Junior problems

J475. Let ABC be a triangle with $\angle B$ and $\angle C$ acute and different from $45°$. Let D be the foot of the altitude from A. Prove that $\angle A$ is right if and only if

$$\frac{1}{AD-BD}+\frac{1}{AD-CD}=\frac{1}{AD}.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

First solution by Albert Stadler, Herrliberg, Switzerland

Let $AD=h, BD=x, CD=y$. The stated equation then reads as $h^2=xy$.
If $\angle A$ is right then by the right triangle altitude theorem or geometric mean theorem we have $h^2=xy$.

Suppose that $h^2=xy$. Clearly, $\cot\angle B=\dfrac{x}{h}, \cot\angle C=\dfrac{y}{h}$. Then

$$\cot(\angle B+\angle C)=\frac{\cot(\angle B)\cot(\angle C)-1}{\cot(\angle B)+\cot(\angle C)}=0,$$

implying that $\angle B+\angle C=\dfrac{\pi}{2}$ and thus $\angle A=\dfrac{\pi}{2}$.

Second solution by Joel Schlosberg, Bayside, NY, USA

$AD\neq BD$ since otherwise $\triangle ABD$ would be an isosceles right triangle with $\angle B=45°$; $AD\neq CD$ since otherwise $\triangle ACD$ would be a right triangle with $\angle C=45°$. Thus $\frac{1}{AD-BD}+\frac{1}{AD-CD}=\frac{1}{AD}$ is equivalent to

$$\begin{aligned}0&=AD(AD-CD)+AD(AD-BD)-(AD-BD)(AD-CD)\\&=AD^2-BD\cdot CD\end{aligned}$$

and thus to $AD:BD=CD:AD$. This in turn is equivalent to $\triangle BAD\sim\triangle ACD$ since both triangles have a right angle at vertex D; for the same reason it is further equivalent to $\angle BAD$ and $\angle ACD$ being complementary (since $\angle ACD$ is complementary to $\angle BAD$ iff it is congruent to $\angle ABD$) and thus to $\angle BAC$ being right.

Also solved by Polyahedra, Polk State College, FL, USA; Daniel Lasaosa, Pamplona, Spain; Yi Won Kim, Taft School in Watertown, CT, USA; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Santosh Kumar, Hyderabad, India; Titu Zvonaru, Comănești, Romania; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Akash Singha Roy, Chennai Mathematical Institute, India; Nikos Kalapodis, Patras, Greece; Joonsoo Lee, Dwight Englewood School, NJ, USA; Min Jung Kim, Boston, MA, USA; Sebastian Foulger, Charters Sixth Form, Sunningdale, England, UK.

J476. Let x, y, z be positive numbers such that $x + y + z \geq S$. Prove that

$$6\left(x^3 + y^3 + z^3\right) + 9xyz \geq S^3.$$

Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Solution by Daniel Lasaosa, Pamplona, Spain

Let $\sigma = x + y + z \geq S$, or it suffices to show that the LHS is not smaller than σ^3. In turn, this is equivalent to

$$5\left(x^3 + y^3 + z^3\right) + 3xyz \geq 3\left(x^2y + y^2z + z^2x + xy^2 + yz^2 + zx^2\right).$$

Schur's inequality may be written in the form

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + y^2z + z^2x + xy^2 + yz^2 + zx^2.$$

Moreover, the AM-GM inequality produces $x^3 + y^3 + z^3 - 3xyz \geq 0$, with equality iff $x = y = z$, which is also a sufficient condition for equality in Schur's inequality. Adding thrice the first inequality and twice the second, results in the proposed result. The conclusion follows, equality holds iff $x = y = z$.

Also solved by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Akash Singha Roy, Chennai Mathematical Institute, India; Nikos Kalapodis, Patras, Greece; Joonsoo Lee, Dwight Englewood School, NJ, USA; Min Jung Kim, Boston, MA, USA; Polyahedra, Polk State College, FL, USA; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; Arkady Alt, San Jose, CA, USA; Albert Stadler, Herrliberg, Switzerland; Aron Ban-Szabo, Budapest, Hungary; Dumitru Barac, Sibiu, Romania; Henry Ricardo, Westchester Area Math Circle, NY, USA; Ioannis D. Sfikas, Athens, Greece; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Pantelis.N, Junior High School Athens, Greece; Titu Zvonaru, Comănești, Romania.

J477. Find the area of a kite $ABCD$ with $AB - CD = \left(\sqrt{2}+1\right)\left(\sqrt{3}+1\right)$ and $11\angle A = \angle C$.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by the author

Let E be the intersection of diagonals AC and BD. Then $\angle EAB = (15/2)^\circ$, so $AE = BE\cot(15/2)^\circ$. But

$$\cot\left(\frac{x}{2}\right) = \frac{\cos(x/2)}{\sin(x/2)} = \frac{2\cos^2(x/2)}{2\sin(x/2)cos(x/2)} = \frac{1+\cos x}{\sin x},$$

so

$$\cot\left(\frac{15}{2}\right)^\circ = \frac{1+\cos(15)}{\sin(15)} = \frac{1+\frac{\sqrt{6}+\sqrt{2}}{4}}{\frac{\sqrt{6}-\sqrt{2}}{4}} = 2+\sqrt{2}+\sqrt{3}+\sqrt{6}.$$

It follows that

$$\frac{1}{2}(\sqrt{2}+1)(\sqrt{3}+1) = AE - BE = (2+\sqrt{2}+\sqrt{3}+\sqrt{6})BE - BE,$$

implying $BE = \dfrac{1}{2}$ and $AE = \dfrac{1}{2}(2+\sqrt{2}+\sqrt{3}+\sqrt{6})$.

Hence the area of the kite is $\dfrac{1}{2}(2+\sqrt{2}+\sqrt{3}+\sqrt{6})$.

Also solved by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland.

J478. Prove that in any triangle ABC the following inequality holds:

$$4\left({\ell_a}^2+{\ell_b}^2+{\ell_c}^2\right)\le(a+b+c)^2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, CA, USA

We have that

$$l_a=\frac{2bc}{b+c}\cos\frac{A}{2}=\frac{2bc}{b+c}\sqrt{\frac{s(s-a)}{bc}}=\frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)}\le\sqrt{s(s-a)}.$$

Therefore,

$$l_a\le\sqrt{s(s-a)}$$

and similarly $l_b\le\sqrt{s(s-b)}$ and $l_c\le\sqrt{s(s-c)}$.

Hence,

$$4(l_a^2+l_b^2+l_c^2)\le 4s(3s-a-b-c)=4s^2=(2s)^2=(a+b+c)^2.$$

Also solved by Telemachus Baltsavias, Keramies Junior High School, Kefalonia, Greece; Polyahedra, Polk State College, USA; Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Akash Singha Roy, Chennai Mathematical Institute, India; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; Arkady Alt, San Jose, CA, USA; Albert Stadler, Herrliberg, Switzerland; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Aron Ban-Szabo, Budapest, Hungary; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Ioan Viorel Codreanu, Satulung, Maramureş, Romania; Ioannis D. Sfikas, Athens, Greece; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania; Kevin Soto Palacios, Huarmey, Perú.

