Junior Problems

J481. Find all triples (p, q, r) of primes such that

$$p^2 + 2q^2 + r^2 = 3pqr.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J482. Find all positive integers less than 10,000 which are palindromic both in base 10 and base 11.

Proposed by Mircea Becheanu, Montreal, Canada

J483. Let a, b, c be real numbers such that $13a + 41b + 13c = 2019$ and

$$\max\left(\left|\frac{41}{13}a - b\right|, \left|\frac{13}{41}b - c\right|, |c - a|\right) \leq 1.$$

Prove that $2019 \leq a^2 + b^2 + c^2 \leq 2020$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J484. Let a and b positive real numbers such that $a^2 + b^2 = 1$. Find the minimum value of

$$\frac{a+b}{1+ab}.$$

Proposed by Marius Stănean, Zalău, Romania

J485. Find the maximum and minimum of

$$\frac{1}{\sin^4 x + \cos^2 x} + \frac{1}{\sin^2 x + \cos^4 x}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J486. Let a, b, c be positive numbers. Prove that

$$\frac{bc}{(2a+b)(2a+c)} + \frac{ca}{(2b+c)(2b+a)} + \frac{ab}{(2c+a)(2c+b)} \geq \frac{1}{3}$$

Proposed by An Zhenping, Xianyang Normal University, China

Senior Problems

S481. Let n be a positive integer. Evaluate

$$\sum_{k=1}^{n} \frac{(n+k)^4}{n^3+k^3}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S482. Prove that in any regular 31-gon $A_0A_1\dots A_{30}$ the following equality holds:

$$\frac{1}{A_0A_1} = \frac{1}{A_0A_2} + \frac{1}{A_0A_4} + \frac{1}{A_0A_8} + \frac{1}{A_0A_{15}}$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

S483. For any real number a let $\lfloor a \rfloor$ and $\{a\}$ be the greatest integer less than or equal to a and the fractional part of a, respectively. Solve the equation

$$16x\lfloor x \rfloor - 10\{x\} = 2019.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

S484. Let a, b, c be positive real numbers such that $a+b+c=2$. Prove that

$$a^2\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) + b^2\left(\frac{1}{c}-1\right)\left(\frac{1}{a}-1\right) + c^2\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right) \geq \frac{1}{3}$$

Proposed by An Zhenping, Xianyang Normal University, China

S485. Find all positive integers n for which there is a real constant c such that

$$(c+1)(\sin^{2n} x + \cos^{2n} x) - c(\sin^{2(n+1)} x + \cos^{2(n+1)} x) = 1,$$

for all real numbers x.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S486. Let ABC be an acute triangle. Let B_1, C_1 be the midpoint of AC and AB, respectively and B_2, C_2 be the foot of altitude from B, C, respectively. Let B_3, C_3 be the reflection of B_2, C_2 across the line B_1C_1. The lines BB_3 and CC_3 intersect in X. Prove that $XB = XC$.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Undergraduate Problems

U481. Evaluate

$$\lim_{n\to\infty}\frac{1}{n}\left(\lfloor e^{\frac{1}{n}}\rfloor+\lfloor e^{\frac{2}{n}}\rfloor+\cdots+\lfloor e^{\frac{n}{n}}\rfloor\right)$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U482. For any positive integer n consider the polynomial $f_n=x^{2n}+x^n+1$. Prove that for any positive integer m there is a positive integer n such that f_n has exactly m irreducible factors in $\mathbb{Z}[X]$.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

U483. Evaluate

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j<k\le n}\cot^{-1}\left(\frac{i}{n}\right)\cot^{-1}\left(\frac{j}{n}\right)\cot^{-1}\left(\frac{k}{n}\right)$$

Proposed by Nicuşor Zlota, Focşani, Romania

U484. Find all polynomials $P(x)$ for which:

$$P(a+b)=6(P(a)+P(b))+15a^2b^2(a+b),$$

for all complex numbers a and b such that $a^2+b^2=ab$.

Proposed by Titu Andreescu, USA and Mircea Becheanu, Canada

U485. Let $f:[0,1]\longrightarrow(0,\infty)$ be a continuous function and let A be the set of all positive integers n for which there is a real number x_n such that

$$\int_{x_n}^{1}f(t)\mathrm{d}t=\frac{1}{n}.$$

Prove that the set $\{x_n\}_{n\in A}$ is an infinite sequence and find

$$\lim_{n\to\infty}n(x_n-1).$$

Proposed by Florin Rotaru, Focşani, Romania

U486. Let $\lfloor x\rfloor$ be the floor function and let $k\ge 3$ be a positive integer. Evaluate

$$\int_0^\infty\frac{\lfloor x\rfloor}{x^k}dx$$

Proposed by Metin Can Aydemir, Ankara, Turkey

Olympiad Problems

O481. Prove that

$$\prod_{k=1}^{n}\left(1-4\sin\frac{\pi}{5^k}\sin\frac{3\pi}{5^k}\right)=-\sec\frac{\pi}{5^n}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O482. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=1$. Prove that

$$\frac{a^2}{c^3}+\frac{b^2}{a^3}+\frac{c^2}{b^3}\geq(a+b+c)^3$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

O483. Find all integers n for which $(4n^2-1)(n^2+n)+2019$ is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O484. Let ABC be a triangle with $AB=AC$. Points E and F lie on AB and AC, respectively so that EF passes through the circumcenter of ABC. Let M be the midpoint of AB, let N be the midpoint of AC and set $P=FM\cap EN$. Prove that the lines AP and EF are perpendicular.

Proposed by Tovi Wen, USA

O485. Prove that any infinite set of positive integers contains two numbers whose sum has a prime divisor greater than 10^{2020}.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O486. Let a, b, c be positive real numbers. Prove that

$$a^2+b^2+c^2\geq a\sqrt[3]{\frac{b^3+c^3}{2}}+b\sqrt[3]{\frac{c^3+a^3}{2}}+c\sqrt[3]{\frac{a^3+b^3}{2}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam