jiho Lee, Canterbury School, CT, USA.

J465. Let x, y be real numbers such that $xy \geq 1$. Prove that

$$\frac{1}{1+x^2}+\frac{1}{1+xy}+\frac{1}{1+y^2} \geq \frac{3}{1+\left(\frac{x+y}{2}\right)^2}.$$

Proposed by Anish Ray, Institute of Mathematics, Bhubaneswar, India

Solution by Polyahedra, Polk State College, USA

It suffices to consider $x, y > 0$. By the AM-GM inequality,

$$\frac{1}{1+xy} \geq \frac{1}{1+\left(\frac{x+y}{2}\right)^2}.$$

Moreover,

$$\frac{1}{1+x^2}+\frac{1}{1+y^2}-\frac{2}{1+\left(\frac{x+y}{2}\right)^2} = \frac{\left(2+x^2+y^2\right)\left(4+x^2+2xy+y^2\right)-8\left(1+x^2+y^2+x^2y^2\right)}{(1+x^2)(1+y^2)(4+x^2+2xy+y^2)}$$

$$= \frac{\left(x^2-y^2\right)^2+2(x-y)^2(xy-1)}{(1+x^2)(1+y^2)(4+x^2+2xy+y^2)} \geq 0,$$

from which the claimed inequality follows.

Also solved by Ioannis D. Sfikas, Athens, Greece; Daniel Lasaosa, Pamplona, Spain; Young Joon Kim, Horace Mann School, Bronx, NY, USA; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Joehyun Kim, Fort Lee High School, NJ, USA; Andrew Lee, Choate Rosemary Hall, Wallingford, CT, USA; Bryant Hwang, Korea International School, South Korea; Bazar Tumurkhan, National University of Mongolia, Mongolia; Arkady Alt, San Jose, CA, USA; Daniel Vacaru, Pitești, Romania; Maalav Mehta; Nick Iliopoulos, Greece; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Sarah B. Seales, Prescott, AZ, USA.

J466. Let ABC be a triangle and P a point on segment AB. Prove that

$$\frac{PA}{BC^2}+\frac{PB}{AC^2}\geq\frac{AB}{PA\cdot PB+PC^2}.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Polyahedra, Polk State College, USA

By the law of cosines,

$$PA\cdot BC^2=PA\left(PB^2+PC^2-2PB\cdot PC\cos\angle BPC\right)$$

and

$$PB\cdot AC^2=PB\left(PA^2+PC^2+2PA\cdot PC\cos\angle BPC\right).$$

Adding the two equations, we get $PA\cdot BC^2+PB\cdot AC^2=AB\left(PA\cdot PB+PC^2\right)$. Then by the Cauchy-Schwarz inequality,

$$\begin{aligned}AB\left(PA\cdot PB+PC^2\right)\left(PA\cdot AC^2+PB\cdot BC^2\right)&=\left(PA\cdot BC^2+PB\cdot AC^2\right)\left(PA\cdot AC^2+PB\cdot BC^2\right)\\&\geq(PA\cdot BC\cdot AC+PB\cdot AC\cdot BC)^2\\&=(AB\cdot AC\cdot BC)^2\end{aligned}$$

from which the claimed inequality follows.

Also solved by Daniel Lasaosa, Pamplona, Spain; Young Joon Kim, Horace Mann School, Bronx, NY, USA; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Joehyun Kim, Fort Lee High School, NJ, USA; Andrew Lee, Choate Rosemary Hall, Wallingford, CT, USA; Nikos Kalapodis, Patras, Greece; Anderson Torres, Sao Paulo, Brazil; Taeyang David Park, Peddie School, Hightstown, NJ, USA; C.R.Aditya, Bangalore, India; Arkady Alt, San Jose, CA, USA; Yi Won Kim, Taft School, Watertown, CT, USA; Daniel Vacaru, Pitești, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

J467. Find all pairs (x, y) of positive real numbers such that

$$\frac{\sqrt{x}}{3x+y} + \frac{\sqrt{y}}{x+3y} = \sqrt{x} + \sqrt{y} = 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

$$\frac{\sqrt{y}(3x+y) + \sqrt{x}(x+3y)}{(3x+y)(x+3y)} = 1,$$

implying

$$(\sqrt{x} + \sqrt{y})^3 = (3x+y)(x+3y).$$

It follows that $1 = (3x+y)(x+3y)$, hence

$$3(x+y)^2 - 6xy + 10xy = 1.$$

But $x + y + 2(\sqrt{x})(\sqrt{y}) = 1$, implying $3(x+y)^2 + [1 - (x+y)]^2 = 1$.
It follows that $[2(x+y) - 1]^2 = 0$, hence $x + y = \frac{1}{2}$ and then $xy = \frac{1}{16}$, implying $x = y = \frac{1}{4}$.

Also solved by Ioannis D. Sfikas, Athens, Greece; Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, USA; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Joehyun Kim, Fort Lee High School, NJ, USA; Nikos Kalapodis, Patras, Greece; Anderson Torres, Sao Paulo, Brazil; Taeyang David Park, Peddie School, Hightstown, NJ, USA; C.R.Aditya, Bangalore, India; Evangelos Panagiotakis, Evangeliki Model High School of Smyrna, Athens, Greece; George Theodoropoulos, 2nd High School of Agrinio, Greece; Sebastian Foulger, Charters Sixth Form, Sunningdale, England, UK; Spyros Kalloniatis, SHAPE American High School, Belgium; Arkady Alt, San Jose, CA, USA; Yi Won Kim, Taft School, Watertown, CT, USA; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Aenakshee Roy, Mumbai, India; Daniel Vacaru, Pitești, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nick Iliopoulos, Greece; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paul Kim, Seoul International School, South Korea; Pradyumna Atreya, India; Sarah B. Seales, Prescott, AZ, USA; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy.

J468. Let a, b, c positive numbers. Prove that

$$\sqrt{\frac{a}{b}} + \sqrt[3]{\frac{b}{c}} + \sqrt[5]{\frac{c}{a}} > 2.$$

Proposed by Florin Rotaru, Focşani, România

Solution by Nikos Kalapodis, Patras, Greece

From the AM-GM inequality we get

$$\begin{aligned}\sqrt{\frac{a}{b}} + \sqrt[3]{\frac{b}{c}} + \sqrt[5]{\frac{c}{a}} &= 2 \cdot \frac{1}{2}\sqrt{\frac{a}{b}} + 3 \cdot \frac{1}{3}\sqrt[3]{\frac{b}{c}} + 5 \cdot \frac{1}{5}\sqrt[5]{\frac{c}{a}} \\ &\geq 10 \sqrt[10]{\frac{1}{2^2} \cdot \frac{1}{3^3} \cdot \frac{1}{5^5} \cdot \frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}} = 10 \sqrt[10]{\frac{1}{2^2 3^3 5^5}}.\end{aligned}$$

So it suffices to prove that $\frac{1}{2^2 3^3 5^5} > \frac{1}{5^{10}}$ or $5^5 > 2^2 3^3$ or $3125 > 108$ which is clearly true.