J479. Let a, b, c be nonzero real numbers, not all equal, such that

$$\left(\frac{a^2}{bc}-1\right)^3+\left(\frac{b^2}{ca}-1\right)^3+\left(\frac{c^2}{ab}-1\right)^3=3\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}-\frac{bc}{a^2}-\frac{ca}{b^2}-\frac{ab}{c^2}\right).$$

Prove that $a+b+c=0$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, USA

Let $x=\frac{a^2}{bc}$, $y=\frac{b^2}{ca}$, and $z=\frac{c^2}{ab}$. Then x, y, z are not all equal, $xy=\frac{ab}{c^2}$, $yz=\frac{bc}{a^2}$, $zx=\frac{ca}{b^2}$, and $xyz=1$. Thus, the given equation becomes

$$\begin{aligned}0&=x^3+y^3+z^3-3\left(x^2+y^2+z^2\right)-3xyz+3(xy+yz+zx)\\&=\frac{1}{2}(x+y+z-3)\left[(x-y)^2+(y-z)^2+(z-x)^2\right].\end{aligned}$$

Hence,

$$0=x+y+z-3=\frac{a^3+b^3+c^3-3abc}{abc}=\frac{(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]}{2abc},$$

so $a+b+c=0$.

Also solved by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Daniel Lasaosa, Pamplona, Spain; Akash Singha Roy, Chennai Mathematical Institute, India; Nikos Kalapodis, Patras, Greece; Joonsoo Lee, Dwight Englewood School, NJ, USA; Min Jung Kim, Boston, MA, USA; Arkady Alt, San Jose, CA, USA; Albert Stadler, Herrliberg, Switzerland; Aenakshee Roy, Mumbai, India; Dumitru Barac, Sibiu, Romania; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Santosh Kumar, Hyderabad, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania.

J480. Let m, n be integers greater than 1. Find the number of ordered systems $(a_1, a_2, \ldots, a_m)$ where a_i are nonnegative integers less than n and such that

$$a_1 + a_3 + \cdots \equiv a_2 + a_4 + \cdots \pmod{n+1}.$$

Proposed by Mircea Becheanu, Montreal, Canada

Solution by Daniel Lasaosa, Pamplona, Spain

For each ordered system $(a_1, a_2, \ldots, a_m)$, consider the number whose representation in base n is $\overline{a_1 a_2 \ldots a_m}$. Clearly, there is a one-to-one correspondence between the ordered systems of length m and the nonnegative numbers less than n^m. Now, the condition given in the problem statement is clearly equivalent to $\overline{a_1 a_2 \ldots a_m}$ being a multiple of $n+1$, since $n^{2u} \equiv 1 \pmod{n+1}$ and $n^{2u+1} \equiv -1 \pmod{n+1}$ for all nonnegative integer u. The problem is therefore equivalent to finding how many numbers in $\{0, 1, \ldots, n^m - 1\}$ are multiples of $n+1$.

- If m is even, then $n^m - 1$ is a multiple of $n+1$, or since 0 is also a multiple of $n+1$, the total number of such ordered systems is $\frac{n^m-1}{n+1} + 1$.
- If m is odd, then $n^m + 1$ is a multiple of $n+1$, or the total number of such ordered systems is $\frac{n^m+1}{n+1}$.

Both forms may be combined into one, namely that the total number of such ordered system for any m is

$$\left\lfloor \frac{n^m + n}{n+1} \right\rfloor.$$

Also solved by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Akash Singha Roy, Chennai Mathematical Institute, India; Sebastian Foulger, Charters Sixth Form, Sunningdale, England, UK; Albert Stadler, Herrliberg, Switzerland.

Senior problems

S475. Let a and b be positive real numbers such that

$$\frac{a^3}{b^2}+\frac{b^3}{a^2}=5\sqrt{5ab}.$$

Prove that

$$\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}=\sqrt{5}.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan
Let $x=\sqrt{\frac{a}{b}}$ and $y=x+\frac{1}{x}$, then we obtain

$$\begin{aligned}
&\frac{a^3}{b^2}+\frac{b^3}{a^2}=5\sqrt{5ab}\\
&\Leftrightarrow x^5+\frac{1}{x^5}=5\sqrt{5}\\
&\Leftrightarrow \left(x+\frac{1}{x}\right)\left(x^4-x^2+1-\frac{1}{x^2}+\frac{1}{x^4}\right)=5\sqrt{5}\\
&\Leftrightarrow y(y^4-5y^2+5)=5\sqrt{5}\\
&\Leftrightarrow (y-\sqrt{5})(y^4+\sqrt{5}y+5)=0.
\end{aligned}$$

Since $y^4+\sqrt{5}y+5>0$, $y=\sqrt{5}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Akash Singha Roy, Chennai Mathematical Institute, India; Joonsoo Lee, Dwight Englewood School, NJ, USA; Min Jung Kim, Boston, MA, USA; Arkady Alt, San Jose, CA, USA; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Joel Schlosberg, Bayside, NY, USA; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Oana Prajitura, College at Brockport, SUNY, NY, USA; Santosh Kumar, Hyderabad, India; Spyridon Kalloniatis, Shape American High School; Titu Zvonaru, Comănești, Romania; Kevin Soto Palacios, Huarmey, Perú.

S476. Prove that in any triangle ABC,

$$4\cos\frac{A+\pi}{4}\cos\frac{B+\pi}{4}\cos\frac{C+\pi}{4} \ge \sqrt{\frac{r}{2R}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, CA, USA

Noting that

$$4\prod_{cyc}\cos\frac{A+\pi}{4} = 4\prod_{cyc}\sin\left(\frac{\pi-A}{4}\right) = 4\prod_{cyc}\sin\frac{B+C}{4} = \sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-1$$

and

$$\sqrt{\frac{r}{2R}} = \sqrt{2\cdot\frac{r}{4R}} = \sqrt{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

we can rewrite original inequality as follows:

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-1 \ge \sqrt{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}. \qquad (1)$$

Let $\alpha := \dfrac{\pi-A}{2}, \beta := \dfrac{\pi-B}{2}, \gamma := \dfrac{\pi-C}{2}$.

Then $\alpha,\beta,\gamma \in \left(0,\dfrac{\pi}{2}\right), \alpha+\beta+\gamma=\pi$ and (1) $\Longleftrightarrow$

$$\cos\alpha+\cos\beta+\cos\gamma-1 \ge \sqrt{2\cos\alpha\cos\beta\cos\gamma}. \qquad (2)$$

Let $A_1B_1C_1$ be some triangle with angles α,β,γ and let s,R and r from now be, respectively, semiperimeter, circumradius and inradius of $\triangle A_1B_1C_1$.

Then, since $\cos\alpha+\cos\beta+\cos\gamma = 1+\dfrac{r}{R}$ and $\cos\alpha\cos\beta\cos\gamma = \dfrac{s^2-(2R+r)^2}{4R^2}$ we obtain that

$$(2) \iff \frac{r}{R} \ge \sqrt{\frac{s^2-(2R+r)^2}{2R^2}} \iff \frac{r^2}{R^2} \ge \frac{s^2-(2R+r)^2}{2R^2} \iff$$

$$s^2-(2R+r)^2 \le 2r^2 \iff s^2 \le 4R^2+4Rr+3r^2 \text{ (Gerretsen's Inequality).}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Nikos Kalapodis, Patras, Greece; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Akash Singha Roy, Chennai Mathematical Institute, India; Ioannis D. Sfikas, Athens, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Kevin Soto Palacios, Huarmey, Perú.

S477. Let $ABCD$ be a kite inscribed in a circle such that

$$2AB^2 + AC^2 + 2AD^2 = 4BD^2.$$

Prove that $\angle A = 4\angle C$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Clearly, $AC = 2r$ is a diameter of the circle circumscribed to the kite, $\angle ABC = \angle CDA = 90°$, and denoting $\gamma = \angle DCA = \angle ACB$, we have $\angle BAC = \angle CAD = 90° - \gamma$, or denoting by P the intersection of both diagonals,

$$AB = AD = 2r\sin\gamma, \qquad BD = 2PB = 2AB\cos\gamma = 4r\sin\gamma\cos\gamma,$$

and the proposed condition rewrites as

$$0 = 1 - 12\sin^2\gamma + 16\sin^4\gamma.$$

Now, by the de Moivre formulae, any angle γ verifies that 5γ is an odd multiple of $\frac{\pi}{2}$ iff

$$0 = \cos(5\gamma) = \cos\gamma\left(\cos^4\gamma - 10\cos^2\gamma\sin^2\gamma + 5\sin^4\gamma\right) =$$

$$= \cos\gamma\left(1 - 12\sin^2\gamma + 16\sin^4\gamma\right),$$

or since $0 < \gamma < \frac{\pi}{2}$, we have $\gamma \in \left\{\frac{\pi}{10}, \frac{3\pi}{10}\right\}$. If $\gamma = \frac{\pi}{10}$, then $C = 2\gamma = \frac{\pi}{5}$ and $A = \pi - C = \frac{4\pi}{5} = 4C$, or it suffices to rule out the case $\gamma = \frac{3\pi}{10}$. Note that, in this case, $\angle BAD = \pi - 2\gamma = \gamma = \frac{2\pi}{5} < \frac{\pi}{2}$, we have $AB^2 = AD^2 > 2r^2$, or $2AB^2 + AC^2 + 2AD^2 > 12r^2$, whereas since $BD = 2r\sin(2\gamma) < 2r\sin 60° = \sqrt{3}r$, then $4BD^2 < 12r^2$, and the proposed condition is not met when $\gamma = \frac{3\pi}{10}$. The conclusion follows.

Also solved by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Akash Singha Roy, Chennai Mathematical Institute, India; Joonsoo Lee, Dwight Englewood School, NJ, USA; Min Jung Kim, Boston, MA, USA; Albert Stadler, Herrliberg, Switzerland; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Santosh Kumar, Hyderabad, India; Titu Zvonaru, Comănești, Romania.

S478. Let a, b, c be positive numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+4\left(\frac{a}{2a+b+c}+\frac{b}{a+2b+c}+\frac{c}{a+b+2c}\right)\geq\frac{9}{2}.$$

Proposed by Titu Zvonaru, Comăneşti, Romania

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA
Using the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{cyclic}\frac{a}{b+c}+4\sum_{cyclic}\frac{a}{2a+b+c} &= \sum_{cyclic}\frac{a^2}{a(b+c)}+\sum_{cyclic}\frac{4a^2}{2a^2+a(b+c)}\\ &\geq \frac{9(a+b+c)^2}{2(a^2+b^2+c^2)+4(ab+bc+ca)}\\ &= \frac{9(a+b+c)^2}{2(a+b+c)^2}=\frac{9}{2}.\end{aligned}$$

Also solved by Pantelis.N, Junior High School Athens, Greece; Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Akash Singha Roy, Chennai Mathematical Institute, India; Nikos Kalapodis, Patras, Greece; Arkady Alt, San Jose, CA, USA; Albert Stadler, Herrliberg, Switzerland; Adamopoulos Dionysios, 3rd High School, Pyrgos, Greece; Ioan Viorel Codreanu, Satulung, Maramureş, Romania; Ioannis D. Sfikas, Athens, Greece; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Maalav Mehta, Ahmedabad, India; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Kevin Soto Palacios, Huarmey, Perú.

S479. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers and let $(i_1, i_2, \ldots, i_n)$ be a permutation of the numbers $(1, 2, \ldots, n)$ such that $i_k \neq k$, for all $k = 1, 2, \ldots, n$. Prove that

$$a_1^n a_{i_1} + a_2^n a_{i_2} + \cdots + a_n^n a_{i_n} \geq a_1 a_2 \ldots a_n (a_1 + a_2 + \cdots + a_n).$$

Proposed by Mircea Becheanu, Montreal, Canada

Solution by the author

For convenience, we introduce the following notations: $P = a_1 a_2 \ldots a_n$ and $P_i = P/a_i$. The inequality is equivalent to the following:

$$\frac{a_1^n}{P_{i_1}} + \frac{a_2}{P_{i_2}} + \cdots + \frac{a_n}{P_{i_n}} \geq a_1 + a_2 + \cdots + a_n. \tag{1}$$

For every i, P_i is a product of $n-1$ factors and we can write it as

$$P_i = (\sqrt[n-1]{a_1 \ldots a_{i-1} a_{i+1} \ldots a_n})^{n-1}.$$

Now, we use the following lemma:

Lemma. For every positive integer $n > 1$ and every positive numbers $a_1, \ldots, a_k$, $b_1, \ldots, b_k$, the following inequality holds:

$$\frac{a_1^n}{b_1^{n-1}} + \cdots + \frac{a_k^n}{b_k^{n-1}} \geq \frac{(a_1 + \cdots + a_k)^n}{(b_1 + \cdots + b_k)^{n-1}}.$$

This lemma is more or less known and it can be proven elementary by induction on k, or by using Holder inequality.

Using Lemma in (1) we obtain the inequality:

$$\frac{a_1^n}{P_{i_1}} + \frac{a_2}{P_{i_2}} + \cdots + \frac{a_n}{P_{i_n}} \geq \frac{(a_1 + \cdots + a_n)^n}{(\sqrt[n-1]{P_{i_1}} + \cdots + \sqrt[n-1]{P_{i_n}})^{n-1}}. \tag{2}$$

For evrey i we have by AM-GM

$$\sqrt[n-1]{P_i} \leq \frac{a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n}{n-1}.$$

Using these in (2)we obtain

$$\frac{(a_1 + \cdots + a_n)^n}{(\sqrt[n-1]{P_{i_1}} + \cdots + \sqrt[n-1]{P_{i_n}})^{n-1}} \geq \frac{(a_1 + \cdots + a_n)^n}{(a_1 + \cdots + a_n)^{n-1}} = a_1 + \cdots + a_n. \tag{3}$$

Also solved by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan.

S480. Let a, b, c be positive real numbers. Prove that

$$\frac{a\left(a^3+b^3\right)}{a^2+ab+b^2}+\frac{b\left(b^3+c^3\right)}{b^2+bc+c^2}+\frac{c\left(c^3+a^3\right)}{c^2+ca+a^2} \geq \frac{2}{3}\left(a^2+b^2+c^2\right).$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania

First, we guess the equality holds when $a = b = c$. We want to find a result in the following form

$$\frac{a(a^3+b^3)}{a^2+ab+b^2} \geq ma^2+nb^2$$

This inequality is equivalent to

$$a^4+ab^3-(ma^4+ma^3b+ma^2b^2+na^2b^2+nab^3+nb^4) \geq 0 \Leftrightarrow$$

$$(1-m)a^4-ma^3b-(m+n)a^2b^2+(1-n)ab^3-nb^4 \geq 0$$

Consider the polynomial $f(a) = (1-m)a^4-ma^3b-(m+n)a^2b^2+(1-n)ab^3-nb^4$
We will find two real numbers m, n for which $a = b$ is a double root of $f(a)$.
This happens when $f'(a) = 4(1-m)a^3-3ma^2b-2(m+n)ab^2+(1-n)b^3$
$f(b) = 0$ results in $3m+3n = 2$
$f'(b) = 0$ results in $9m+3n = 5$
Therefore, $m = \frac{1}{2}, n = \frac{1}{6}$. We will show that $\frac{a(a^3+b^3)}{a^2+ab+b^2} \geq \frac{1}{2}a^2+\frac{1}{6}b^2$
Indeed, this inequality is equivalent to

$$3a^4-3a^3b-4a^2b^2+5ab^3-b^4 \geq 0 \Leftrightarrow (a-b)^2(3a^2+3ab-b^2) \geq 0$$

$$\frac{b(b^3+c^3)}{b^2+bc+c^2} \geq \frac{1}{2}b^2+\frac{1}{6}c^2$$

Similarly, we have

$$\frac{c(c^3+a^3)}{c^2+ca+a^2} \geq \frac{1}{2}c^2+\frac{1}{6}a^2$$

Summing up these relations concludes the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland; Ioannis D. Sfikas, Athens, Greece; Pantelis.N, Junior High School Athens, Greece.

Undergraduate problems

U475. Evaluate

$$\lim_{x\to 0}\frac{sin(x\sin x)+\sin(x\sin(x\sin x))}{x\sin(\sin x)+\sin(\sin(x\sin x))}$$

Proposed by Mircea Becheanu, Montreal, Canada

First solution by Thiago Landim de Souza Leão, Federal University of Pernambuco, Recife, Brazil
Since $\sin x = x + o(x)$, then $x\sin(\sin x) = \sin(x\sin x) = \sin(\sin(x\sin x)) = x^2 + o(x^2)$ and $\sin(x\sin(x\sin x)) = x^3 + o(x^3) = o(x^2)$. Therefore

$$\lim_{x\to 0}\frac{\sin(x\sin x)+\sin(x\sin(x\sin x))}{x\sin(\sin x)+\sin(\sin(x\sin x))} = \frac{x^2+o(x^2)}{2x^2+o(x^2)} = \frac{1}{2}.$$

Second solution by Mircea Becheanu, Montreal, Canada
We divide the numerator and denominator by x^2 and we obtain the following limit:

$$l = \lim_{x\to 0}\frac{\frac{\sin(x\sin x)}{x^2}+\frac{\sin(x\sin(x\sin x))}{x^2}}{\frac{x\sin(\sin x)}{x^2}+\frac{\sin(\sin(x\sin x))}{x^2}}$$

One evaluates separately:

$$\lim_{x\to 0}\frac{\sin(x\sin x)}{x^2} = \lim_{x\to 0}\frac{\sin(x\sin x)}{x\sin x}.\frac{x\sin x}{x^2} = 1,$$

$$\lim_{x\to 0}\frac{\sin(x\sin(x\sin x))}{x^2} = \lim_{x\to 0}\frac{\sin(x\sin(x\sin x))}{x\sin(x\sin x)}.\frac{x\sin(x\sin x)}{x\sin x}.\frac{x\sin x}{x^2} = 0,$$

$$\lim_{x\to 0}\frac{x\sin(\sin x)}{x^2} = \lim_{x\to 0}\frac{\sin(\sin x)}{x} = \lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}.\frac{\sin x}{x} = 1,$$

$$\lim_{x\to 0}\frac{\sin(\sin(x\sin x))}{x^2} = \lim_{x\to 0}\frac{\sin(\sin(x\sin x))}{\sin(x\sin x)}.\frac{\sin(x\sin x)}{x\sin x}.\frac{x\sin x}{x^2} = 1.$$

When we gather together these limits we have:

$$l = \frac{1+0}{1+1} = \frac{1}{2}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Akash Singha Roy, Chennai Mathematical Institute, India; Joonsoo Lee, Dwight Englewood School, NJ, USA; Min Jung Kim, Boston, MA, USA; Sebastian Foulger, Charters Sixth Form, Sunningdale, England, UK; Kevin Soto Palacios, Huarmey, Perú; Arkady Alt, San Jose, CA, USA; Albert Stadler, Herrliberg, Switzerland; Yi Won Kim, Taft School in Watertown, CT, USA; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Dumitru Barac, Sibiu, Romania; G. C. Greubel, Newport News, VA, USA; Henry Ricardo, Westchester Area Math Circle, NY, USA; Joel Schlosberg, Bayside, NY, USA; Ioannis D. Sfikas, Athens, Greece; Lukas Seier, Charters Sixth Form, Sunningdale, England, UK; Moubinool Omarjee, Lycée Henri IV, Paris, France; Oana Prajitura, College at Brockport, SUNY, NY, USA; Santosh Kumar, Hyderabad, India; Sarah B. Seales, Prescott, AZ, USA; Khakimboy Egamberganov, ICTP.

U476. Evaluate

$$\int \frac{x(x+1)(4x-5)}{x^5+x-1}dx$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note that $x^5+x-1=\left(x^3+x^2-1\right)\left(x^2-x+1\right)$, or

$$\frac{x(x+1)(4x-5)}{x^5+x-1}=\frac{3}{2}\cdot\frac{2x-1}{x^2-x+1}+\sqrt{3}\cdot\frac{2}{\sqrt{3}}\cdot\frac{1}{1+\left(\frac{2x-1}{\sqrt{3}}\right)^2}-\frac{3x^2+2x}{x^3+x^2-1}.$$

Therefore,

$$\int\frac{x(x+1)(4x-5)}{x^5+x-1}dx=\frac{3}{2}\log\left|x^2-x+1\right|+\sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)-\log\left|x^3+x^2-1\right|+C,$$

where C is an integration constant.

Also solved by Pedro Acosta De Leon, Massachusetts Institute of Technology, Cambridge, MA, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Akash Singha Roy, Chennai Mathematical Institute, India; Joonsoo Lee, Dwight Englewood School, NJ, USA; Min Jung Kim, Boston, MA, USA; Sebastian Foulger, Charters Sixth Form, Sunningdale, England, UK; Kevin Soto Palacios, Huarmey, Perú; Khakimboy Egamberganov, ICTP; Arkady Alt, San Jose, CA, USA; Albert Stadler, Herrliberg, Switzerland; Yi Won Kim, Taft School in Watertown, CT, USA; Corneliu Mănescu-Avram, Ploiești, Romania; G. C. Greubel, Newport News, VA, USA; Henry Ricardo, Westchester Area Math Circle, NY, USA; Ioannis D. Sfikas, Athens, Greece; Lukas Seier, Charters Sixth Form, Sunningdale, England, UK; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Oana Prajitura, College at Brockport, SUNY, NY, USA; Sarah B. Seales, Prescott, AZ, USA.

U477. Evaluate

$$\lim_{n\to\infty} \frac{\frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{n^2}}{\log\left(1+\frac{1}{n}\right)}$$

Proposed by Tiago Landim de Sousa Leão, University of Pernambuco, Brasil

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Let L be the proposed limit. We shall show that $L = 1$.

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{n\left(\frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{n^2}\right)}{n\log(1+\frac{1}{n})} \\ &= \lim_{n\to\infty} n\left(\frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{n^2}\right) \\ &= \lim_{n\to\infty} \frac{\frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{n^2}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n} - \frac{1}{n+1}} \quad \text{(by Stolz-Cezaro Lemma)} \\ &= 1 \end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Sebastian Foulger, Charters Sixth Form, Sunningdale, England, UK; Khakimboy Egamberganov, ICTP; Albert Stadler, Herrliberg, Switzerland; Dumitru Barac, Sibiu, Romania; G. C. Greubel, Newport News, VA, USA; Joel Schlosberg, Bayside, NY, USA; Ioannis D. Sfikas, Athens, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

U478. Let n be a positive integer. Prove that

$$\prod_{k=1}^{n}\left(1+\tan^4\frac{k\pi}{2n+1}\right)$$

is a sum of two perfect squares.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

Solution by Daniel Lasaosa, Pamplona, Spain

By the De Moivre formulae, we have

$$\sin\left((2n+1)\alpha\right)=\sin\alpha\cos^{2n}\alpha\sum_{u=0}^{n}\binom{2n+1}{2u+1}(-1)^u\tan^{2u}\alpha=\sin\alpha\cos^{2n}\alpha p\left(\tan^2\alpha\right),$$

where $p(x)$ is a polinomial with integer coefficients. Note that when $\alpha=\frac{k\pi}{2n+1}$ with $k=1,2,\dots,2n$, the LHS vanishes, or $\tan^2\frac{k\pi}{2n+1}$ is a root of $p(x)$ for $k=1,2,\dots,2n$. Moreover, k and $2n+1-k$ produce the same absolute value of $\tan\alpha$ with opposite signs, or $p(x)$ has exactly n roots, this n roots being equal to the n values of $\tan^2\frac{k\pi}{2n+1}$ for $k=1,2,\dots,n$. In other words, and since the coefficient of $\tan^{2n}\alpha$ is $(-1)^n$, we may write

$$p(x)=(-1)^n\prod_{k=1}^{n}\left(x-\tan^2\frac{k\pi}{2n+1}\right).$$

Taking the modulus squared of $p(i)$, where i is the imaginary unit, we obtain

$$|p(i)|^2=\prod_{k=1}^{n}\left|i-\tan^2\frac{k\pi}{2n+1}\right|=\prod_{k=1}^{n}\left(1+\tan^4\frac{k\pi}{2n+1}\right).$$

But $p(x)$ has integer coefficients, or $p(i)=r+is$ where r,s are integers, and indeed

$$\prod_{k=1}^{n}\left(1+\tan^4\frac{k\pi}{2n+1}\right)=|p(i)|^2=r^2+s^2.$$

The conclusion follows.

Also solved by Ioannis D. Sfikas, Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Akash Singha Roy, Chennai Mathematical Institute, India; Khakimboy Egamberganov, ICTP.

U479. Evaluate

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}\frac{x^{2(n+m)}}{(2n+2m)!}.$$

Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Because the series is absolutely convergent, we may sum terms in any order. Summing the terms with $n+m=k$ for increasing k yields

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+n}\frac{x^{2(n+m)}}{(2n+2m)!} &= \sum_{k=2}^{\infty}(-1)^k(k-1)\frac{x^{2k}}{(2k)!}\\
&= \frac{x}{2}\sum_{k=2}^{\infty}(-1)^k\frac{x^{2k-1}}{(2k-1)!}-\sum_{k=2}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}\\
&= -\frac{x}{2}\sum_{k=1}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}-\sum_{k=2}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}\\
&= -\frac{x}{2}(\sin x-x)-\left(\cos x-1+\frac{x^2}{2}\right)\\
&= 1-\cos x-\frac{x}{2}\sin x.
\end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Akash Singha Roy, Chennai Mathematical Institute, India; Joonsoo Lee, Dwight Englewood School, NJ, USA; Min Jung Kim, Boston, MA, USA; Khakimboy Egamberganov, ICTP; Albert Stadler, Herrliberg, Switzerland; G. C. Greubel, Newport News, VA, USA; Joe Simons, Utah Valley University, UT, USA; Ioannis D. Sfikas, Athens, Greece; Thiago Landim de Souza Leão, Federal University of Pernambuco, Recife, Brazil.

U480. Let

$$\begin{pmatrix} 4 & -3 & 2 \\ 15 & -10 & 6 \\ 10 & -6 & 3 \end{pmatrix}$$

Find the least possible n for which one entry of A^n is 2019.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan

$$\begin{aligned}
&\begin{pmatrix} 4 & -3 & 2 \\ 15 & -10 & 6 \\ 10 & -6 & 3 \end{pmatrix}^n \\
&= \frac{1}{4}\begin{pmatrix} 0 & 4 & 0 \\ 2 & 12 & 0 \\ 3 & 8 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}^n\begin{pmatrix} -6 & 2 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 2 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 0 & 4 & 0 \\ 2 & 12 & 0 \\ 3 & 8 & 2 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & (-1)^n & (-1)^{n-1}n \\ 0 & 0 & (-1)^n \end{pmatrix}\begin{pmatrix} -6 & 2 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 5(-1)^{n-1}n + (-1)^n & 3(-1)^n n & 2(-1)^{n-1}n \\ 15(-1)^{n-1}n & (-1)^n - 9(-1)^{n-1}n & 6(-1)^{n-1}n \\ 10(-1)^{n-1}n & -6(-1)^{n-1}n & 4(-1)^{n-1}n + (-1)^n \end{pmatrix}.
\end{aligned}$$

Since $2 \nmid 2019$ and $5 \nmid 2019$, it is sufficient to investigate 4 entries, $5(-1)^{n-1}n + (-1)^n, 3(-1)^n n, (-1)^n - 9(-1)^{n-1}n$ and $4(-1)^{n-1}n + (-1)^n$. After annoying calculation, we obtain that only when $n = 505$, 2019 appears as an entry of the matrix. Therefore the desired $n = 505$.

Second solution by the author

Let $A = B - I_3$. We notice that $B^2 = 0_3$, hence

$$A^n = (B - I_3)^n = (-1)^{(n-1)}nB + (-1)^n I_3.$$

The conclusion follows as in the first solution.

Also solved by Daniel Lasaosa, Pamplona, Spain; Joonsoo Lee, Dwight Englewood School, NJ, USA; Min Jung Kim, Boston, MA, USA; Sebastian Foulger, Charters Sixth Form, Sunningdale, England, UK; Khakimboy Egamberganov, ICTP; Albert Stadler, Herrliberg, Switzerland; Dumitru Barac, Sibiu, Romania; G. C. Greubel, Newport News, VA, USA; Joe Simons, Utah Valley University, UT, USA; Ioannis D. Sfikas, Athens, Greece; Lukas Seier, Charters Sixth Form, Sunningdale, England, UK; Oscar Brown, Charters School, Sunningdale, UK.

O475. Let a, b, c be nonnegative real numbers such that $\frac{a}{b+c} \geq 2$. Prove that

$$(ab+bc+ca)\left[\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right] \geq \frac{49}{18}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by Daniel Lasaosa, Pamplona, Spain

Taking the first derivative of the LHS with respect to a we find it to be

$$\frac{1}{b+c}-\frac{ab+bc+ca-b^2}{(a+b)^3}-\frac{ab+bc+ca-c^2}{(c+a)^3}.$$

Now, when $\frac{a}{b+c} \geq 2$, the first term is at least $\frac{2}{a} \geq \frac{1}{a+b}+\frac{1}{c+a}$, or the first derivative is at least

$$\frac{1}{a+b}-\frac{ab+bc+ca-b^2}{(a+b)^3}+\frac{1}{c+a}-\frac{ab+bc+ca-c^2}{(c+a)^3}=$$

$$=\frac{(a+b)(a-c)+2b^2}{(a+b)^3}+\frac{(c+a)(a-b)+2c^2}{(c+a)^3},$$

clearly non negative, being zero only when $a=b=c=0$, which results in the proposed expression not having a real value. It thus follows that we just need to prove the inequality when $a=2(b+c)$, and a necessary condition for equality is $a=2(b+c)$.

When $a=2(b+c)$, and denoting $s=b+c$ and $p=bc$, the inequality rewrites as

$$\frac{98s^6+69ps^4+12p^2s^2+p^3}{36s^6+12ps^4+p^2s^2} \geq \frac{49}{18},$$

which rewrites as

$$p\left(654s^4+167ps^2+18p^2\right) \geq 0,$$

clearly true and with equality iff $p=0$, ie iff $bc=0$. Note that since $a=2(b+c)$, if $b=c=0$ the proposed expression does not have a real value, or equality holds iff either $(a,b,c)=(2k,k,0)$ or $(a,b,c)=(2k,0,k)$, for any positive real k. The conclusion follows.

Also solved by Albert Stadler, Herrliberg, Switzerland; Akash Singha Roy, Chennai Mathematical Institute, India; Ioannis D. Sfikas, Athens, Greece.

O476. Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) + 11abc \le 12.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Corneliu Mănescu-Avram, Ploiești, Romania and Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania

Denote $a + b + c = p = 3, ab + bc + ca = q, abc = r$. After expanding, we can rewrite the given inequality as

$$P = (a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) = a^2b^2c^2 - abc\sum a^3 + \sum a^2b^2(a^2 + b^2) - \sum a^3b^3$$

$$P = r^2 - r(p^3 - 3pq + 3r) + (p^2 - 2q)(q^2 - 2pr) - 3r^2 - (q^3 - 3pqr + 3r^2) =$$
$$-8r^2 - 810r + 30qr - 3q^3 + 9q^2.$$

The inequality equivalent to

$$-8r^2 - 810r + 30qr - 3q^3 + 9q^2 + 11r - 12 \le 0 \text{ or}$$

$$8r^2 + 10r(7 - 3q) + 3(q^3 - 3q^2 + 4) \ge 0$$

Putting

$$f(r) = 8r^2 + 10r(7 - 3q) + 3(q^3 - 3q^2 + 4)$$

Consider two cases:

$$0 \le q \le \frac{35}{8} \Rightarrow r_{ct} \le 0. \tag{1}$$

Then $f(0) = 3(q^3 - 3q^2 + 4) = 3(q - 2)^2(q + 1) \ge 0$

$$\Delta = 100(7 - 3q)^2 - 96(q^3 - 3q^2 + 4) = 4(-24q^3 + 297q^2 - 1050q + 1127) < 0. \tag{2}$$

So, $f(r) \ge 0$ and the conclusion follows.

Also solved by Albert Stadler, Herrliberg, Switzerland; Akash Singha Roy, Chennai Mathematical Institute, India; Ioannis D. Sfikas, Athens, Greece.

O477. Solve in integer numbers the equation

$$(x^2-3)(y^3-2)+x^3=2(x^3y^2+2)+y^2.$$

Proposed by Konstantinos Metaxas, Athens, Greece

Solution by the author

The only solutions of the Diophantine equation

$$(x^2-3)(y^3-2)+x^3=2(x^3y^2+2)+y^2. \tag{1}$$

are $(x,y)=(-2,-1)$ and $(x,y)=(1,-1)$.

Assume y is even. Then x is even by (3), which according to (3) implies $2^2|2$. This contradiction means y is odd. Equation (1) is equivalent to

$$(2y^2-1)x^3+(2-y^3)x^2+3y^3+y^2-2=0 \tag{2}$$

and

$$(x^2-3)y^3-(2x^3+1)y^2+x^3-2x^2+2=0. \tag{3}$$

If $x=0$, then $3y^3+y^2=2$, yielding $y^2|2$, implying $y=\pm1$ and $2=3(\pm1)^3+(\pm1)^2=\pm3+1$, which is impossible. Hence $x\neq0$, meaning $|x|,|y|\geq1$ (since y is odd). According to (2) and (3) we have

$$x^2|3y^3+y^2-2 \tag{4}$$

and

$$y^2|x^3-2x^2+2. \tag{5}$$

Apllying (2) - (5), we

nd that the only solutions of equation (1) when $|x|,|y|\leq4$ are $(x,y)=(-2,-1)$ and $(x,y)=(1,-1)$. Assume $|x|,|y|\geq5$. By (4) and (5) we obtain

$$x^2\leq|3y^3+y^2-2|<3|y|^3+|y|^2<4|y|^3$$

and

$$y^2\leq|x^3-2x^2+2|<|x|^3+2|x|^2<2|x|^3$$

respectively, implying

$$|x|<2|y|^{\frac{3}{2}} \tag{6}$$

and

$$|y|<\sqrt{2}|x|^{\frac{3}{2}}. \tag{7}$$

Now, equation (1) can be expressed as

$$(y-2x)(xy)^2=-x^3+2x^2+3y^3+y^2-2. \tag{8}$$

Dividing both sides of equation (8) with $(xy)^2$, the result is

$$y-2x=-\frac{x}{y^2}+\frac{3y}{x^2}+\frac{1}{x^2}+\frac{2}{y^2}-\frac{2}{(xy)^2}$$

yielding

$$|y-2x|<\frac{|x|}{|y|^2}+\frac{3|y|}{|x|^2}+\frac{1}{|x|^2}+\frac{2}{|y|^2} \tag{9}$$

yielding

$$|y-2x|<3. \tag{10}$$

Now, $y-2x$ is odd since y is odd. Hence $y-2x=\pm1$ by (10), i.e. $y=2x\pm1$, which inserted in (8) result in

$$\pm x^2(2x\pm1)^2=-x^3+2x^2+3(2x\pm1)^3+(2x\pm1)^2-2 \tag{11}$$

Consequently by (11)

$$x|\pm3+1-2=-1\pm3$$

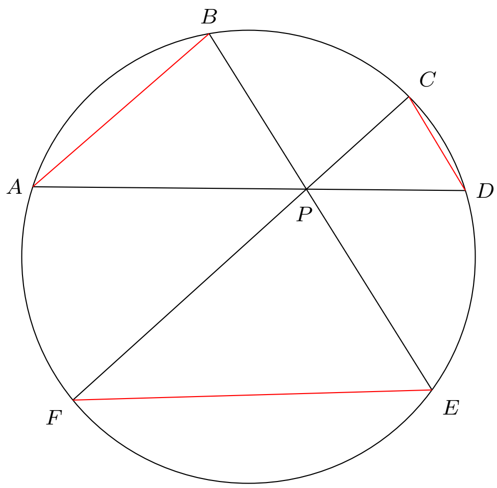
meaning $|x|\leq3+1=4$. This contradiction (since $|x|\geq5$) means there are no solutions to equation (1) when $|x|,|y|\geq5$. Conclusion: Equation (1) has exactly two solutions, namely $(x,y)=(-2,-1)$ and $(x,y)=(1,-1)$.

O478. Let $ABCDEF$ be a cyclic hexagon inscribed in a circle of radius 1. Suppose that the diagonals AD, BE, CF are concurrent in the point P. Prove that

$$AB + CD + EF \leq 4.$$

Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

Solution by the author



Suppose w.l.o.g. that $EF \geq \max\ \{AB, CD\}$. Then $EF \geq AB$ and $EF \geq CD$.

Consider the case $EF \parallel AD$; see Figure 1. Since $AB \leq EF$, we have arc $AB \leq$ arc EF implying $\alpha \leq \gamma$ and hence

$$\begin{aligned} \delta &= \alpha + \beta \quad \text{(by the external angle theorem applied to } \triangle AEP \text{ at } P) \\ &\leq \gamma + \beta. \end{aligned}$$

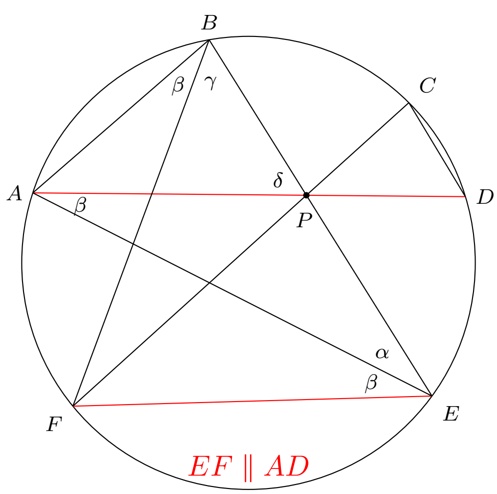


Figure 1

Thus

$$\delta \leq \gamma + \beta$$

which implies (in $\triangle ABP$)

$$AB \leq AP.$$

Similarly,

$$CD \leq PD.$$

Therefore,

$$AB + CD \le AP + PD = AD \le 2,$$

where the last inequality holds because $AD \le$ diameter.

Next consider that the points A, B, C, and D are fixed and points P, E, and F are variable; see Figure 2. Then EF has its maximum value when $EF \parallel AD$.

Indeed, we observe that the length of EF is maximum when θ is maximum i.e., when ϕ $(= \theta + \angle BFC)$ is maximum, that is, when AD is tangent at P to the circumcircle of $\triangle BPC$. When this happens we have $\beta = \gamma$, and therefore $\alpha = \beta = \gamma = \delta$.

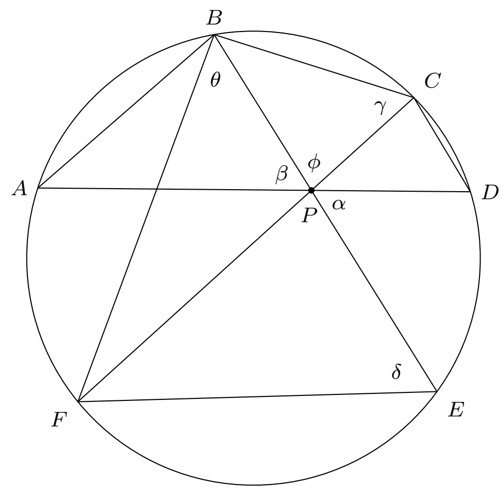


Figure 2

Finally we may choose points E' and F' on the given unit circle so that $E'F'$ is parallel to AD and $E'B$ meets $F'C$ at P' on AD.

By the above, it follows that

$$EF \le E'F'.$$

Because $E'F' \le$ diameter, we have the desired result

$$AB + CD + EF \le (AB + CD) + E'F' \le 2 + 2 = 4.$$

O479. Let $ABCD$ be a quadrilateral with $AB = CD = 4$, $AD^2 + BC^2 = 32$ and $\angle(ABD) + \angle(BDC) = 51°$. If $BD = \sqrt{6} + \sqrt{5} + \sqrt{2} + 1$, find AC.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Daniel Lasaosa, Pamplona, Spain

First, note that

$$\cos^2(15°) = \frac{1+\cos(30°)}{2} = \frac{2+\sqrt{3}}{4} = \frac{8+4\sqrt{3}}{16} = \frac{\left(\sqrt{6}\right)^2 + \left(\sqrt{2}\right)^2 + 2\sqrt{6}\sqrt{2}}{16} =$$
$$= \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2.$$

Note also that for $\alpha = 18°$, we have by the de Moivre formulae that

$$0 = \cos(90°) = \cos(5\alpha) = \cos\alpha\left(1 - 12\sin^2\alpha + 16\sin^4\alpha\right),$$

$$\sin^2\alpha = \frac{6 \pm 2\sqrt{5}}{16} = \frac{\left(\sqrt{5}\right)^2 + 1^2 \pm 2\sqrt{5}}{16} = \left(\frac{\sqrt{5}\pm 1}{4}\right)^2.$$

Now, of both solutions of $\cos(5\alpha) = 0$ with positive sine, the least angle, and thus the least sine, corresponds to $\alpha = 18°$, or

$$\cos(36°) = 1 - 2\sin^2(18°) = 1 - 2\frac{3-\sqrt{5}}{8} = \frac{\sqrt{5}+1}{4}.$$

It follows that $BD = 4\cos(36°) + 4\cos(15°)$.

