Junior problems

J469. Let a and b be distinct real numbers. Prove that

$$(3a+1)(3b+1)=3a^2b^2+1$$

if and only if

$$\left(\sqrt[3]{a}+\sqrt[3]{b}\right)^3=a^2b^2.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

First solution by by the author
The first condition is equivalent to $a+b-a^2b^2+3ab=0$.

The identity $x^3+y^3+z^3-3xyz=\frac{1}{2}(x+y+z)[(x-y)^2+(y-z)^2+(z-x)^2]$
shows that if x,y,z are real numbers, not all equal, then $x^3+y^3+z^3-3xyz=0$ if and only if $x+y+z=0$.
In our problem, $x=\sqrt[3]{a}, y=\sqrt[3]{b},$ and $z=-\sqrt[3]{a^2b^2}$, so

$$\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{a^2b^2}=0,$$

implying

$$\sqrt[3]{a}+\sqrt[3]{b}=\sqrt[3]{a^2b^2}.$$

Hence the conclusion.

Second solution by Takuji Imaiida, Fujisawa, Kanagawa, Japan
Let $x=\sqrt[3]{a}+\sqrt[3]{b}$ and $y=\sqrt[3]{ab}$, then it suffices to show that

$$3y^3+x(x^2-3y)=y^6 \tag{1}$$

if and only if

$$x=y^2. \tag{2}$$

Since $3y^3+x(x^2-3y)=y^6 \Leftrightarrow (x-y^2)(x^2+xy^2+y^4-3y)=0$, clearly, (2) $\Rightarrow$ (1). Conversely, assume that (1) holds. Since a and b are distinct, we obtain $x^2-4y>0$. Therefore

$$\begin{aligned}x^2+xy^2+y^4-3y &> x^2+xy^2+y^4-\frac{3}{4}x^2\\ &=\frac{1}{4}x^2+xy^2+y^4=(\frac{1}{2}x^2+y^2)^2\geq 0.\end{aligned}$$

$x-y^2=0$, (2) holds and we are done.

Also solved by Ioannis D. Sfikas, Athens, Greece; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; G. C. Greubel, Newport News, VA, USA; Santosh Kumar Mvrk, Hyderabad, India; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Pantelis.N, Athens, Greece; Pradyumna Atreya, Mumbai, India; Saloni Gole, FIITJEE Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Polyahedra, Polk State College, USA; Bryant Hwang, Korea International School, South Korea; Dumitru Barac, Sibiu, Romania; Prajnanaswaroopa S, Amrita University, Coimbatore, India.

J470. Solve in real numbers the equation

$$\left(x^3-2\right)^3+\left(x^2-2\right)^2=0.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Polyahedra, Polk State College, USA

Denote by $f(x)$ the left-hand side of the equation. Then $f(1)=0$ and we show that f has no other real zero.

If $x>\sqrt[3]{2}$ then $f(x)>0$.

If $1<x\leq\sqrt[3]{2}$, then $0\leq 2-x^3<2-x^2<1$, so

$$\left(2-x^3\right)^3\leq\left(2-x^3\right)^2<\left(2-x^2\right)^2,$$

thus $f(x)>0$.

Finally, consider $x<1$. Then $x^2-2\geq x^3-2$ and

$$\left(x^2-2\right)+\left(x^3-2\right)=(x-1)\left(x^2+2x+2\right)-2<0,$$

thus $\left|x^2-2\right|\leq 2-x^3$. Therefore,

$$\left(x^2-2\right)^2\leq\left(2-x^3\right)^2<\left(2-x^3\right)^3,$$

that is, $f(x)<0$.

Also solved by Takuji Imaiida, Fujisawa, Kanagawa, Japan; Bryant Hwang, Korea International School, South Korea; Albert Stadler, Herrliberg, Switzerland; G. C. Greubel, Newport News, VA, USA; Ioannis D. Sfikas, Athens, Greece; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania; Pantelis.N, Athens, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Akash Singha Roy, Chennai Mathematical Institute, India; Titu Zvonaru, Comănești, Romania.

J471. Find all real numbers a for which the equation

$$\left(\frac{x}{x-1}\right)^2 + \left(\frac{x}{x+1}\right)^2 = a$$

has four distinct real roots.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by the author

Clearly, $a \geq 0$. Completing the square gives $\frac{x}{x-1} + \left(\frac{x}{x+1}\right)^2 - \frac{2x^2}{x^2-1} = a$, which rewrites

$$\left(\frac{2x^2}{x^2-1}\right)^2 - \frac{2x^2}{x^2-1} = a.$$

With the substitution $\frac{2x^2}{x^2-1} = t$, this is equivalent to $t^2 - t + \frac{1}{4} = a + \frac{1}{4}$, that is,

$$t - \frac{1}{2} = b \text{ or } t - \frac{1}{2} = -b,$$

where $b = \sqrt{a + \frac{1}{4}}$.

It follows that $(2b-3)x^2 = 2b+1$, implying $b > \frac{3}{2}$, or $(2b+3)x^2 = 2b-1$, which has two distinct solutions for $b > \frac{1}{2}$ (which is always true for a diff from 0). We cannot have $b < \frac{1}{2}$ because that would imply $a < 0$, a contradiction. It follows that $a + \frac{1}{4} > \frac{9}{4}$, and so the answer is $a > 2$. The four solutions are distinct as $b \neq 0$ implies

$$\frac{2b+1}{2b-3} \neq \frac{2b-1}{2b+3}.$$

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Takuji Imaiida, Fujisawa, Kanagawa, Japan; Polyahedra, Polk State College, USA; Dumitru Barac, Sibiu, Romania; Akash Singha Roy, Chennai Mathematical Institute, India; Albert Stadler, Herrliberg, Switzerland; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; G. C. Greubel, Newport News, VA, USA; Ioannis D. Sfikas, Athens, Greece; Konstantinos Kritharidis, American College of Greece - Pierce, Athens, Greece; Lukas Seier, Charters Sixth Form, Sunningdale, England, UK; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania; Oana Prajitura, College at Brockport, SUNY, NY, USA; Santosh Kumar Mvrk, Hyderabad, India; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA; Sebastian Foulger, Charters Sixth Form, Sunningdale, England, UK.

J472. Let a, b, c be positive numbers such that $ab + bc + ca = 1$. Prove that

$$a\sqrt{b^2+1} + b\sqrt{c^2+1} + c\sqrt{a^2+1} \geq 2.$$

Proposed by An Zhenping, Xianyang Normal University, China

First solution by Polyahedra, Polk State College, USA
By the Cauchy-Schwarz inequality, $\sqrt{(b^2+1)(c^2+1)} \geq bc + 1$, etc. Hence,

$$\begin{aligned}
\left(a\sqrt{b^2+1} + b\sqrt{c^2+1} + c\sqrt{a^2+1}\right)^2 &= a^2\left(b^2+1\right) + b^2\left(c^2+1\right) + c^2\left(a^2+1\right) \\
&\quad +2ab\sqrt{(b^2+1)(c^2+1)} + 2bc\sqrt{(c^2+1)(a^2+1)} + 2ca\sqrt{(a^2+1)(b^2+1)} \\
\geq\ & (ab)^2 + (bc)^2 + (ca)^2 + a^2 + b^2 + c^2 + 2ab(bc+1) + 2bc(ca+1) + 2ca(ab+1) \\
=\ & (ab+bc+ca)^2 + (a+b+c)^2 \geq 1 + 3(ab+bc+ca) = 4,
\end{aligned}$$

completing the proof.

Second solution by Polyahedra, Polk State College, USA
As a branch of the hyperbola $y^2 - x^2 = 1$, the function $f(x) = \sqrt{x^2+1}$ is convex. By Jensen's inequality,

$$\begin{aligned}
a\sqrt{b^2+1} + b\sqrt{c^2+1} + c\sqrt{a^2+1} &\geq (a+b+c)\sqrt{\left(\frac{ab+bc+ca}{a+b+c}\right)^2 + 1} \\
= \ & \sqrt{1+(a+b+c)^2} \geq \sqrt{1+3(ab+bc+ca)} = \sqrt{4} = 2.
\end{aligned}$$

Also solved by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Bryant Hwang, Korea International School, South Korea; Dumitru Barac, Sibiu, Romania; Prajnanaswaroopa S, Amrita University, Coimbatore, India; Akash Singha Roy, Chennai Mathematical Institute, India; Jamal Gadirov, Ishik University, Iraq; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Sarah B. Seales, Prescott, AZ, USA; Ioannis D. Sfikas, Athens, Greece; Soumyadeep Paul, D.A.V. Public School, Haldia, India; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Loreta Arzumanyan, "Quantum" College, Armenia; Arkady Alt, San Jose, CA, USA.

J473. Let a, b, c be distinct real numbers. Prove that

$$\left(\frac{a}{b-a}\right)^2 + \left(\frac{b}{c-b}\right)^2 + \left(\frac{c}{a-c}\right)^2 \geq 1.$$

Proposed by Anish Ray, Institute of Mathematics, Bhubaneswar, India

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Let

$$\frac{a}{b-a} = x, \frac{b}{c-b} = y, \frac{c}{a-c} = z$$

then easily to see that

$$xyz = (x+1)(y+1)(z+1).$$

This implies

$$xy + yz + zx + x + y + z + 1 = 0.$$

Using this relation we get

$$\begin{aligned} x^2 + y^2 + z^2 &= (x+y+z)^2 - 2(xy+yz+zx) \\ &= (x+y+z)^2 + 2(x+y+z+1) \\ &= (x+y+z+1)^2 + 1 \\ &\geq 1 \end{aligned}$$

and we are done.

Also solved by Polyahedra, Polk State College, USA; Dumitru Barac, Sibiu, Romania; Akash Singha Roy, Chennai Mathematical Institute, India; Albert Stadler, Herrliberg, Switzerland; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Ioannis D. Sfikas, Athens, Greece; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania; Pantelis.N, Athens, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA.

J474. Let k be a positive integer. Suppose x and y are positive integers such that for every positive integer n, $n > k$,

$$x^{n-k} + y^n \mid x^n + y^{n+k}.$$

Prove that $x = y$.

Proposed by Valentio Iverson, Medan, North Sumatra, Indonesia

Solution by Polyahedra, Polk State College, USA

First, consider $x > y$. Since $x^n + y^{n+k} = \left(x^{n-k} + y^n\right) x^k + \left(y^k - x^k\right) y^n$, $x^{n-k} + y^n$ must divide $\left(y^k - x^k\right) y^n$ for all $n > k$. But as $n \to \infty$,

$$0 > \frac{\left(y^k - x^k\right) y^n}{x^{n-k} + y^n} = \frac{y^k - x^k}{\frac{1}{x^k}\left(\frac{x}{y}\right)^n + 1} \to 0,$$

thus the ratio cannot be an integer for sufficiently large n.

Next, consider $x < y$. Since $x^n + y^{n+k} = \left(x^{n-k} + y^n\right) y^k + \left(x^k - y^k\right) x^{n-k}$, $x^{n-k} + y^n$ must divide $\left(x^k - y^k\right) x^{n-k}$ for all $n > k$. But as $n \to \infty$,

$$0 > \frac{\left(x^k - y^k\right) x^{n-k}}{x^{n-k} + y^n} = \frac{x^k - y^k}{1 + x^k\left(\frac{y}{x}\right)^n} \to 0,$$

thus the ratio cannot be an integer for sufficiently large n.

Also solved by Akash Singha Roy, Chennai Mathematical Institute, India; Sarah B. Seales, Prescott, AZ, USA; Albert Stadler, Herrliberg, Switzerland.

Senior problems

S469. Let $ABCD$ be a kite with $\angle A = 5\angle C$ and $AB \cdot BC = BD^2$. Find $\angle B$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

If $\angle D = x$, then $\angle A = 5x$, and since the angles of $\triangle ABD$ add up to $180°$, we have $5x < 180°$, implying $x < 36°$.

From isosceles triangles ABD and BCD, we have

$$BD = 2 \cdot AB \cdot \sin\frac{5x}{2} \qquad \text{and} \qquad BD = 2 \cdot BC \cdot \sin\frac{x}{2},$$

respectively.

Hence

$$BD^2 = 4 \cdot AB \cdot BC \cdot \sin\frac{5x}{2}\sin\frac{x}{2}.$$

Dividing both sides by $BD^2 = AB \cdot BC$ gives

$$\sin\frac{5x}{2}\sin\frac{x}{2} = \frac{1}{4},$$

which is equivalent to

$$\cos 3x - \cos 2x = -\frac{1}{2}.$$

This, in turn, can be written as a cubic in $\cos x$

$$8\cos^3 x - 4\cos^2 x - 6\cos x + 3 = 0$$

which factors immediately so that its solutions $\left(\cos x = \frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$ can be read off.

From $x < 36°$, the only admissible solution is $\cos x = \frac{\sqrt{3}}{2}$ and $x = 30°$. Thus

$$\angle D = 30°$$

and

$$\angle A = 150°.$$

Since the angles of kite $ABCD$ add up to $360°$ and $\angle B = \angle C$, we conclude that $\angle B = 90°$.

Also solved by Santosh Kumar Mvrk, Hyderabad, India; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania; Pradyumna Atreya, Mumbai, India; Ioannis D. Sfikas, Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Kevin Soto Palacios, Huarmey, Perú.

S470. Let x, y, z be positive real numbers such that $xyz(x+y+z) = 4$. Prove that

$$(x+y)^2 + 3(y+z)^2 + (z+x)^2 \geq 8\sqrt{7}.$$

Proposed by An Zhenping, Xianyang Normal University, China

First solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam
We know that in any triangle ABC and for all real numbers u, v, w such that $uv + vw + wu \geq 0$

$$ua^2 + vb^2 + wc^2 \geq 4S\sqrt{uv + vw + wu}$$

where S denotes the are of the triangle. Using Ravi's substitutions $a = y+z, b = z+x, c = x+y$ with $x, y, z > 0$ then the above result becomes

$$u(y+z)^2 + v(z+x)^2 + w(x+y)^2 \geq 4\sqrt{xyz(x+y+z)(uv+vw+wu)}.$$

Now we apply this result for $(u, v, w) = (3, 1, 1)$ and note that the condition $xyz(x+y+z) = 4$ to obtain

$$(x+y)^2 + 3(y+z)^2 + (z+x)^2 \geq 8\sqrt{7}$$

as desired.

Second solution by Arkady Alt, San Jose, CA, USA
Let $p := y+z, q := yz$.Then $p^2 \geq 4q,\ 4 = xyz\,(x+y+z) = qx^2 + pqx$ and, therefore,

$$(x+y)^2 + 3(y+z)^2 + (z+x)^2 = 2\left(x^2 + x\,(y+z) + 4\,(y+z)^2 - 2yz\right) =$$

$$2\left(4p^2 + x^2 + px - 2q\right) = 2\left(4p^2 + \frac{qx^2 + pqx}{q} - 2q\right) = 2\left(4p^2 + \frac{4}{q} - 2q\right) \geq$$

$$2\left(4p^2 + \frac{4}{p^2/4} - 2\cdot\frac{p^2}{4}\right) = 2\left(4p^2 + \frac{7}{p^2}\right) \geq$$

$$2\cdot 2\sqrt{4p^2\cdot\frac{7}{p^2}} = 8\sqrt{7}.$$

Also solved by Haosen Chen, Zhejiang, China; Dumitru Barac, Sibiu, Romania; Ioannis D. Sfikas, Athens, Greece; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland.

S471. Prove that the following inequality holds for all positive real numbers a, b, c:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{9(a+b+c)}{ab+bc+ca} \geq 8\left(\frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}\right)$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, CA, USA
Since for any positive a, b, c holds inequality
(*Vasile Cirtoaje, Algebraic Inequalities, Old and New Methods, Inequality 59,p.13.*)

$$\sum_{cyc}\frac{a}{a^2+bc} \leq \sum_{cyc}\frac{1}{b+c} \tag{1}$$

remains to prove inequality $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{9(a+b+c)}{ab+bc+ca} \geq 8\sum_{cyc}\frac{1}{b+c} \iff$

$$\frac{ab+bc+ca}{abc}+\frac{9(a+b+c)}{ab+bc+ca} \geq 8\sum_{cyc}\frac{1}{b+c}. \tag{2}$$

Let $p := ab+bc+ca, q := abc$. Also we may assume that $a+b+c=1$ (due homogeneity of (2)).
Then $\sum_{cyc}\frac{1}{b+c} = \frac{8(1+p)}{p-q}$ and since

$$3p = 3(ab+bc+ca) \leq (a+b+c)^2 = 1, 3q = 3abc(a+b+c) \leq (ab+bc+ca)^2 = p^2$$

we obtain

$$\frac{ab+bc+ca}{abc}+\frac{9(a+b+c)}{ab+bc+ca}-8\sum_{cyc}\frac{1}{b+c} = \frac{p}{q}+\frac{9}{p}-\frac{8(1+p)}{p-q} \geq$$

$$\frac{12}{p}-\frac{8(1+p)}{p-\frac{p^2}{3}} = \frac{12(1-3p)}{p(3-p)} \geq 0.$$

Also solved by Dumitru Barac, Sibiu, Romania; Akash Singha Roy, Chennai Mathematical Institute, India; Loreta Arzumanyan, "Quantum" College, Armenia; Albert Stadler, Herrliberg, Switzerland; Ioannis D. Sfikas, Athens, Greece; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania; Titu Zvonaru, Comănești, Romania.

S472. Let ABC be a triangle with $\angle B$ and $\angle C$ acute and let D be the foot of the altitude from A. Prove that $\angle A$ is right if and only if

$$\frac{BD}{AB^2} + \frac{CD}{AC^2} = \frac{2}{BC}.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Albert Stadler, Herrliberg, Switzerland

Let $x = BD, y = CD, b = AB, c = AC$. Assume first that $\angle A = \pi/2$. Then, $b^2 = x(x+y)$ and $c^2 = y(x+y)$. Therefore,

$$\frac{x}{b^2} + \frac{y}{c^2} = \frac{2}{x+y},$$

as required.

Assume next that $\frac{x}{b^2} + \frac{y}{c^2} = \frac{2}{x+y}$. Clearly, $b^2 - x^2 = c^2 - y^2$. Solving these two equations for x and y yields

$$x = \frac{b^2}{\sqrt{b^2+c^2}}, \ y = \frac{c^2}{\sqrt{b^2+c^2}}.$$

Then $b^2 + c^2 - (x+y)^2 = 0$, which proves that $\triangle ABC$ is a right triangle.

Also solved by Dumitru Barac, Sibiu, Romania; Akash Singha Roy, Chennai Mathematical Institute, India; Kevin Soto Palacios, Huarmey, Perú; Corneliu Mănescu-Avram, Ploiești, Romania; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Ioannis D. Sfikas, Athens, Greece; Telemachus Baltsavias, Keramies Junior High School, Kefallonia, Greece; Titu Zvonaru, Comănești, Romania.