Also solved by Daniel Lasaosa, Pamplona, Spain; Young Joon Kim, Horace Mann School, Bronx, NY, USA; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Polyahedra, Polk State College, USA; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Joehyun Kim, Fort Lee High School, NJ, USA; Andrew Lee, Choate Rosemary Hall, Wallingford, CT, USA; Bryant Hwang, Korea International School, South Korea; Bazar Tumurkhan, National University of Mongolia, Mongolia; Anderson Torres, Sao Paulo, Brazil; Taeyang David Park, Peddie School, Hightstown, NJ, USA; Daniel Vacaru, Pitești, Romania; Idamia Abdelhamid, Casablanca, Morocco; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sarah B. Seales, Prescott, AZ, USA; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Jiho Lee, Canterbury School, CT, USA.

Senior problems

S463. Solve in real numbers the equation:

$$\sqrt[3]{x^3+3x^2-4}-x=\sqrt[3]{x^3-3x+2}-1.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Daniel Lasaosa, Pamplona, Spain
First, note that

$$(x-1)=\sqrt[3]{(x-1)(x+2)^2}-\sqrt[3]{(x-1)^2(x+2)},$$

or denoting $u=\sqrt[3]{\frac{x-1}{x+2}}$, and since $x+2\neq 0$ because $x=-2$ is clearly not a root, we have $u^3=u-u^2$, with roots

$$u=0,\qquad u=\frac{-1+\sqrt{5}}{2},\qquad u=\frac{-1-\sqrt{5}}{2},$$

which result respectively in

$$x=1,\qquad x=\frac{1+3\sqrt{5}}{4},\qquad x=\frac{1-3\sqrt{5}}{4}.$$

Also solved by Ioannis D. Sfikas, Athens, Greece; Young Joon Kim, Horace Mann School, Bronx, NY, USA; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Joehyun Kim, Fort Lee High School, NJ, USA; Andrew Lee, Choate Rosemary Hall, Wallingford, CT, USA; Anderson Torres, Sao Paulo, Brazil; Taeyang David Park, Peddie School, Hightstown, NJ, USA; C.R.Aditya, Bangalore, India; Evangelos Panagiotakis, Evangeliki Model High School of Smyrna, Athens, Greece; George Theodoropoulos, 2nd High School of Agrinio, Greece; Sebastian Foulger, Charters Sixth Form, Sunningdale, England, UK; Stanislav Chobanov, USA and Ravi Kumar B, Hyderabad, India; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; G. C. Greubel, Newport News, VA, USA; Nick Iliopoulos, Greece; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Pradyumna Atreya, India; Sachin Kumar, India; Saloni Gole, India; Sarah B. Seales, Prescott, AZ, USA; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Jiho Lee, Canterbury School, CT, USA; Pedro Acosta De Leon, Massachusetts Institute of Technology, Cambridge, MA, USA.

S464. Prove that in any regular 31-gon, $A_0A_1\dots A_{30}$ the following inequality holds:

$$\frac{1}{A_0A_1} < \frac{1}{A_0A_2} + \frac{1}{A_0A_3} + \cdots + \frac{1}{A_0A_{15}}.$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

Solution by Daniel Lasaosa, Pamplona, Spain

We will prove a more general result, namely that for any regular $2n$-gon or $(2n+1)$-gon for $n \geq 4$, the following inequality holds:

$$\frac{1}{A_0A_1} < \frac{1}{A_0A_2} + \frac{1}{A_0A_3} + \cdots + \frac{1}{A_0A_n}.$$

Denote respectively $\alpha = \frac{\pi}{2n}$ and $\alpha = \frac{\pi}{2n+1}$, and note that for any angle $x < \frac{\pi}{4}$, we have

$$\frac{2}{\sin(2x)} - \frac{1}{\sin x} = \frac{2(1-\cos x)}{\sin(2x)} > 0,$$

or

$$\frac{1}{\sin\alpha} < \frac{2}{\sin(2\alpha)} < \frac{1}{\sin(2\alpha)} + \frac{2}{\sin(4\alpha)} < \frac{1}{\sin(2\alpha)} + \frac{1}{\sin(3\alpha)} + \frac{1}{\sin(4\alpha)} \leq$$
$$\leq \frac{1}{\sin(2\alpha)} + \frac{1}{\sin(3\alpha)} + \cdots + \frac{1}{\sin(n\alpha)}.$$

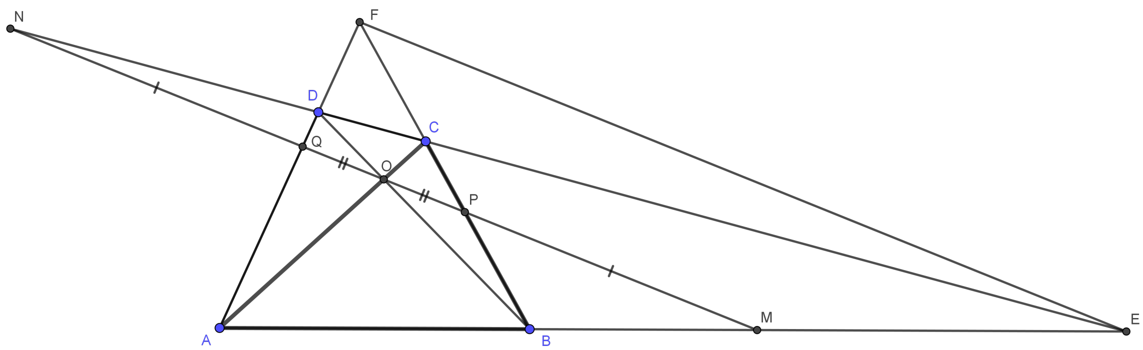
It suffices then to realize that for any regular $2n$-gon or $(2n+1)$-gon with circumradius R, we have $A_0A_k = 2R\sin(k\alpha)$ for $k = 1, 2, \dots, n$. The conclusion follows.

Also solved by Albert Stadler, Herrliberg, Switzerland; Young Joon Kim, Horace Mann School, Bronx, NY, USA; Ravi Kumar B, India; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

S465. Let $ABCD$ be a quadrilateral which has no parallel sides. The sides AB and CD meet in the point E, the sides BC and AD meet in the point F and the diagonals AC and BD meet in the point O. The line l which passes through O and is parallel to EF intersects the lines AB, BC, CD and AD in the points M, P, N and Q, respectively. Prove that $OM = ON$ and $OP = OQ$.

