Joining the Incenter and Orthocenter Configurations: Properties Associated with a Tangential Quadrilateral

Andrew Wu

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Abstract

Normally, it is quite rare to see olympiad geometry problems pertaining to both the incenter and orthocenter configurations. We discuss properties pertaining to a configuration related to both these centers with the condition that a certain quadrilateral is tangential.

1 The Configuration

Displayed below is a triangle $ABC$ with orthocenter $H$ such that $E$ and $F$ are the feet of the altitudes from $B$ and $C$, respectively; furthermore, $I$ is the incenter, and the incircle meets sides $BC, CA,$ and $AB$ at $P, Q,$ and $R$, respectively. This triangle has the special property that quadrilateral $BFEC$ is tangential—that is, it has an inscribed circle, the incircle of $\triangle ABC$.

In general, it is not true that $BFEC$ will have an inscribed circle, but when it does we can find many interesting properties.

2 Properties Associated with This Configuration

You may have already observed that it seems as if $R, H,$ and $Q$ are collinear. Indeed, this follows immediately from a degenerate case of Brianchon’s Theorem. However, there are many more nontrivial properties we can show, beginning with the following:
**Property 1.** \( AH = PI, \) and in particular, \( AHPI \) is a parallelogram.

*Proof.* Of course, as \( \overline{AH} \parallel \overline{TP}, \) showing that \( AH = PI \) will immediately imply that \( AHPI \) is a parallelogram. We begin by applying Pitot’s Theorem to quadrilateral \( BFEC; \) we have \( BC + EF = BF + EC. \) As triangles \( AEF \) and \( ABC \) are similar with scale factor \( \cos \angle A, \) we know that \( EF = BC \cos \angle A; \) this motivates us to express \( BF \) and \( EF \) in terms of \( BC, \) yielding the following equation:

\[
BC + BC \cos \angle A = BC \cos \angle B + BC \cos \angle C.
\]

Of course, we divide through by \( BC \) and get \( 1 + \cos \angle A = \cos \angle B + \cos \angle C. \) Now we will apply the identity \( \cos \angle A + \cos \angle B + \cos \angle C = 1 + \frac{r}{R}, \) where \( r \) is the length of the inradius and \( R \) is the length of the circumradius. (This identity can be proven through some mechanical expansion with identities, or by a clever application of Carnot’s theorem.) Simplification with the identity yields

\[
2 \cos \angle A = \frac{r}{R}
\]

which yields \( 2R \cos \angle A = r, \) and as \( AH = 2R \cos \angle A, \) it follows that \( AH = r = PI. \) We are done.

**Property 2.** The circumcircles of triangles \( AEF, APQ, \) and \( ABC \) are coaxial, and consequently \( \overrightarrow{HI} \) passes through the midpoint \( M \) of \( BC. \)

*Proof.* Using **Property 1**, we deduce that \( \overrightarrow{PH} \perp \overrightarrow{QR}. \) Let \( K \) be the intersection of the circumcircles of \( \triangle AQR \) and \( \triangle ABC. \)
We claim that $I$, $H$, and $K$ are collinear; as $H$ is the foot of the $P$-altitude in $\triangle PQR$, it lies on the nine-point circle of $\triangle PQR$. Furthermore, upon an inversion about the incircle, the circumcircle of $\triangle ABC$ swaps with the nine-point circle of $\triangle PQR$, as $A$, $B$, and $C$ map to the midpoints of intouch chords $QR$, $RP$, and $PQ$, respectively. Thus under this inversion, $H$ maps to a point on the circumcircle of $\triangle ABC$, and as $H$ lies on $QR$, it must also map to a point on the circumcircle of $\triangle IQR$. Thus $H$ maps to $K$ and this claim is proven.

The rest is easy: we know that $\angle AKI = 90^\circ$, so therefore $\angle AKH = 90^\circ$ and $K$ lies on the circle with diameter $AH$, as do $E$ and $F$. As a consequence of this property, it follows that $\overrightarrow{HI}$ passes through the midpoint $M$ of $BC$ because it is well-known that $K$, $H$, and $M$ are collinear (say, by orthocenter reflections.)

There is an alternative method of proving that $H$, $I$, and $M$ are collinear without involving $K$; all we need is Property 1. For simplicity, in the diagram below, we have included only the essential components:
Here we let $M_A$ be the midpoint of arc $\widehat{BC}$ not containing $A$, so that $A, I,$ and $M_A$ are collinear; to show that $H, I,$ and $M$ are collinear, it suffices to show that $\frac{IA}{AH} = \frac{IM}{MA}$.

By the Incenter-Excenter Lemma, we know that $M_AI = M_AB$, so $\frac{IM_A}{MA} = \frac{MA}{MA} = \csc\left(\frac{1}{2}\angle A\right)$. But $\frac{IA}{AH} = \frac{IA}{r}$ by Property 1, and it is clear that $\frac{IA}{r} = \csc\left(\frac{1}{2}\angle A\right)$, so therefore $H, I,$ and $M$ are collinear. From here we may let $K$ be the intersection of the circle with diameter $\overline{AH}$ and the circumcircle of $\triangle ABC$, such that $\angle AKH = \angle AKI = 90^\circ$, once again showing the desired coaxiality.

**Property 3.** If $H_A$ is the reflection of the orthocenter $H$ in $\overline{BC}$ and if $L$ is the midpoint of arc $\widehat{BAC}$, then $L, P,$ and $H_A$ are collinear.

To prove this property, we will use two separate lemmas, one relying on the fact that $K$ is the Miquel point of $BFEC$, and the other relying on the fact that $K$ is the Miquel point of $BRQC$. (Of course, this is true because $K$ is the intersection of the circumcircles of $\triangle AEF$, $\triangle AQR$, and $\triangle ABC$.)

**Lemma 1.** In any arbitrary triangle $ABC$ with $D, E,$ and $F$ as the feet of the $A, B,$ and $C$-altitudes, respectively, if $K$ is the Miquel Point of quadrilateral $BFEC$ and $H_A$ is the intersection of $\overrightarrow{AD}$ and the circumcircle of $\triangle ABC$, then $KBH_A$ is a harmonic quadrilateral.
Proof of Lemma 1. Letting $G$ be the intersection of $\overrightarrow{BC}$ and $\overrightarrow{AK}$, we firstly observe that $G$ lies on $\overrightarrow{EF}$ by an application of the radical axis theorem. Next, it follows that $(G, D; B, C) = -1$, and thus $(G, D; B, C) \overset{A}{=} (K, H_A; B, C)$, so $KBH_AC$ is a harmonic quadrilateral.

Lemma 2. In any arbitrary triangle $ABC$ such that its incircle meets sides $BC, CA,$ and $AB$ at points $P, Q,$ and $R$, respectively, if $K$ is the Miquel Point of quadrilateral $BRQC$ and $L$ is the midpoint of arc $\hat{BAC}$, then $\overrightarrow{LK}, \overrightarrow{QR},$ and $\overrightarrow{BC}$ are concurrent. Furthermore, if $\overrightarrow{LP}$ meets the circumcircle of $\triangle ABC$ again at $J$, then $KBJC$ is a harmonic quadrilateral.

Proof of Lemma 2. We first show that $\overrightarrow{KP}$ bisects $\angle BKC$, which will show that it passes through $M_A$, the midpoint of arc $\overline{BC}$ not containing $A$. As $K$ is the center of the spiral similarity mapping $\overrightarrow{BR}$ to $\overrightarrow{CQ}$, it follows that $\frac{KB}{BR} = \frac{KC}{CQ}$; by equal tangents, $BR = BP$ and $CQ = CP$, so $\frac{KB}{BP} = \frac{KC}{CP}$, implying that $K, P,$ and $M_A$ are collinear by the Angle Bisector theorem.
Next, we know that \((L, M_A; B, C) = -1\) and \((S, P; B, C) = -1\), where \(S\) denotes the intersection of \(\overrightarrow{QR}\) and \(\overrightarrow{BC}\). It follows from projecting through \(K\) that \(L, K,\) and \(S\) are collinear. Furthermore, \(KBJC\) is a harmonic quadrilateral from projecting \((S, P; B, C)\) through \(L\). The lemma is proven.