S473. Let a, b, c be positive real numbers. Prove that

$$(a-b)^4 + (b-c)^4 + (c-a)^4 \leq 6\left(a^4 + b^4 + c^4 - abc(a+b+c)\right).$$

Proposed by Nicușor Zlota, Focșani, Romania

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

The desired inequality is equivalent to

$$\sum_{\text{cyc}}(a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4) \leq 6(a^4 + b^4 + c^4 - abc(a+b+c)),$$

$$3(a^2b^2 + b^2c^2 + c^2a^2) + 3abc(a+b+c) \leq 2(a^4 + b^4 + c^4) + 2ab(a^2+b^2) + bc(b^2+c^2) + ca(c^2+a^2).$$

We have

$$\begin{aligned} 2(a^4 + b^4 + c^4) + \sum_{\text{cyc}} 2ab(a^2+b^2) &\geq 2(a^2b^2 + b^2c^2 + c^2a^2) + \sum_{\text{cyc}} 4a^2b^2 \\ &= 6(a^2b^2 + b^2c^2 + c^2a^2) \\ &\geq 3(a^2b^2 + b^2c^2 + c^2a^2) + 3abc(a+b+c). \end{aligned}$$

The proof is completed.

Also solved by Dumitru Barac, Sibiu, Romania; Akash Singha Roy, Chennai Mathematical Institute, India; Loreta Arzumanyan, "Quantum" College, Armenia; Albert Stadler, Herrliberg, Switzerland; Haosen Chen, Zhejiang, China; Ioannis D. Sfikas, Athens, Greece; Arkady Alt, San Jose, CA, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Titu Zvonaru, Comănești, Romania.

S474. Let a, b, c, d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 12$. Prove that

$$a^3 + b^3 + c^3 + d^3 + 9(a + b + c + d) \le 84$$

Proposed by Marius Stănean, Zalău, Romania

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy

$$\sum_{\text{cyc}} (a^3 + 9a) \le 84$$

$a^3 + 9a$ is convex, therefore, the maximum of the above sum occurs when at least three variables are equal.

Set $b = c = d = x$. The inequality becomes

$$a^3 + 3x^3 + 9(a + 3x) \le 84$$

provided $a^2 + 3x^2 = 12$. $a = \sqrt{12 - 3x^2}$, $0 \le x \le 2$. The inequality becomes

$$\sqrt{12 - 3x^2} + 3x^3 + 9\sqrt{12 - 3x^2} + 27x - 84 \le 0$$

if and only if

$$\left(\sqrt{12 - 3x^2} + 9\sqrt{12 - 3x^2}\right)^2 - (84 - 3x^3 - 27x)^2 \le 0$$

The inequality becomes

$$36(x^4 + 2x^3 - 6x^2 - 28x + 49)(x - 1)^2 \ge 0$$

Let $x = 2t/(1 + t)$, $t \ge 0$, the quantity $x^4 + 2x^3 - 6x^2 - 28x + 49$ becomes

$$\frac{t^4 - 4t^3 + 102t^2 + 140t + 49}{(1 + t)^4} \ge 0$$

because

$$t^4 + 4t^2 \ge 4t^3$$

and the result follows.

Also solved by Haosen Chen, Zhejiang, China; Ioannis D. Sfikas, Athens, Greece; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland.

U469. Let $x > y > z > t > 1$ be real numbers. Prove that

$$(x-1)(z-1)\ln y \ln t > (y-1)(t-1)\ln x \ln z.$$

Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Solution by Oana Prajitura, College at Brockport, SUNY, NY, USA

All quantities in the inequality are positive. Therefore the inequality is equivalent to

$$\frac{x-1}{\ln x} \cdot \frac{z-1}{\ln z} > \frac{y-1}{\ln y} \cdot \frac{t-1}{\ln t}$$

Let $f : (1, \infty) \to \mathbb{R}$ given by

$$f(x) = \frac{x-1}{\ln x}.$$

To prove the inequality it suffices to show that f is a strictly increasing function.

$$f'(x) = \frac{\ln x - \frac{x-1}{x}}{(\ln x)^2} = \frac{x ln x - x + 1}{x(\ln x)^2}$$

The denominator of the last fraction is strictly greater than 0. I need to show that the same is true for the numerator.

Let $g : (0, \infty) \to \mathbb{R}$, given by $g(x) = x \ln x - x + 1$. Then $g'(x) = \ln x + 1 - 1 = \ln x > 0$ on $(1, \infty)$. Therefore for $x \in (1, \infty)$

$$g(x) > \lim_{x \to 1} g(x) = g(1) = 0$$

which completes the proof.

Also solved by Dumitru Barac, Sibiu, Romania; Akash Singha Roy, Chennai Mathematical Institute, India; Ioannis D. Sfikas, Athens, Greece; Thiago Landim de Souza Leão, Federal University of Pernambuco, Recife, Brazil; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Arkady Alt, San Jose, CA, USA.

U470. Let n be a positive integer. Evaluate

$$\lim_{x\to 0}\frac{1-\cos^n x\cos nx}{x^2}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Thiago Landim de Souza Leão, Federal University of Pernambuco, Recife, Brazil
Since $\cos x = 1-\frac{x^2}{2}+O(x^4)$ and since $(1+x)^n = 1+nx+O(x^2)$,

$$\cos nx = 1-\frac{n^2x^2}{2}+O(x^4)$$

and

$$\cos^n x = (1-\frac{x^2}{2}+O(x^4))^n = 1-n\frac{x^2}{2}+O(x^4).$$

Therefore,

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos^n x\cos nx}{x^2} &= \lim_{x\to 0}\frac{1-(1-n\frac{x^2}{2}+O(x^4)(1-\frac{n^2x^2}{2}+O(x^2)}{x^2}\\ &= \lim_{x\to 0}\frac{1-1+\frac{n(n+1)}{2}x^2+O(x^4)}{x^2}\\ &= \frac{n(n+1)}{2}.\end{aligned}$$

Also solved by Daniel López-Aguayo, Instituto Tecnológico y de Estudios Superiores de Monterrey, Campus Monterrey, Mexico; Dumitru Barac, Sibiu, Romania; Prajnanaswaroopa S, Amrita University, Coimbatore, India; Akash Singha Roy, Chennai Mathematical Institute, India; Lukas Seier, Charters Sixth Form, Sunningdale, England, UK; Albert Stadler, Herrliberg, Switzerland; Aenakshee Roy, FIITJEE Chembur, Mumbai, India; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Corneliu Mănescu-Avram, Ploiești, Romania; G. C. Greubel, Newport News, VA, USA; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Matthew Too, College at Brockport, SUNY, NY, USA; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania; Oana Prajitura, College at Brockport, SUNY, NY, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Pradyumna Atreya, Mumbai, India; Santosh Kumar Mvrk, Hyderabad, India; Sebastian Foulger, Charters Sixth Form, Sunningdale, England, UK.

U471. Let $f(x) = ax^2 + bx + c$, where $a < 0 < b$ and $b\sqrt[3]{c} \geq \frac{3}{8}$. Prove that

$$f\left(\frac{1}{\Delta^2}\right) \geq 0$$

where $\Delta = b^2 - 4ac$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

So b, c, and Δ are positive and it suffices to prove that

$$a + b(\Delta^2) + c(\Delta^2)^2 >= 0.$$

Let $g(x) = cx^2 + bx + a$, having the same discriminant Δ and roots x_1 and $x_2, x_1 \leq x_2$. It suffices to prove that

$$\Delta^2 \geq x_2 = \frac{-b + \sqrt{\Delta}}{2c}.$$

This inequality rewrites

$$\Delta^2 + \frac{b}{6c} + \frac{b}{6c} + \frac{b}{6c} \geq \frac{\sqrt{\Delta}}{2c}$$

and follows by the AM-GM Inequality and the condition $b\sqrt[3]{c} \geq \dfrac{3}{8}$.

Second solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy

$$f\left(\frac{1}{\Delta^2}\right) = \frac{c\Delta^4 + b\Delta^2 + a}{\Delta^4} \geq 0 \iff c\Delta^4 + b\Delta^2 + a$$

that is

$$\Delta^2 \leq \frac{-b - \sqrt{b^2 - 4ac}}{c} \quad \text{or} \quad \Delta^2 \geq \frac{-b + \sqrt{b^2 - 4ac}}{c}$$

Since $c > 0$ by $b > 0$ and $b\sqrt[3]{c} \geq \frac{3}{8}$, $\Delta^2 \leq \frac{-b-\sqrt{b^2-4ac}}{c}$ is impossible. Instead

$$\Delta^2 \geq \frac{-b + \sqrt{b^2 - 4ac}}{c} \iff 2c\Delta^2 + b \geq \sqrt{\Delta}$$

$$2c\Delta^2 + b = 2c\Delta^2 + \frac{b}{3} + \frac{b}{3} + \frac{b}{3} \underbrace{\geq}_{AGM} \frac{4}{3^{\frac{3}{4}}}\left(2c\Delta^2 b^3\right)^{\frac{1}{4}} \geq \sqrt{\Delta}$$

if and only if

$$2^{\frac{9}{4}} c^{\frac{1}{4}} b^{\frac{3}{4}} \frac{1}{3^{\frac{3}{4}}} \geq 1 \iff 2^3 c^{\frac{1}{3}} \frac{b}{3} \geq 1$$

that is the result.

Also solved by Ioannis D. Sfikas, Athens, Greece; Albert Stadler, Herrliberg, Switzerland.

U472. If $f, g, h : \mathbb{R} \to \mathbb{R}$ are derivatives, find whether or not the function $\max\{f, g, h\}$ is the derivative of a function.

Proposed by Dorin Andrica, Babeș-Bolyai University, Cluj-Napoca, Romania

Solution by Albert Stadler, Herrliberg, Switzerland

We claim that the function $\max\{f, g, h\}$ is not necessarily the derivative of a function. We will construct a counterexample as follows:

Let $F(x) = x^2 \sin \frac{1}{x^2}$, if $x \neq 0$, and $F(0) = 0$. F is a differentiable function, since $F'(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}$, if $x \neq 0$, and $F'(0) = \lim_{t\to 0} \frac{F(t) - F(0)}{t} = \lim_{t\to 0} tsin\frac{1}{t^2} = 0$.

Put $f(x) = F'(x), g(x) = -F'(x), h(x) = 0$. Then

$$\max\{f, g, h\} = |f|$$

.

Suppose that $|f|$ is the derivative of a (differentiable) function V. Then $V(x)$ is given (up to a constant) by

$$V(x) = \int_1^x |f(t)|dt,$$

since $|f(x)|$ is continuous for $x \neq 0$. We will prove that $V(0) = \int_1^0 |f(t)|dt = -\infty$ which cannot hold if V is throughout differentiable. Indeed,

$$|f(x)| \geq \frac{2}{|x|} \left|\cos \frac{1}{x^2}\right| - 2|x|,$$

and therefore,

$$\int_0^1 |f(t)|dt \geq \sum_{k=1}^{\infty} \int_{\frac{1}{\sqrt{\pi k + \frac{\pi}{4}}}}^{\frac{1}{\sqrt{\pi k - \frac{\pi}{4}}}} |f(t)|dt \geq -1 + \sqrt{2} \sum_{k=1}^{\infty} \int_{\frac{1}{\sqrt{\pi k + \frac{\pi}{4}}}}^{\frac{1}{\sqrt{\pi k - \frac{\pi}{4}}}} \frac{dt}{t} = -1 + \frac{\sqrt{2}}{2} \sum_{k=1}^{\infty} \ln\left(\frac{\pi k + \frac{\pi}{4}}{\pi k - \frac{\pi}{4}}\right) =$$

$$-1 + \frac{\sqrt{2}}{2} \sum_{k=1}^{\infty} \ln\left(1 + \frac{1}{2k - \frac{1}{2}}\right) = \frac{\sqrt{2}}{2} \sum_{k=1}^{\infty} \frac{1}{2k - \frac{1}{2}} + O(1) = \infty.$$

Also solved by Ioannis D. Sfikas, Athens, Greece.

U473. For each continuous function $f : [0,1] \mapsto [0,\infty)$, let

$$I_f = \int_0^1 (2f(x) + 3x) f(x) dx$$

and

$$J_f = \int_0^1 (4f(x) + x)\sqrt{xf(x)} dx.$$

Find the minimum of $I_f - J_f$ over all such functions f.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Albert Stadler, Herrliberg, Switzerland

Set $g(x) = \sqrt{f(x)}$. We need to find the minimum of

$$\int_0^1 \left(2g^4(x) + 3xg^2(x) - 4\sqrt{x}g^3(x) - x\sqrt{x}g(x)\right) dx$$

over all continuous functions $g : [0,1] \mapsto [0,\infty)$. This is a variational problem whose solution is derived from the Euler-Lagrange-equation:

$$0 = \frac{\partial}{\partial g}\left(2g^4(x) + 3xg^2(x) - 4\sqrt{x}g^3(x) - x\sqrt{x}g(x)\right) =$$

$$8g^3(x) + 6xg(x) - 12\sqrt{x}g^2(x) - x\sqrt{x} = (2g(x) - \sqrt{x})^3.$$

We conclude that $g(x) = \frac{1}{2}\sqrt{x}$ and finally $f(x) = \frac{x}{4}$.

Also solved by Ioannis D. Sfikas, Athens, Greece.

U474. Let $f\colon[0,1]\to\mathbb{R}$ be a differentiable function such that $f(1)=0$ and

$$\int_0^1 x^n f(x)dx = 1$$

Prove that

$$\int_0^1 \left(f'(x)\right)^2 dx \geq (2n+3)(n+1)^2$$

When does the equality occur?

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy
Integrating by parts

$$1 = \int_0^1 x^n f(x)dx = \frac{x^{n+1}}{n+1}f(x)\Big|_0^1 - \int_0^1 \frac{x^{n+1}}{n+1}f'(x)dx = -\int_0^1 \frac{x^{n+1}}{n+1}f'(x)dx$$

Cauchy–Schwarz yields

$$1 = -\int_0^1 \frac{x^{n+1}}{n+1}f'(x)dx \leq \int_0^1 \left(\frac{x^{2n+2}}{(n+1)^2}dx\right)^{\frac{1}{2}} \left(\int_0^1 \left(f'(x)\right)^2 dx\right)^{\frac{1}{2}}$$

whence

$$\int_0^1 \left(f'(x)\right)^2 dx \geq (2n+3)(n+1)^2$$

Also solved by Stroe Octavian and Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Dumitru Barac, Sibiu, Romania; Akash Singha Roy, Chennai Mathematical Institute, India; Albert Stadler, Herrliberg, Switzerland; Jamal Gadirov, Ishik University, Iraq; Ioannis D. Sfikas, Athens, Greece; Joshua Siktar, Carnegie Mellon University, PA, USA; Thiago Landim de Souza Leão, Federal University of Pernambuco, Recife, Brazil; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania.

O469. Find the greatest constant k such that the following inequality holds for all positive real numbers a and b

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{k}{a^3+b^3}\geq\frac{16+4k}{(a+b)^3}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, CA, USA

Assuming $a+b=1$ (due homogeneity) and denoting $t:=ab\in(0,1/4]$ we obtain

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{k}{a^3+b^3}\geq\frac{16+4k}{(a+b)^3}\iff\frac{1-3t}{t^3}+\frac{k}{1-3t}\geq16+4k\iff$$

$$\frac{1-3t}{t^3}-16-4k+\frac{k}{1-3t}\geq0\iff\frac{(1-4t)\left(4t^2+t+1\right)}{t^3}-3k\cdot\frac{1-4t}{1-3t}\geq0\iff$$

$$\frac{(1-4t)\left((1-3t)\left(4t^2+t+1\right)-3kt^3\right)}{(1-3t)\,t^3}\geq0$$

Since $\dfrac{1-4t}{t^3(1-3t)}>0$ for any $t\in\left(0,\dfrac{1}{4}\right)$ then

$$k\leq\frac{(1-3t)\left(4t^2+t+1\right)}{3t^3},\forall t\in(0,1/4)\iff k\leq\inf_{t\in(0,1/4)}\frac{(1-3t)\left(4t^2+t+1\right)}{3t^3}=8$$

because

$$\frac{(1-3t)\left(4t^2+t+1\right)}{3t^3}=\frac{1}{3t}\left(\frac{1}{t}-1\right)^2-4\leq\frac{1}{3\cdot1/4}\left(\frac{1}{1/4}-1\right)^2-4=8.$$

Thus greatest constant $k=8$ is maximal value of constant k such that inequality $\dfrac{1}{a^3}+\dfrac{1}{b^3}+\dfrac{k}{a^3+b^3}\geq\dfrac{16+4k}{(a+b)^3}$ holds for all positive a,b.

Also solved by Santosh Kumar Mvrk, Hyderabad, India; Dumitru Barac, Sibiu, Romania; Akash Singha Roy, Chennai Mathematical Institute, India; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Ploiești, Romania; Ioannis D. Sfikas, Athens, Greece; M.A,Prasad, Mumbai, India; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania; Titu Zvonaru, Comănești, Romania.

O470. Let a, b, c, x, y, z be nonnegative real numbers such that $a \geq b \geq c$, $x \geq y \geq z$ and

$$a + b + c + x + y + z = 6.$$

Prove that

$$(a + x)(b + y)(c + z) \leq 6 + abc + xyz.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

Rewrite the above inequality as

$$abz + bcx + cay + xyc + yza + zxb \leq 6.$$

I will prove the following inequality

$$[(a + b)z + (b + c)x + (c + a)y]^2 \geq 4(abz + bcx + cay)(x + y + z). \tag{1}$$

This is equivalent to

$$(a - b)^2 z^2 + (b - c)^2 x^2 + (c - a)^2 y^2 \geq 2(a - b)(b - c)zx + 2(b - c)(c - a)xy + 2(c - a)(a - b)yz,$$

or

$$[(a - b)z - (b - c)x + (c - a)y]^2 \geq 4(c - a)(a - b)yz$$

which is obviously true.

Similarly, if in (1) we take $(a, b, c, x, y, z) \longleftrightarrow (x, y, z, a, b, c)$, we get the following inequality

$$[(x + y)c + (y + z)a + (z + x)b]^2 \geq 4(xyc + yza + zxb)(a + b + c). \tag{2}$$

By (1) and (2) it follows that

$$\begin{aligned}
&abz + bcx + cay + xyc + yza + zxb \leq \\
\leq &\frac{[(a + b)z + (b + c)x + (c + a)y]^2}{4(x + y + z)} + \frac{[(x + y)c + (y + z)a + (z + x)b]^2}{4(a + b + c)} \\
= &\frac{[(a + b)z + (b + c)x + (c + a)y]^2 (a + b + c + x + y + z)}{4(x + y + z)(a + b + c)} \\
= &\frac{6\left[(a + b)z + (b + c)x + (c + a)y\right]^2}{4(x + y + z)(a + b + c)}.
\end{aligned}$$

But,

$$3\left[(a + b)z + (b + c)x + (c + a)y\right] \leq 2(a + b + c)(x + y + z)$$

because it reduces to

$$(a + b - 2c)z + (b + c - 2a)x + (c + a - 2b)y \leq 0,$$

or

$$(b - c)z - (c - a)z + (c - a)x - (a - b)x + (a - b)y - (b - c)y \leq 0,$$

or

$$(a - b)(x - y) + (b - c)(y - z) + (c - a)(z - x) \geq 0,$$

obviously true.

Hence, using this result and AM-GM Inequality,

$$\begin{aligned} abz + bcx + cay + xyc + yza + zxb &\le \frac{2(a+b+c)(x+y+z)}{3} \\ &\le \frac{2(a+b+c+x+y+z)^2}{12} \\ &= 6. \end{aligned}$$

Equality holds if and only if $a = b = c$ and $x = y = z$.

Also solved by M.A,Prasad, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

O471. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that for all real numbers x, y, z the following inequality holds

$$ayz + bzx + cxy \le x^2 + y^2 + z^2.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

From the relation

$$a^2 + b^2 + c^2 + abc = 4$$

we deduce that there exist an acute triangle ABC such that

$$a = 2\cos A, b = 2\cos B, c = 2\cos C.$$

Then the inequality becomes

$$2yz\cos A + 2zx\cos B + 2xy\cos C \le x^2 + y^2 + z^2.$$

This is equivalent to

$$(z - y\cos A - x\cos B)^2 + (y\sin A - x\sin B)^2 \ge 0$$

which is obvious and we are done.

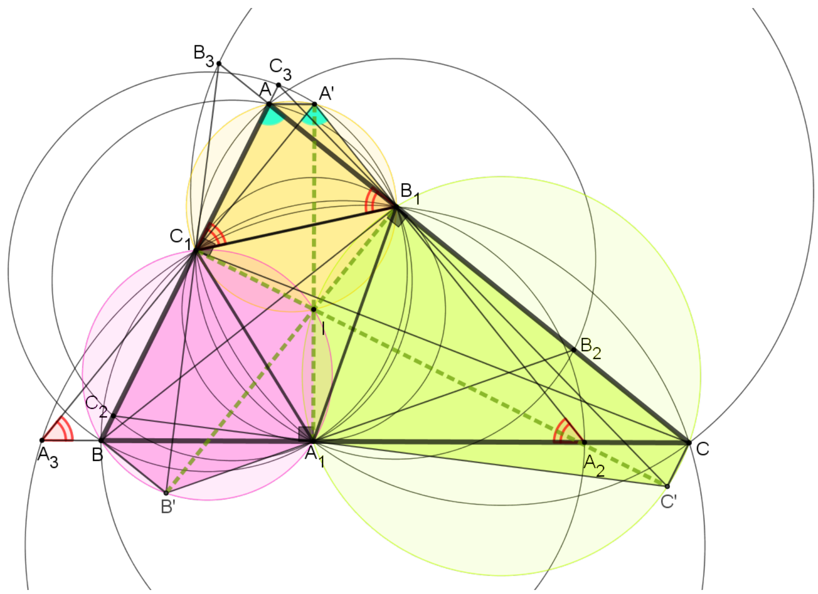
Also solved by Dumitru Barac, Sibiu, Romania; Akash Singha Roy, Chennai Mathematical Institute, India; Ioannis D. Sfikas, Athens, Greece; M.A,Prasad, Mumbai, India; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Nicusor Zlota ,"Traian Vuia" Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

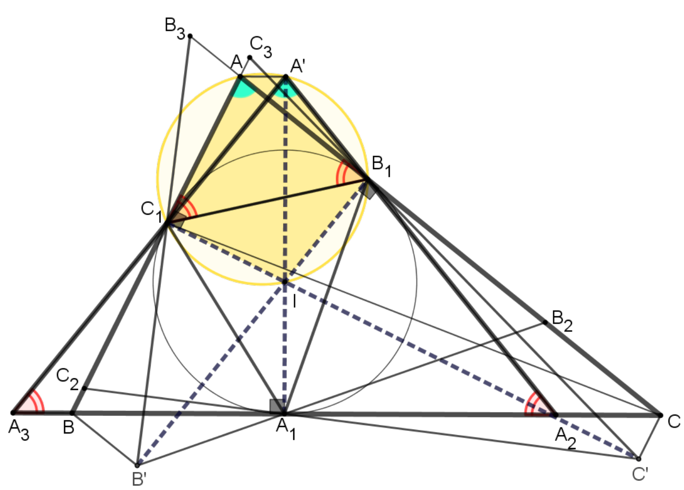
O472. Let $\triangle ABC$ be an acute triangle and A_1, B_1, C_1 the tangency points between BC, AC, AB and ABC incircle. Circumcircles of $\triangle BB_1C_1, \triangle CB_1C_1$ cut BC in A_2, respectively A_3, analogously, circumcircles of $\triangle AB_1A_1, \triangle BA_1B_1$ cut AB in C_2, C_3 and circumcircles of $\triangle AC_1A_1, \triangle CC_1A_1$ cut AC in C_2, C_3. If $A_2B_1 \cap A_3C_1 = (A'\}, B_2A_1 \cap B_3C_1 = \{B'\}, C_2A_1 \cap C_3B_1 = \{C'\}$, show that $A_1A',\ B_1B',\ C_1C'$ are concurrent lines.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution by the author

Denote by I the triange ABC incentre





$IA_1 \perp BC,\ IB_1 \perp AC,\ IC_1 \perp AB \Rightarrow IB_1AC_1$ is inscribed in the circle with AI diameter.