Denote $\beta = \angle ABD$ and $\delta = \angle BDC$, or by the Cosine Law,

$$AD^2 = 16 + BD^2 - 8BD\cos\beta, \qquad BC^2 = 16 + BD^2 - 8BD\cos\delta,$$

or adding and using that $AD^2 + BC^2 = 32$, we have

$$2\cos\frac{\beta+\delta}{2}\cos\frac{\beta-\delta}{2} = \cos\beta + \cos\delta = \frac{BD}{4} = \cos(36°) + \cos(15°) = 2\cos\frac{51°}{2}\cos\frac{21°}{2},$$

and since wlog $\beta \geq \delta$ (because we may interchange A, C and simultaneously B, D without altering the problem), and as $\beta + \delta = 36° + 15°$, we have $\beta - \delta = 36° - 15°$, or $\beta = 36°$ and $\delta = 15°$.

Denote $\alpha = \angle ADB$. By the Sine Law and the Cosine Law,

$$\sin\alpha = \frac{4\sin(36°)}{AD}, \qquad \cos\alpha = \frac{BD^2 + AD^2 - 16}{2AD \cdot BD},$$

while at the same time $\sin(15°) = \frac{\sqrt{6}-\sqrt{2}}{4}$ so that $\sin^2(15°) + \cos^2(15°) = 1$. Therefore,

$$\cos\angle ADC = \cos(15°)\cos\alpha - \sin(15°)\sin\alpha = \frac{2+\sqrt{3}}{AD} - \frac{\left(\sqrt{3}-1\right)\sqrt{5-\sqrt{5}}}{2AD},$$

where we have used that $\sin(36°) = \frac{\sqrt{10-2\sqrt{5}}}{4}$, and that by the Cosine Law,

$$AD^2 = AB^2 + BD^2 - 2AB \cdot BD\cos\beta = 18 + 4\sqrt{3} - 2\sqrt{5}.$$

Now, using again the Cosine Law, we finally obtain after some algebra

$$AC^2 = AD^2 + 16 - 8AD\cos\angle ADC = 18 - 4\sqrt{3} - 2\sqrt{5} + 2\left(\sqrt{6}-\sqrt{2}\right)\sqrt{10-2\sqrt{5}} =$$
$$= \left(\sqrt{6}-\sqrt{2}\right)^2 + \left(\sqrt{10-2\sqrt{5}}\right)^2 + 2\left(\sqrt{6}-\sqrt{2}\right)\sqrt{10-2\sqrt{5}} =$$
$$= \left(\sqrt{6}-\sqrt{2}+\sqrt{10-2\sqrt{5}}\right)^2,$$

and finally,

$$AC = \sqrt{6} - \sqrt{2} + \sqrt{10-2\sqrt{5}}.$$

Second solution by the author

We have $AB^2 + CD^2 = AD^2 + BC^2$, so diagonals AC and BD are perpendicular. If we denote the point of intersection of these diagonals by E, then

$$BE + ED = AB\cos u + CD\cos v = 4(\cos u + \cos v),$$

where $u = \angle ABD$ and $v = \angle BDC$.

But $BE + ED = (\sqrt{6} + \sqrt{2}) + (\sqrt{5} + 1) = 4\cos(36) + 4\cos(15)$. It follows that

$$\cos u + \cos v = \cos(36) + \cos(15),$$

implying

$$2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = 2\cos\left(\frac{51}{2}\right)\cos\left(\frac{21}{2}\right).$$

The condition $u + v = 51$ implies $|u - v| = 21$ so, assuming $u > v$ we get $u = 36$ and $v = 15$. Hence $BE = \sqrt{5} + 1$ and $ED = \sqrt{6} + \sqrt{2}$, implying

$$AC = AE + EC = \sqrt{10 - 2\sqrt{5}} + \sqrt{6} - \sqrt{2}.$$

Also solved by Lukas Seier, Charters Sixth Form, Sunningdale, England, UK; Albert Stadler, Herrliberg, Switzerland; Akash Singha Roy, Chennai Mathematical Institute, India; Sebastian Foulger, Charters Sixth Form, Sunningdale, England, UK.

O480. At a party, some people shake hands. The followings are known:

- Each person shakes hands with exactly 20 persons.
- For each pair of persons who shake hands with each other, there is exactly one other person who shakes hands with both of them.
- For each pair of persons who do not shake hands with each other, there are exactly six other persons who shake hands with both of them.

Determine the number of people in the party.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Adamopoulos Dionysios, 3rd High School, Pyrgos, Greece

Suppose we have N students.

Denote by S, the set of the triples (s_1, s_2, s_3), such as s_1 shake hands with both s_2 and s_3 (think of s_2 and s_3 as an unordered pair).

Note that $(a,b,c) \equiv (a,c,b)$, but $(a,b,c) \neq (b,a,c)$.

We wish to count $|S|$.

Each of the N students shake hands with exactly 20 people, who form $\binom{20}{2} = 190$ pairs.

So $|S| = 190N$.

On the other hand, we have a total of $\binom{N}{2}$ pairs of students. We can distinguish them into two categories. The pairs that shake hands, call them F, and the pairs that do not shake hands, call them E.

Clearly, $F + E = \binom{N}{2}$.

Moreover there is a total of $\frac{20N}{2} = 10N$, shaking of hands (each person shakes hands with 20 other people, but we count twice!).

As a result $F = 10N$ and hence $E = \binom{N}{2} - 10N$.

From the last two conditions of the problem we have that

$$|S| = F + 6E = 10N + 6\left[\binom{N}{2} - 10N\right] = 3N(N-1) - 50N$$

We have just counted $|S|$ in two ways and now we can get that

$$3N(N-1) - 50N = 190N \Leftrightarrow 3N(N-1) = 240N \Leftrightarrow N - 1 = 80 \Leftrightarrow N = 81$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Akash Singha Roy, Chennai Mathematical Institute, India; Joonsoo Lee, Dwight Englewood School, NJ, USA; Min Jung Kim, Boston, MA, USA; Albert Stadler, Herrliberg, Switzerland.