Linear Approximation Implies Unique Solution

Titu Andreescu, Marian Tetiva

The following (folkloric) problem was proposed in a Romanian TST in the year 1983:

Problem. Find all pairs of real numbers (p, q) such that the inequality

$$|\sqrt{1-x^2} - (px+q)| \leq \frac{\sqrt{2}-1}{2}$$

holds for any $x \in [0, 1]$.

The answer is $(p, q) = \left(-1, \frac{\sqrt{2}+1}{2}\right)$, that is, there exists precisely one solution. We considered this an interesting, worth to study fact, and tried to put it in a general context. So, in this note, we prove the following statement that generalizes the contest problem:

Proposition. Let a and b be two real numbers with $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ that has a second order derivative f'' on (a, b), such that $f''(x) < 0$ for every $x \in (a, b)$. Let c be the unique point in (a, b) for which

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

and let

$$A = \frac{(c-b)f(a) + (a-c)f(b) + (b-a)f(c)}{2(b-a)}.$$

Then there exists precisely one pair of real numbers (p, q), having the property that

$$|f(x) - (px+q)| \leq A$$

for every $x \in [a, b]$.

We start with two helping results.

Lemma 1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is also differentiable on (a, b), and let $c \in (a, b)$ be such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(that is, c is a point whose existence is ensured by Lagrange's mean value theorem). Also, let

$$A = \frac{(c-b)f(a) + (a-c)f(b) + (b-a)f(c)}{2(b-a)}$$

be defined as in the proposition, and let p be some real number (not necessarily connected with the statement of the proposition). Finally, define g by

$$g(x) = f(x) - px$$

for every $x \in [a, b]$. Then the following identities hold:

(a) $f(c) - f(a) - (c-a)f'(c) = f(c) - f(b) - (c-b)f'(c) = 2A$;

(b) $g(c) - g(a) - (c-a)f'(c) = g(c) - g(b) - (c-b)g'(c) = 2A$.

Proof. Of course, g is also differentiable on (a, b), with derivative $g'(x) = f'(x) - p$ for all $x \in (a, b)$. All equalities follow by straightforward computations. For (a) we have

$$f(c) - f(a) - (c-a)f'(c) = f(c) - f(a) - (c-a)\frac{f(b) - f(a)}{b - a} =$$

$$= \frac{(b-a)f(c)-(b-a)f(a)-(c-a)f(b)+(c-a)f(a)}{b-a} = 2A.$$

We proceed similarly with the second equality. Now (b) follows from (a) and the fact that

$$g(c) - g(a) - (c-a)g'(c) = f(c) - f(a) - (c-a)f'(c)$$

(and $g(c) - g(b) - (c-b)g'(c) = f(c) - f(b) - (c-b)f'(c)$), due to the definition of g and to $g'(c) = f'(c) - p$.

Lemma 2. Let $f : [a,b] \to \mathbb{R}$ be a continuous function having a negative second derivative on (a,b) (as in the statement of the proposition), and let c be the unique point in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Also, let A be defined as in the statement of the proposition, and let g_0 be defined by

$$g_0(x) = f(x) - xf'(c),$$

for all $x \in [a,b]$. Because g_0 (as f) is continuous, we can consider

$$m_0 = \min_{x\in[a,b]} g_0(x) \text{ and } M_0 = \max_{x\in[a,b]} g_0(x).$$

(a) We have

$$m_0 = g_0(c) \text{ and } M_0 = g(a) = g(b),$$

and, moreover,

$$M_0 - m_0 = 2A.$$

(b) Define

$$p_0 = f'(c) \text{ and } q_0 = M_0 - A = A + m_0.$$

Then

$$|f(x) - p_0x - q_0| \leq A$$

for every $x \in [a,b]$.

Proof. (a) The function g_0 (a particular case of the function g from lemma 1, of course) is twice differentiable on (a,b) and has the first and second order derivatives

$$g_0'(x) = f'(x) - f'(c), \ \ \forall x \in (a,b),$$

and

$$g_0''(x) = f''(x) < 0, \ \ \forall x \in (a,b),$$

respectively. Thus, the first derivative of g_0 is strictly decreasing and has c as a unique zero in (a,b). It follows that g_0' is positive on the interval (a,c) and negative on (c,b), which implies that g_0 is strictly increasing on (a,c) and strictly decreasing on (c,b). By continuity, the monotony actually extends to $[a,c]$ and $[c,b]$, and we conclude that $M_0 = g_0(c)$, while $m_0 = \min\{g_0(a), g_0(b)\}$. However, we have $g_0(a) = f(a) - af'c) = f(b) - bf'(c) = g_0(b)$ by the very definition of c, hence $m_0 = g_0(a) = g_0(b)$.

Consequently we have

$$M_0 - m_0 = f(c) - f(a) - (c-a)f'(c) = 2A,$$

according to part (a) from lemma 1.