$AB_1 \equiv AC_1$ (tangents from A to ABC incircle)$\Rightarrow$ $\triangle AC_1B_1$ is an isosceles triangle $\Rightarrow$ $\measuredangle AC_1B_1 \equiv \measuredangle AB_1C_1 = \dfrac{180^\circ - \measuredangle A}{2} = 90^\circ - \dfrac{\measuredangle A}{2}$

- Show that $A'A_2A_3$ is an isisceles triangle. (Analogously $\triangle B'B_2B_3$ and $\triangle C'C_2C_3$ are isosceles triangles).

$$\left.\begin{array}{l} B, A_2, B_1, C_1 = \text{concyclic points} \Rightarrow \measuredangle A_3A_2B_1 \equiv \measuredangle B_1C_1A(\text{suplement} \measuredangle B_1C_1B) \\ C, B_1, C_1, A_3 = \text{concyclic points} \Rightarrow \measuredangle C_1A_2A_3 \equiv \measuredangle C_1B_1A(\text{suplement} \measuredangle C_1B_1C) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \measuredangle A'A_3A_2 \equiv \measuredangle A'A_2A_3 = 90^\circ - \frac{\measuredangle BAC}{2} \Rightarrow \measuredangle A_3A'A_2 \equiv \measuredangle BAC$$

So, A', A, C_1, I, B_1 are concyclic points on the circle with AI diameter. $\Rightarrow IC_1 = IB_1 \Rightarrow \measuredangle IA'C_1 = \measuredangle IA'B_1$

- Show that A', I, A_1 are collinear points (analogously B', I, B_1 and C', I, C_1)
In the isosceles triangle $A_3A'A_2$, $A'I$ is angle bisector $\Rightarrow A'I \perp BC$
From $A'I \perp BC$ and $IA_1 \perp BC \Rightarrow A', I, A_1$ = collinear points.

- Show that $A_1A' \cap B_1B' \cap C_1C' = \{I\}$

$$\left.\begin{array}{l} I \in A_1A' \\ I \in B_1B' \\ I \in C_1C' \end{array}\right\} \Rightarrow A_1A' \cap B_1B' \cap C_1C' = \{I\}$$

Also solved by Shuborno Das, Ryan International School, Bangalore, India; Loreta Arzumanyan, "Quantum" College, Armenia.

O473. Let x, y, z be positive real numbers such that $x^6 + y^6 + z^6 = 3$. Prove that

$$x + y + z + 12 \geq 5(x^6y^6 + y^6z^6 + z^6x^6).$$

Proposed by Hoan Le Nhat Tung, Hanoi, Vietnam

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

We rewrite the inequality as

$$x + y + z + 12 \geq \frac{5}{2}[(x^6 + y^6 + z^6)^2 - (x^{12} + y^{12} + z^{12})]$$

or equivalently

$$5(x^{12} + y^{12} + z^{12}) + 2(x + y + z) \geq 21.$$

Now we use the AM-GM inequality to obtain

$$\underbrace{x^{12} + \cdots + x^{12}}_{15} + \underbrace{x + \cdots + x}_{6} + \underbrace{1 + \cdots + 1}_{10} \geq 31\sqrt[31]{(x^{12})^{15}x^6} = 31x^6.$$

Similarly

$$15y^{12} + 6y + 10 \geq 31y^6,$$

$$15z^{12} + 6z + 10 \geq 31z^6.$$

Adding these three inequalities we get

$$15(x^{12} + y^{12} + z^{12}) + 6(x + y + z) + 30 \geq 31(x^6 + y^6 + z^6) = 93.$$

This yields

$$5(x^{12} + y^{12} + z^{12}) + 2(x + y + z) \geq 21.$$

The proof is completed. Equality occurs if and only if $x = y = z = 1$.

Also solved by Dumitru Barac, Sibiu, Romania; Loreta Arzumanyan, "Quantum" College, Armenia; Lukas Seier, Charters Sixth Form, Sunningdale, England, UK; M.A,Prasad, Mumbai, India; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland.

O474. Let $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_2x^2 + a_0$ be a polynomial with positive integer coefficients of degree $d \geq 2$. We define the sequence $(b_n)_{n\geq 1}$, where $b_1 = a_0$ and $b_n = P(b_{n-1})$, for all $n \geq 2$. Prove that for all $n \geq 2$, there is a prime p such that $p \mid b_n$ and p does not divide $b_1 \dots b_{n-1}$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

We shall argue by contradiction. Let there be a positive integer $n \geq 2$ that appears in prime decomposition of $b_1 \dots b_{n-1}$. Let q be an arbitrary prime divisor of b_n, then $b_n = q^r l$ where $\gcd(l, q) = 1$. Now, one can find that

$$b_{n+1} = P(b_n) = a_d(q^r l)^d + a_{d-1}(q^r l)^{d-1} + \cdots + a_2(q^r l)^2 + a_0 \equiv a_0 = b_1 \pmod{q^{1+r}}$$

Hence, one can inductively prove that for all i we have $b_{n+i} \equiv b_i \pmod{q^{1+r}}$. That is,

$$b_{n+i+1} = P(b_{n+i}) \equiv P(b_i) = b_{i+1} \pmod{q^{1+r}}.$$

Now, we find that $b_n \equiv b_{2n} \equiv \cdots \equiv b_{kn} \pmod{q^{1+r}}$. Since $v_p(b_n) = r$, we find that:

$$v_p(b_n) = v_p(b_{2n}) = \cdots = v_p(b_{kn}) = r.$$

By our assumption, there must exist an index $i, 1 \leq i \leq n-1$ such that b_i must be divisible by q. Repeating the same procedure yields to

$$v_p(b_i) = v_p(b_{2i}) = \dots$$

This implies that $v_p(b_i) = v_p(b_{ni}) = v_p(b_n) = r$. Then for any prime divisor of b_n, the exponent of it would be the same as exponent of p in prime decomposition of b_i for some $1 \leq i \leq n-1$. Hence,

$$b_n \mid b_1 \dots b_{n-1}.$$

Therefore, $b_n \leq b_1 \dots b_{n-1}$. But $b_n = P(b_{n-1}) > b_{n-1}^2$, thus, $b_{n-1} < \sqrt{b_n}$. Now,

$$b_{n-k} < \sqrt{b_{n-k+1}} < \sqrt[4]{b_{n-k+2}} < \cdots < b_n^{\frac{1}{2^k}}.$$

Yielding $0 < b_1 \dots b_{n-1} < b_n^{\frac{1}{2^{n-1}}} b_n^{\frac{1}{2^{n-2}}} \dots b_n^{\frac{1}{2}} = b_n^{\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}} < b_n$. Contradiction!

Note that the last inequality is true, because $\dfrac{1}{2} + \cdots + \dfrac{1}{2^{n-1}} < 1$.

Also solved by Prajnanaswaroopa S, Amrita University, Coimbatore, India; M.A,Prasad, Mumbai, India.