Proposed by Mihai Miculiţa, Oradea, România

Solution by Andrea Fanchini, Cantù, Italy



We use barycentric coordinates with reference to the triangle ABC.
We denote $D(u, v, w)$ where u, v, w are parameters with $u + v + w = 1$.
Then we have

$$AD : wy - vz = 0, \qquad BD : wx - uz = 0, \qquad CD : vx - uy = 0$$

$$E = CD \cap AB = (u : v : 0), \qquad F = AD \cap BC = (0 : v : w), \qquad O = BD \cap AC = (u : 0 : w)$$

The line $EF : vwx - uwy + uvz = 0$ has infinite point $EF_\infty\,(-u(v+w) : v(u-w) : w(u+v))$ so the line l which passes through O and is parallel to EF is

$$l : vw(w-u)x - uw(1+v)y + uv(u-w)z = 0$$

and therefore the points M, P, N and Q have absolute coordinates

$$M\left(\frac{u(1+v)}{u+vw}, \frac{v(w-u)}{u+vw}, 0\right), \qquad P\left(0, \frac{v(u-w)}{w+uv}, \frac{w(1+v)}{w+uv}\right)$$

$$N\left(\frac{u(u-w)}{u+vw}, \frac{v(u-w)}{u+vw}, \frac{2w(u+v)}{u+vw}\right), \qquad Q\left(\frac{2u(v+w)}{w+uv}, \frac{v(w-u)}{w+uv}, \frac{w(w-u)}{w+uv}\right)$$

Then the midpoint of MN and PQ is $O(u : 0 : w)$.

Also solved by Alibi Zhenis, Nurorda High School, Astana, Kazakhstan; Daniel Lasaosa, Pamplona, Spain; Young Joon Kim, Horace Mann School, Bronx, NY, USA; Titu Zvonaru, Comănești, Romania.

S466. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 6$. Find all possible values of the expression

$$\left(\frac{a+b+c}{3} - a\right)^5 + \left(\frac{a+b+c}{3} - b\right)^5 + \left(\frac{a+b+c}{3} - c\right)^5.$$

Proposed by Marius Stănean, Zalău, România

Solution by Takuji Imaiida, Fujisawa, Kanagawa, Japan

Define $x = (-2a+b+c)/3, y = (a-2b+c)/3, z = (a+b-2c)/3$ and $f(x,y,z) = x^5 + y^5 + z^5$. We find maximum value of f. Since $x+y+z = 0$, we may assume that $x \geq y \geq z$ and $y = -x/2 + t, z = -x/2 - t$ for $0 \leq t \leq x/2$. Then

$$\begin{aligned} &f(x, -x/2+t, -x/2-t) = \frac{15}{16}x^5 - \frac{5}{2}x^3t^2 - 5xt^4 \\ &\leq \frac{15}{16}x^5 = f(x, -x/2+0, -x/2-0). \end{aligned}$$

Since $a^2+b^2+c^2 = 6$, we obtain $x \leq 2\sqrt{6}/3$ and $f(x,y,z) \leq f(2\sqrt{6}/3, -\sqrt{6}/3, -\sqrt{6}/3) = 40\sqrt{6}/9$. f is continuous and therefore

$$-\frac{40\sqrt{6}}{9} \leq f(x,y,z) \leq \frac{40\sqrt{6}}{9}.$$

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

S467. Let a, b, c be real numbers such that $a, b, c \geq -\frac{1}{3}$ and $a+b+c=2$. Prove that

$$\left(a^3-2ab+b^3+\frac{8}{27}\right)\left(b^3-2bc+c^3+\frac{8}{27}\right)\left(c^3-2ca+a^3+\frac{8}{27}\right) \leq \left[\frac{10}{3}\left(\frac{4}{3}-ab-bc-ca\right)\right]^3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Takuji Imaiida, Fujisawa, Kanagawa, Japan

$$\begin{aligned}
&a^3-2ab+b^3+\frac{8}{27}\\
&=\left(a+b+\frac{2}{3}\right)\left(a^2+b^2+\left(\frac{2}{3}\right)^2-ab-\frac{2a}{3}-\frac{2b}{3}\right)
\end{aligned}$$

Since $a, b, c \geq -1/3$, using AM-GM twice,

$$\prod_{cyc}\left(a+b+\frac{2}{3}\right) \leq \left(\frac{2(a+b+c)+6\cdot\frac{1}{3}}{3}\right)^3 = 2^3$$

and

$$\begin{aligned}
&\prod_{cyc}\left(a^2+b^2+\left(\frac{2}{3}\right)^2-ab-\frac{2a}{3}-\frac{2b}{3}\right)\\
&\leq \left(\frac{1}{3}\left(2(a^2+b^2+c^2)+\frac{4}{3}-(ab+bc+ca)-\frac{4}{3}(a+b+c)\right)\right)^3\\
&=\left(\frac{1}{3}\left(2(a+b+c)^2-\frac{4}{3}-5(ab+bc+ca)\right)\right)^3\\
&=\left(\frac{5}{3}\left(\frac{4}{3}-(ab+bc+ca)\right)\right)^3.
\end{aligned}$$

Multiplying two inequalities, we obtain the desired result.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Spain; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

S468. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{a}{a^2 + bc + 1} + \frac{b}{b^2 + ca + 1} + \frac{c}{c^2 + ab + 1} \le 1.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Jamal Gadirov, Istanbul, Turkey

According to the AM-GM inequality we have $a^2 + 1 \ge 2a$, $b^2 + 1 \ge 2b$ and $c^2 + 1 \ge 2c$. Hence left hand side of the original inequality is less than or equal to

$$\frac{a}{2a + bc} + \frac{b}{2b + ac} + \frac{c}{2c + ab}$$

Now we claim that

$$\sum_{cyc} \frac{a}{2a + bc} \le 1$$

To prove this multiply both side with $(2a + bc)(2b + ca)(2c + ab)$ then the last inequality turns

$$a(2b + ac)(2c + ab) + b(2a + bc)(2c + ab) + c(2a + bc)(2b + ac) \le$$

$$(2a + bc)(2b + ca)(2c + ab)$$

After some simplifications we see that last inequality is equivalent to

$$4 \le abc + a^2 + b^2 + c^2 = 9 - 2(ab + bc + ca) + abc$$

Now let introduce new variables $p = a + b + c = 3$, $q = ab + bc + ca$ and $r = abc$ then we need to show $4 \le 9 - 2q + r \iff r \ge 2q - 5$. According to the new variables p, q, r well-known Schur's inequality has form as

$$p^3 - 4pq + 9r \ge 0 \iff r \ge \frac{4q - 9}{3}$$

Finally,

$$\frac{4q - 9}{3} \ge 2q - 5 \iff q \le 3$$

completes the proof since the last inequality holds immediately from $p^2 \ge 3q$ which is well-known inequality. So, we have done! Equality holds iff $a = b = c = 1$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland; Arkady Alt, San Jose, CA, USA; Nikos Kalapodis, Patras, Greece; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Titu Zvonaru, Comănești, Romania.

Undergraduate problems

U463. Let x_1, x_2, x_3, x_4 be the roots of the polynomial $P(X) = 2X^4 - 5X + 1$. Find the sum

$$\frac{1}{(1-x_1)^3} + \frac{1}{(1-x_2)^3} + \frac{1}{(1-x_3)^3} + \frac{1}{(1-x_4)^3}.$$

Proposed by Mircea Becheanu, Montreal, Canada

First solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

Substitute $1-y$ for x, so that $x = 1-y$, then the equation $2x^4 - 5x + 1 = 0$ becomes

$$2(1-y)^4 - 5(1-y) + 1 = 0$$

or

$$2y^4 - 8y^3 + 12y^2 - 3y - 2 = 0$$

whose roots are $1-x_1$, $1-x_2$, $1-x_3$, $1-x_4$.

Write $\frac{1}{z}$ for y, multiply by z^4, and change all the signs; then the resulting equation

$$2z^4 + 3z^3 - 12z^2 + 8z - 2 = 0$$

has for its roots $\frac{1}{1-x_1}$, $\frac{1}{1-x_2}$, $\frac{1}{1-x_3}$, $\frac{1}{1-x_4}$ and the given expression is equal to the value of the power sum $S_3 = \sum_{i=1}^4 \frac{1}{(1-x_i)^3}$ in this equation.

Denoting by σ_1, σ_2, σ_3 the elementary symmetric polinomials on four variables $\left\{\frac{1}{1-x_1}, \frac{1}{1-x_2}, \frac{1}{1-x_3}, \frac{1}{1-x_4}\right\}$, here

$$\sigma_1 = -\frac{3}{2} \qquad \sigma_2 = \frac{-12}{2} = -6 \qquad \sigma_3 = -\frac{8}{2} = -4$$

and

$$S_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3,$$

whence we obtain

$$S_3 = \frac{-339}{8}.$$

Second solution by Albert Stadler, Herrliberg, Switzerland
All zeros of the polynomial $P(x)$ lie in $|x|<2$, for $x\geq 2$ and

$$|P(x)|\geq 2|x|^4-5|x|-1\geq 16|x|-5|x|-1=11|x|-1\geq 21.$$

Let $R\geq 2$. By the Residue Theorem:

$$\frac{1}{2\pi i}\int_{|x|=R}\frac{1}{(1-x)^3}\cdot\frac{P'(x)}{P(x)}dx=\sum_{j=1}^{4}\frac{1}{(1-x_j)^3}-\frac{1}{2}\frac{d^2}{dx^2}\frac{P'(x)}{P(x)}\bigg|_{x=1},$$

where $\int_{|x|=R}...dx$ is a complex contour integral, the contour being a circle with radius R and center at 0 that is run through once in the positive direction. Letting R tend to infinity we actually see that the integral equals to 0. Therefore,

$$\begin{aligned}\sum_{j=1}^{4}\frac{1}{(1-x_j)^3}=\frac{1}{2}\frac{d^2}{dx^2}\frac{P'(x)}{P(x)}\bigg|_{x=1}&=\frac{1}{2}\frac{d^2}{dx^2}\frac{8x^3-5}{2x^4-5x+1}\bigg|_{x=1}\\&=\frac{32x^9+360x^6-192x^5+300x^3-60x^2+24x-125}{(2x^4-5x+1)^3}\bigg|_{x=1}=-\frac{339}{8}.\end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Anderson Torres, Sao Paulo, Brazil; Arkady Alt, San Jose, CA, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Sarah B. Seales, Prescott, AZ, USA; Titu Zvonaru, Comănești, Romania.

U464. Evaluate

$$\sum_{k=1}^{n} \cot^{-1}\left(\frac{k^2+k}{2}+\frac{1}{k}\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Albert Stadler, Herrliberg, Switzerland

Let $0 < x < y < \frac{\pi}{2}$. Then,

$$\cot(y-x) = \frac{\cos(y-x)}{\sin(y-x)} = \frac{\cos y \cos x + \sin y \sin x}{\sin y \cos x - \cos y \sin x} = \frac{\cot y \cot y + 1}{\cot x - \cot y}.$$

Set $a := \cot x, b := \cot y$. Then $a > b$ and

$$\cot^{-1} b - \cot^{-1} a = y - x = \cot^{-1}\left(\frac{\cot y \cot x + 1}{\cot x - \cot y}\right) = \cot^{-1}\left(\frac{ab+1}{a-b}\right).$$

In particular, if we set $a := k^2+k+1 = (k+1)^2-(k+1)+1, b := k^2-k+1$,

$$\cot^{-1}(k^2-k+1) - \cot^{-1}((k+1)^2-(k+1)+1) = \cot^{-1}\left(\frac{(k^2+1)^2-k^2+1}{2k}\right) =$$

$$= \cot^{-1}\left(\frac{k^3+k}{2}+\frac{1}{k}\right).$$

Hence,

$$\sum_{k=1}^{n} \cot^{-1}\left(\frac{k^3+k}{2}+\frac{1}{k}\right) = \sum_{k=1}^{n} \cot^{-1}(k^2-k+1) - \sum_{k=1}^{n} \cot^{-1}((k+1)^2-(k+1)+1) =$$

$$= \cot^{-1}(1) - \cot^{-1}(n^2+n+1) = \frac{\pi}{4} - \cot^{-1}(n^2+n+1).$$

Also solved by Young Joon Kim, Horace Mann School, Bronx, NY, USA; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Joehyun Kim, Fort Lee High School, NJ, USA; Taeyang David Park, Peddie School, Hightstown, NJ, USA.

U465. Let n be an odd positive integer. Prove that

$$\int_1^n (x-1)(x-2)\cdots(x-n)dx = 0$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Henry Ricardo, Westchester Area Math Circle, NY, USA

Let $x - \frac{n+1}{2} = u$. Then $x - \frac{n+1}{2} - k = u - k$ and $x - \frac{n+1}{2} + k = u + k$ for $k = 1, 2, \ldots, (n-1)/2$, so that $(x-1)(x-2)\cdots(x-n)$ becomes $(u-(n-1)/2)\cdots(u-1)u(u+1)\cdots(u+(n-1)/2)$. Now

$$\int_1^n (x-1)(x-2)\cdots(x-n)\,dx = \int_{\frac{1-n}{2}}^{-\frac{1-n}{2}} u(u^2-1)\cdots\left(u^2-((n-1)/2)^2\right)\,du = 0$$

since the last integrand is an odd function, and it is well known (and easily shown) that $\int_{-a}^{a} f(x)dx = 0$ when f is an odd integrable function.

Also solved by Daniel Lasaosa, Pamplona, Spain; Joehyun Kim, Fort Lee High School, NJ, USA; Andrew Lee, Choate Rosemary Hall, Wallingford, CT, USA; Othmane Hadi, Casablanca, Morocco; Bazar Tumurkhan, National University of Mongolia, Mongolia; Anderson Torres, Sao Paulo, Brazil; Taeyang David Park, Peddie School, Hightstown, NJ, USA; Sebastian Foulger, Charters Sixth Form, Sunningdale, England, UK; Jiho Lee, Canterbury School, CT, USA; Pedro Acosta De Leon, Massachusetts Institute of Technology, Cambridge, MA, USA; Arkady Alt, San Jose, CA, USA; Yi Won Kim, Taft School, Watertown, CT, USA; Adarsh Kumar, India; Anish Ray, Institute of Mathematics, Bhubaneswar, India; Akash Singha Roy, Chennai Mathematical Institute, India; Daniel López-Aguayo, Instituto Tecnológico y de Estudios Superiores de Monterrey, Campus Monterrey, Mexico; Lukas Seier, Charters Sixth Form, Sunningdale, England, UK; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Pradyumna Atreya, India; Sachin Kumar, BARC, Mumbai, India; Sarah B. Seales, Prescott, AZ, USA; Albert Stadler, Herrliberg, Switzerland; Thiago Landim de Souza Leão, Federal University of Pernambuco, Brazil.

U466. Let a, b, c positive real numbers. Prove that

$$\left(1+\frac{b}{a}\right)^{\frac{a^2}{b}}\left(1+\frac{c}{b}\right)^{\frac{b^2}{c}}\left(1+\frac{a}{c}\right)^{\frac{c^2}{a}} \geq 2^{a+b+c}.$$

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece

First,

$$\left(1+\frac{b}{a}\right)^{\frac{a^2}{b}}+\left(1+\frac{c}{b}\right)^{\frac{b^2}{c}}+\left(1+\frac{a}{c}\right)^{\frac{c^2}{a}} \geq 2^{a+b+c} \Leftrightarrow$$

$$\frac{a^2}{b}\ln\left(1+\frac{b}{a}\right)+\frac{b^2}{c}\ln\left(1+\frac{c}{b}\right)+\frac{c^2}{a}\ln\left(1+\frac{a}{c}\right) \geq (a+b+c)\ln 2 \Leftrightarrow$$

$$a\cdot\frac{a}{b}\ln\left(1+\frac{b}{a}\right)+b\cdot\frac{b}{c}\ln\left(1+\frac{c}{b}\right)+c\cdot\frac{c}{a}\ln\left(1+\frac{a}{c}\right) \geq (a+b+c)\ln 2$$

Consider the function $f(x)=\frac{1}{x}\ln(1+x)$, $x>0$.

Now,

$$f''(x)=\frac{2\ln(x+1)(x+1)^2-3x^2-2x}{x^3(x+1)^2},\ x>0$$

Let $g(x)=2\ln(x+1)(x+1)^2-3x^2-2x$, $x\geq 0$.

Notice that $g(0)=0$ and

$$g'(x)=4\ln(x+1)(x+1)+2(x+1)-6x-2=4(\ln(x+1)(x+1)-x),\ x>0.$$

Use the well-known inequality $\ln x > 1-\frac{1}{x}$, $x>1$ and get $\ln(x+1)>\frac{x}{x+1}$, $x>0$ or

$$\ln(x+1)(x+1)-x>0.$$

Next, $g'(x)>0$, $x>0$ and $g(0)=0$, so g is increasing and $g(x)>0$ for $x>0$.

As a result $f''(x)>0$, $x>0$ and thus f is convex.

From Jensen's weighted inequality we get that

$$a\cdot\frac{a}{b}\ln\left(1+\frac{b}{a}\right)+b\cdot\frac{b}{c}\ln\left(1+\frac{c}{b}\right)+c\cdot\frac{c}{a}\ln\left(1+\frac{a}{c}\right)=a\cdot f\left(\frac{b}{a}\right)+b\cdot f\left(\frac{c}{b}\right)+c\cdot f\left(\frac{a}{c}\right)\geq$$

$$(a+b+c)f\left(\frac{a\cdot\frac{b}{a}+b\cdot\frac{c}{b}+c\cdot\frac{a}{c}}{a+b+c}\right)=(a+b+c)f(1)=(a+b+c)\ln 2$$

and the conclusion follows.

Also solved by Ioannis D. Sfikas, Athens, Greece; Daniel Lasaosa, Pamplona, Spain; Young Joon Kim, Horace Mann School, Bronx, NY, USA; Joehyun Kim, Fort Lee High School, NJ, USA; Taeyang David Park, Peddie School, Hightstown, NJ, USA; Arkady Alt, San Jose, CA, USA; Henry Ricardo, Westchester Area Math Circle, NY, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy; Sarah B. Seales, Prescott, AZ, USA; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Thiago Landim de Souza Leão, Federal University of Pernambuco, Brazil.

U467. Let A and B be square matrices of dimension 2018×2018 with real entries such that $A^2 + B^2 = AB$. Prove that the matrix $AB - BA$ is singular.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, România

Solution by Li Zhou, Polk State College, USA

Let $\omega = e^{2\pi i/3}$ and $M = A + \omega B$. Then $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$, so

$$M\overline{M} = (A + \omega B)(A + \omega^2 B) = A^2 + \omega BA + \omega^2 AB + B^2 = \omega(BA - AB).$$

Hence, $0 \le |\det(M)|^2 = \omega^{2018} \det(BA - AB) = \omega^2 \det(BA - AB)$, thus $\det(BA - AB) = 0$.

Also solved by Young Joon Kim, Horace Mann School, Bronx, NY, USA; Akash Singha Roy, Chennai Mathematical Institute, India.

U467. Let $a < b$ be real numbers and $f : [a, b] \to [a, b]$ be a function with the following properties:

(a) f has left and right limits at any point $x \in (a, b)$ and $f(x-0) \le f(x+0)$;

(b) there exist limits $f(a+0)$ and $f(b-0)$.

Prove that there exists a point $x_0 \in [a, b]$ such that

$$\lim_{x \to x_0} f(x) = x_0.$$

Proposed by Mihai Piticari and Dan Stefan Marinescu, România

Solution by Daniel Lasaosa, Pamplona, Spain

Assume that the proposed result is not true, or $f(a+0) > a$ and $f(b-0) < b$. By the definition of $f(a+0)$ and taking $\epsilon = \frac{f(a)-a}{2} > 0$, there exists δ such that $f(x) > \frac{f(a)+a}{2}$ for all $x \in (a, a+\delta)$. Let $\delta_a = \min\left\{\delta, \frac{f(a)-a}{2}\right\}$, or $f(x+0) > x$ for all $x \in A = [a, a+\delta_a)$. Define similarly δ_b using $f(b-0)$ and taking $\epsilon = \frac{b-f(b)}{2}$, and note that $f(x+0) < x$ for all $x \in B = (b-\delta_b, b)$. Now, the set $X \subset [a, b]$ such that $x \in X$ iff $f(x+0) \le x$ has an infimum c with $b > c > a$, since $B \subset X$ or $X \ne \varnothing$, and all elements in X are larger than $a + \delta_a > a$.

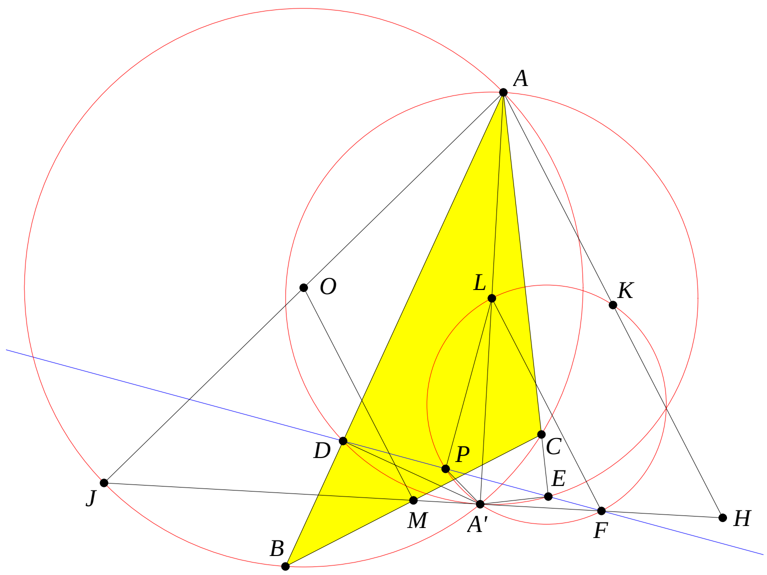
Assume that $f(c+0) > c$. Then, there exists $\delta > 0$ such that $f(x) \ge \frac{f(c+0)+c}{2} > c$ for all $x \in (c, c+\delta)$. But then, $f(x+0) \ge \frac{f(c+0)+c}{2}$ for all $x \in (c, c+\delta)$, in contradiction with the definition of c. Therefore, $f(c+0) \le c$. Moreover, $f(c-0) \le f(c+0) \le c$. Assume that $f(c-0) < c$. Then, there exists $\delta > 0$ such that for all $x \in (c-\delta, 0)$, we have $f(x) \le \frac{f(c-0)+c}{2}$, or $f(x+0) \le \frac{f(c-0)+c}{2}$. Therefore, for all $\frac{f(c-0)+c}{2} < x < c$, we have $f(x) < x$, and $f(x+0) \le x$ for all such x, in contradiction with the definition of c. Therefore, $f(c-0) = c = f(c+0)$. The conclusion follows.

Also solved by Albert Stadler, Herrliberg, Switzerland; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy.

O463. Let ABC ($AB \neq AC$) be an acute triangle with circumcircle $\Gamma(O)$ and let M be the midpoint of the side BC. The circle with diameter AM intersects Γ in a second point A'. Let D and E be the feet of the perpendiculars from A' to AB and AC, respectively. Prove that the line through M and parallel to AO bisects the segment DE.

Proposed by Marius Stănean, Zalău, România

Solution by Li Zhou, Polk State College, USA



There is no need to assume that $\triangle ABC$ is acute. Suppose that AJ is the diameter of Γ. Then it is easy to see that A' is on JM. Let H be the orthocenter of $\triangle ABC$, then it is well known that M is the midpoint of JH. Let F, K, and L be the midpoints of $A'H$, AH, and AA', respectively. Then it is well known that F is on the Simson line DE of $\triangle ABC$ from A'. Since $AA' \perp JH$, $A'FKL$ is a rectangle. Suppose that the circumcircle of $A'FKL$ intersects DE at a second point P. Then $A'P \perp PK$ and $LP \perp DE$. Since L is the center of the circumcircle of $ADA'E$, P is the midpoint of DE. Since $MK \parallel OA$, it then suffices to prove that M, P, K are collinear. This will follows immediately from the claim that $A'P \perp MK$. Indeed,

$$\begin{aligned}\angle LA'P &= \angle LFP = \angle AED - \angle EAH = \angle AA'D - \angle CAH \\ &= 90^\circ - \angle BAA' - \angle OAB = 90^\circ - \angle OAA',\end{aligned}$$

so $A'P \perp OA$, thus $A'P \perp MK$, completing the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Ravi Kumar B, Hyderabad, India and Stanislav Chobanov, USA; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Andrea Fanchini, Cantù, Italy.

O464. Let a, b, c be nonnegative real numbers such that $\frac{a}{b+c} \ge 2$. Prove that

$$5\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \ge \frac{a^2+b^2+c^2}{ab+bc+ca}+10.$$

Proposed by Marius Stănean, Zalău, România

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Let $a+b+c=1$. $a/(b+c) \ge 2$ if and only if $a \ge 2/3$. Moreover we can set $a \ge b \ge c$ or $a \ge c \ge b$. Let's use the first one so $a \ge 2/3$, $b \le 1/3$, $c \le 1/3$. Let

$$f(a,b,c) = 5\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) - \frac{a^2+b^2+c^2}{ab+bc+ca} - 10 \tag{1}$$

$$\frac{\partial f}{\partial b} = -\frac{5a}{(b+c)^2}+\frac{5}{a+c}-\frac{5c}{(a+b)^2}+\frac{2b}{ab+bc+ca}-\frac{(a+c)(a^2+b^2+c^2)}{(ab+bc+ca)^2} \le 0$$

Indeed

$$\frac{2b}{ab+bc+ca}-\frac{(a+c)(a^2+b^2+c^2)}{(ab+bc+ca)^2} \le 0$$

that is

$$(a^2+b^2+c^2)(a+c) \ge 2b(ab+bc+ca)$$

by $a^2+b^2+c^2 \ge ab+bc+ca$ and $a+c \ge 2b+2c+c \ge 2b$.