(b) Indeed, with $p_0 = f'(c)$ we have $g_0(x) = f(x) - p_0x \geq m_0 = q_0 - A$ and $g_0(x) = f(x) - p_0x \leq M_0 = q_0 + A$, that is

$$-A \leq f(x) - p_0x - q_0 \leq A \Leftrightarrow |f(x) - p_0x - q_0| \leq A$$

for every $x \in [a,b]$, as desired. It is time now to see the

Proof of the proposition. Basically, part (b) of lemma 2 shows that there exists a pair (p_0, q_0) of real numbers that satisfies the required inequality. We still need to show that this is the only possible such pair. Suppose that, for some real numbers p and q, we have

$$|f(x) - px - q| \leq A \Leftrightarrow q - A \leq g(x) \leq q + A$$

for all $x \in [a, b]$, where g is defined by $g(x) = f(x) - px$, for all $x \in [a, b]$. As g is continuous on the compact interval $[a, b]$, we can consider

$$m = \min_{x\in[a,b]} g(x) \text{ and } M = \max_{x\in[a,b]} g(x).$$

Note that $|g(u) - g(v)| \leq M - m$, in particular $g(u) - g(v) \leq M - m$ for every $u, v \in [a, b]$. Clearly, the above required inequalities hold for any $x \in [a, b]$ if and only if we also have

$$q - A \leq m \leq M \leq q + A.$$

These imply $M - m \leq (q + A) - (q - A) = 2A$, therefore we must have

$$g(u) - g(v) \leq M - m \leq 2A$$

for every u and v in $[a, b]$. In particular

$$g(c) - g(a) \leq 2A = g(c) - g(a) - (c - a)g'(c)$$

(remember part (b) of lemma 1), which leads to

$$(c - a)g'(c) \leq 0 \Rightarrow g'(c) \leq 0.$$

Similarly

$$g(c) - g(b) \leq 2A = g(c) - g(b) - (c - b)g'(c)$$

implies $g'(c) \geq 0$. Thus we necessarily have $g'(c) = 0$, and this means $p = f'(c) = p_0$. So, g becomes the function g_0 from lemma 2, m becomes m_0, and M becomes M_0. Consequently we must have

$$q - A \leq m_0 \leq M_0 \leq q + A \Leftrightarrow M_0 - A \leq q \leq m_0 + A.$$

But $M_0 - A = A - m_0$ (lemma 2), thus the last inequalities are actually equalities:

$$q = M_0 - A = m_0 + A,$$

that is, q is forced to be q_0–and the proof is now complete.

Note that the winning pair (p_0, q_0) is given by

$$p_0 = f'(c) = \frac{f(b) - f(a)}{b - a}$$

and

$$q_0 = m_0 + A = \frac{bf(a) - af(b)}{b - a} + A,$$

as

$$m_0 = f(a) - af'(c) = f(a) - a\frac{f(b) - f(a)}{b - a} = \frac{bf(a) - af(b)}{b - a}.$$

Since we also have $q_0 = M_0 - A$, and $M_0 = f(c) - cf'(c)$, the equality

$$f(c) - cf'(c) - A = \frac{bf(a) - af(b)}{b - a} + A$$

follows (and it can be verified by direct computations, too).

Remarks. 1) Of course, the point c from the proposition exists due to Lagrange's mean value theorem, while its uniquenes is ensured by the strict monotony of the first derivative of

f–which follows from the fact that the second derivative is negative on (a, b). The number A is positive (as it should be), due to Jensen's inequality.

2) There is also a geometric explanation for the existence of only one pair (p, q) with the required properties. Namely, the inequality $|f(x) - (px + q)| \le A$ is equivalent to

$$f(x) - A \le px + q \le f(x) + A, \quad \forall x \in [a, b].$$

Thus the graph of the function $w(x) = px + q$ (a line segment) has to be situated in the closed region delimited by the graphs of $u(x) = f(x) - A$ and $v(x) = f(x) + A$, and the vertical lines $x = a$ and $x = b$. Since the line joining the endpoints of the graph of v (that is, the points having coordinates $(a, f(a) + A)$ and $(b, f(b) + A)$) has the equation

$$y = \frac{f(b) - f(a)}{b - a}x + \frac{bf(a) - af(b)}{b - a} + A,$$

one sees immediately that it coincides with the tangent to the graph of u at the point $(c, f(c))$, whose equation is

$$y = f'(c)x + f(c) - cf'(c) - A.$$

We keep, of course, the same notations as above, that is c is the point from Lagrange's mean value theorem (for the function f), and the equality

$$f(c) - cf'(c) - A = \frac{bf(a) - af(b)}{b - a} + A$$

has been checked before. Because f is a strictly concave function (having a negative second order derivative), so are u and v, and it is clear (geometrically speaking) that only

$$p = \frac{f(b) - f(a)}{b - a} = f'(c) \text{ and } q = \frac{bf(a) - af(b)}{b - a} + A$$

can be chosen in order to have

$$f(x) - A \le px + q \le f(x) + A \Leftrightarrow |f(x) - (px + q)| \le A,$$

for every $x \in [a, b]$. That is, the graph of w for $p = p_0$ and $q = q_0$ is the only line segment that can be fit between the graphs of u and v.

3) One can see that, in the original problem, we have $a = 0$, $b = 1$, and $f(x) = \sqrt{1 - x^2}$. The point c whose existence is ensured by Lagrange's theorem is $c = \frac{1}{\sqrt{2}}$, as one can easily check, thus $A = \frac{\sqrt{2} - 1}{2}$ follows (as it appears in the statement of the problem). The first and second derivatives of f are

$$f'(x) = \frac{-x}{\sqrt{1 - x^2}}$$

and

$$f''(x) = \frac{-1}{(1 - x^2)\sqrt{1 - x^2}}$$

respectively, and we have, indeed, $f''(x) < 0$ for all $x \in (0, 1)$. By applying the above formulae, one immediately sees that the only solution in this particular case is $p_0 = -1$, $q_0 = \frac{\sqrt{2} + 1}{2}$, as we said in the beginning.

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Titu Andreescu, University of Texas at Dallas
Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bîrlad, România