Moreover

$$\frac{a}{(b+c)^2}+\frac{c}{(a+b)^2} \ge \frac{1}{a+c} \iff (a+c)\left(\frac{a}{(b+c)^2}+\frac{c}{(a+b)^2}\right) \ge 1$$

and Cauchy–Schwarz yields

$$(a+c)\left(\frac{a}{(b+c)^2}+\frac{c}{(a+b)^2}\right) \ge \left(\frac{a}{b+c}+\frac{c}{a+b}\right)^2 \ge \left(\frac{a}{b+c}\right)^2 \ge 4$$

It follows $f'(b) \le 0$ and then by Lagrange mean–value theorem

$$f(b)-f(1/3) = f'(\xi)(b-1/3) \ge 0$$

because $b \le 1/3$. We get

$$f(a,b,c) \ge f(a,\frac{1}{3},c,) = 5\left(\frac{a}{\frac{1}{3}+c}+\frac{\frac{1}{3}}{c+a}+\frac{c}{a+\frac{1}{3}}\right) - \frac{a^2+\frac{1}{9}+c^2}{\frac{a}{3}+\frac{c}{3}+ca} - 10$$

Since $a \ge 2/3$, $f(a,\frac{1}{3},c) = f(\frac{2}{3},\frac{1}{3},0) = 0$ and this completes the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Chennai Mathematical Institute, India; Albert Stadler, Herrliberg, Switzerland.

O465. Let $C_0 = \{i_1, i_2, \dots, i_n\}$ be an ordered set of n positive integers. A transformation of C_0 is the sequence of positive integers

$$C_1 = \{1, 2, \dots, i_1 - 1, 1, 2, \dots, i_2 - 1, \dots, 1, 2, \dots, i_n - 1\},$$

i.e, each $i_k > 1$ is replaced by the sequence $1, 2, \dots, i_k - 1$. Similarly, the sequence C_i is obtained by a transformation from C_{i-1}. (For example, if $C_0 = \{1, 2, 6, 3\}$, then $C_1 = \{1, 1, 1, 2, 3, 4, 5, 1, 2\}$).

a) Assuming that $C_0 = \{1, 2, \dots, n\}$, find the number of occurrences of i in C_j.

b) Let $C_F = \{1, 1, 1, 1, \dots, 1\}$ be the final sequence obtained after performing maximum possible number of transformations to $C_0 = \{1, 2, \dots, n\}$. Find the number of occurrences of 1 in C_F.

Proposed by Anish Ray, Institute of Mathematics, Bhubaneswar, India

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that there is no interaction between the elements in the set when the successive transformations take place, and that the transformation does not rely on the ordering of the elements inside the set. Note also that any set of the form $\{1, 1, \dots, 1\}$ transforms into itself. We will use these observations in the following solution.

Denote by $D_{n,0}$ the set $\{1, 2, \dots, n\}$, and by $D_{n,j}$ the result of applying j times the transformation to $D_{n,0}$. Note that $D_{n,1} = D_{n-1,1} \cup D_{n-1,0}$, and after trivial induction, $D_{n,j} = D_{n-1,j} \cup D_{n-1,j-1}$. Denote now by $N_{i,n,j}$ the number of occurrences of i when the transformation is applied to $D_{n,0}$ j times. Clearly, $N_{i,n,j} = N_{i,n-1,j} + N_{i,n-1,j-1}$ by the previous observation. This allows us to formulate the following

Claim: For all $u \geq 0$, we have

$$N_{i,n,j} = \sum_{v=0}^{u} \binom{u}{v} N_{i,n-u,j-v}.$$

Proof: For $u = 0$ the Claim is clearly the identity, and for $u = 1$ it trivially holds by the previous observation. If it holds for u, then for $u + 1$ we have

$$N_{i,n,j} = \sum_{v=0}^{u} \binom{u}{v} N_{i,n-u-1,j-v} + \sum_{v'=1}^{u+1} \binom{u}{v'-1} N_{i,n-u-1,j-v'} =$$

$$= \binom{u}{0} N_{i,n-(u+1),j} + \binom{u}{u} N_{i,n-(u+1),j-(u+1)} + \sum_{v=1}^{u} \left(\binom{u}{v} + \binom{u}{v-1} \right) N_{i,n-(u+1),j-v} =$$

$$= \sum_{v=0}^{u+1} \binom{u+1}{v} N_{i,n-(u+1),j-v},$$

where we have performed the substitution $v' = v + 1$, and we have used the well-known results

$$\binom{u}{0} = \binom{u}{u} = 1 = \binom{u+1}{0} = \binom{u+1}{u+1}, \qquad \binom{u}{v} + \binom{u}{v-1} = \binom{u+1}{v}.$$

The Claim thus follows by induction.

a) Note now that, unless $D_{n,j} = \{1,1,\ldots,1\}$, the maximum of set $D_{n,j}$ disappears when it transforms into $D_{n,j+1}$. Moreover, $N_{n,n,0} = 1$ and $N_{n,n,j} = 0$ for all $j \ge 1$ clearly hold, and when $i > n$, we have $N_{i,n,j} = 0$ for all $j \ge 0$. Therefore, for all $i \ge 2$, and taking $u = n - i$ in the Claim, we have

$$N_{i,n,j} = \sum_{v=0}^{n-i} \binom{n-i}{v} N_{i,i,j-v} = \binom{n-i}{j} N_{i,i,0} = \binom{n-i}{j}.$$

b) For $i = 1$ the boundary conditions are different, since the 1's remain unchanged after the transformation. This results in $N_{1,1,j} = 1$ for all $j \ge 0$. Moreover, the final condition occurs for $j \ge n-1$, since it takes exactly $n-1$ turns to decrease the maximum of $D_{n,j}$ from n to 1, decreasing by 1 at each transformation. Or taking $u = j = n-1$ in the Claim produces

$$N_{1,n,n-1} = \sum_{v=0}^{n-1} \binom{n-1}{v} N_{1,1,n-1-v} = \sum_{v=0}^{n-1} \binom{n-1}{v} = 2^{n-1}.$$

Also solved by Sebastian Foulger, Charters Sixth Form, Sunningdale, England, UK; Albert Stadler, Herrliberg, Switzerland.

O466. Let $n \geq 2$ be an integer. Prove that there exists a set S of $n-1$ real numbers such that whenever $a_1, \dots, a_n$ are mutually different real numbers satisfying

$$a_1 + \frac{1}{a_2} = a_2 + \frac{1}{a_3} = \dots = a_{n-1} + \frac{1}{a_n} = a_n + \frac{1}{a_1}$$

then the common value of all these sums is a number from S.

Proposed by Josef Tkadlec, Vienna, Austria

Solution by Li Zhou, Polk State College, USA

We show that $S = \left\{2\cos\frac{i\pi}{n} : i = 1, \dots, n-1\right\}$. For any nonzero a_1, define $r_0(x) = a_1$ and $r_k(x) = 2x - \frac{1}{r_{k-1}(x)}$ for $k \geq 1$. Let $U_k(x)$ be Chebyshev's polynomial of the second kind defined by $U_{-1}(x) = 0$, $U_0(x) = 1$, and $U_k(x) = 2xU_{k-1}(x) - U_{k-2}(x)$ for $k \geq 1$. We claim that for $k \geq 1$,

$$r_k(x) = a_1 + \frac{\left(2a_1x - a_1^2 - 1\right)U_{k-1}(x)}{a_1U_{k-1}(x) - U_{k-2}(x)}.$$

The claim is clearly true for $k = 1$. As an induction hypothesis, assume it is true for some $k \geq 1$. Then

$$\begin{aligned} r_{k+1}(x) &= 2x - \frac{1}{r_k(x)} = 2x - \frac{1}{a_1 + \frac{(2a_1x - a_1^2 - 1)U_{k-1}(x)}{a_1U_{k-1}(x) - U_{k-2}(x)}} = 2x - \frac{a_1U_{k-1}(x) - U_{k-2}(x)}{a_1U_k(x) - U_{k-1}(x)} \\ &= a_1 + \frac{(2x - a_1)[a_1U_k(x) - U_{k-1}(x)] - a_1U_{k-1}(x) + U_{k-2}(x)}{a_1U_k(x) - U_{k-1}(x)} \\ &= a_1 + \frac{\left(2a_1x - a_1^2 - 1\right)U_k(x)}{a_1U_k(x) - U_{k-1}(x)}, \end{aligned}$$

completing the induction. Now if $a_1, \dots, a_n$ are mutually different real numbers satisfying

$$a_1 + \frac{1}{a_2} = a_2 + \frac{1}{a_3} = \dots = a_{n-1} + \frac{1}{a_n} = a_n + \frac{1}{a_1} = 2x,$$

then $2a_1x - a_1^2 - 1 \neq 0$ and

$$a_n = 2x - \frac{1}{a_1} = r_1(x) \;\Rightarrow\; a_{n-1} = 2x - \frac{1}{a_n} = r_2(x) \;\Rightarrow\; \dots \;\Rightarrow\; a_1 = 2x - \frac{1}{a_2} = r_n(x).$$

Hence, $U_{n-1}(x) = 0$. Finally, it is well known that $U_{n-1}(x) = 2^{n-1}\prod_{i=1}^{n-1}\left(x - \cos\frac{i\pi}{n}\right)$, completing the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain.

O467. Let ABC be a triangle with $\angle A > \angle B$. Prove that $\angle A = 3\angle B$ if and only if

$$\frac{AB}{BC - CA} = \sqrt{1 + \frac{BC}{CA}}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Nikos Kalapodis, Patras, Greece

By the law of cosines $c^2 = a^2 + b^2 - 2ab\cos C$ we get that $c^2 = (a-b)^2 + 4ab\sin^2\dfrac{C}{2}$ or
$\dfrac{c^2}{(a-b)^2} = 1 + \dfrac{4ab\sin^2\frac{C}{2}}{(a-b)^2}$ (1).
Using (1) and the law of sines we have:
$\dfrac{AB}{BC - CA} = \sqrt{1 + \dfrac{BC}{CA}} \Leftrightarrow \dfrac{c^2}{(a-b)^2} = 1 + \dfrac{a}{b} \Leftrightarrow 1 + \dfrac{4ab\sin^2\frac{C}{2}}{(a-b)^2} = 1 + \dfrac{a}{b} \Leftrightarrow 2b\sin\dfrac{C}{2} = a - b \Leftrightarrow.$
$2\sin B\sin\dfrac{C}{2} = \sin A - \sin B \Leftrightarrow 2\sin B\sin\dfrac{C}{2} = 2\sin\dfrac{A-B}{2}\cos\dfrac{A+B}{2} \Leftrightarrow \sin B = \sin\dfrac{A-B}{2} \overset{*}{\Leftrightarrow}$
$B = \dfrac{A-B}{2} \Leftrightarrow A = 3B.$
(* Since $\angle A > \angle B$ we have that $\angle B, \angle\dfrac{A-B}{2} \in \left(0, \dfrac{\pi}{2}\right)$ and the function $\sin x : \left(0, \dfrac{\pi}{2}\right) \to (0,1)$ is a bijection).

Also solved by Daniel Lasaosa, Pamplona, Spain; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Joehyun Kim, Fort Lee High School, NJ, USA; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

O468. Let A_n be the number of entries in the n-th row of Pascal's triangle that are 1 modulo 3. Let B_n be the number of entries in the n-th row which are 2 modulo 3. Prove that $A_n - B_n$ is a power of 2 for all positive integers n.

Proposed by Enrique Trevino, Lake Forest College, USA

Solution by Albert Stadler, Herrliberg, Switzerland

The entries in the n^{th} row of Pascal's triangle are the binomial coefficients $\binom{n}{m}, 0 \le m \le n$. Let

$$n = n_k 3^k + n_{k-1} 3^{k-1} + \cdots + n_1 3 + n_0$$

and

$$m = m_k 3^k + m_{k-1} 3^{k-1} + \cdots + m_1 3 + n_0$$

be the base 3 expansions of n and m respectively, where $0 \le n_j, m_j \le 2$. Then, by Lucas's theorem

$$\binom{n}{m} \equiv \prod_{j=0}^{k} \binom{n_j}{m_j} \pmod 3.$$

We not that $\binom{n_j}{m_j} = 2$, if $n_j = 2$ and $m_j = 1$, $\binom{n_j}{m_j} = 0$, if $m_j > n_j$ and $\binom{n_j}{m_j} = 1$ in all other cases (i.e. when $n_j = m_j$ or $m_j = 0$).
Let r be the number of n_j with $n_j = 1$. We claim that $A_n - B_n = 2^r$.
Let s be the number of n_j with $n_j = 2$.
Clearly, $r + s \le k$. If $n_j = 1$ then there are exactly 2 values of m_j such that $\binom{n_j}{m_j} \equiv 1 \pmod 3$, namely $m_j = 0$ or $m_j = 1$. If $n_j = 2$ then there are exactly 2 values of m_j such that $\binom{n_j}{m_j} \equiv 1 \pmod 3$, namely $m_j = 0$ or $m_j = 2$, and there are exactly onevalue of m_j such that $\binom{n_j}{m_j} \equiv -1 \pmod 3$, namely $m_j = 1$. Therefore,

$$A_n - B_n = 2^r \left(\sum_{t=0}^{s} \binom{s}{t} 2^t (-1)^{s-t} \right) = 2^r (2-1)^s = 2^r.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Sebastian Foulger, Charters Sixth Form, Sunningdale, England, UK.