Having proven these two lemmas, we may return to the main proof, which is now quite easy. Refer to the diagram below:

Proof of Property 3. Let \(H'_A\) be the intersection of \(\overrightarrow{LP}\) and the circumcircle of \(\triangle ABC\). As \(K\) is the Miquel Point of quadrilateral \(BRQC\), it follows that \((K, H'_A; B, C)\) is harmonic. But as \(K\) is the Miquel Point of quadrilateral \(BFEC\), we know that \((K, HA; B, C)\) is harmonic. Thus \(H'_A = HA\) and \(L, P,\) and \(HA\) are indeed collinear.

Property 4. \(L, I,\) and \(D\) are collinear, where \(L\) is the midpoint of arc \(\widehat{BAC}\) and \(D\) is the foot of the \(A\)-altitude.
Proof. As $H, I,$ and $M$ are collinear, it follows by similar triangles that $\frac{IP}{HD} = \frac{MP}{MD}$. Orthocenter reflections yield that $HD = H_A D$, and as triangles $MPL$ and $DPH_A$ are similar, we know that $\frac{LP}{LH_A} = \frac{MP}{MD}$. It follows that $\frac{LP}{LH_A} = \frac{MP}{MD} = \frac{IP}{HD} = \frac{IP}{H_A D}$.

Therefore $L, I,$ and $D$ are collinear by similar triangles.

**Property 5.** $AI$ is tangent to the circumcircle of $\triangle HID$.

Proof. It suffices to show that $\angle AIH = \angle ADI$. Let $U$ be the antipode of $P$ with respect to the incircle. Our first claim is that $HI$ is parallel to the $A$-extouch cevian; let $X$ be the reflection of $P$ in $M$, such that $A, U,$ and $X$ are collinear and lie on the $A$-extouch cevian. (This is true by homothety at $A$ mapping the incircle to the $A$-excircle.) Then $\overline{MI}$ is the $P$-midline of $\triangle PUX$, so $\overline{MI} \parallel \overline{AX}$ and $\angle AIH = \angle IAX$. 
Now let $T$ be the $A$-mixtilinear touchpoint, such that by the well-known isogonality of $\overrightarrow{AT}$ and $\overrightarrow{AX}$ we have $\angle AIH = \angle IAX = \angle IAT$. But it is well-known that $L, I, \text{ and } T$ are collinear, so $\angle IAT = \frac{1}{2} \overrightarrow{TM}_A$. As $LHA_M$ is an isosceles trapezoid, we have $\frac{1}{2} \overrightarrow{TM}_A = \frac{1}{2} \overrightarrow{TH}_A + \frac{1}{2} \overrightarrow{AL} = \angle ADI$, and the conclusion follows.

Alternatively, we can provide a proof using ratios. It suffices to show that $AI^2 = AH \cdot AD$, or that $\frac{AI}{AD} = \frac{AH}{AI}$. But we know that $AH = r$, the length of the inradius, so therefore $\frac{AH}{AI} = \sin \frac{1}{2} \angle A$. Similar triangles and Property 4 yield that $\frac{AI}{AD} = \frac{M_AI}{M_AL}$. By the Incenter-Excenter lemma, we know that $M_AI = M_AB$, so we get $\frac{M_AI}{M_AL} = \frac{M_AL}{M_AL} = \cos \angle BLM_A = \sin \frac{1}{2} \angle A$, so we may conclude.

**Property 6.** Lines $\overrightarrow{MH}_A$ and $\overrightarrow{LI}$ concur on the circumcircle of $\triangle BIC$.

**Proof.** Suppose that $\overrightarrow{LI}$ meets the circumcircle of $\triangle BIC$ again at $V$. By the Incenter-Excenter lemma, $\overrightarrow{LB}$ and $\overrightarrow{LC}$ are tangent to the circumcircle of $\triangle BIC$. 
It follows that $\overrightarrow{BC}$ is the polar of $L$ with respect to the circumcircle of $\triangle BIC$, and thus $(L, D; I, V) = -1$. But projecting this quadruplet from $M$ onto the $A$-altitude yields that $M, H_A, V$ are collinear, so we are done.

**Property 7.** Again, let $T$ be the $A$-mixtilinear touchpoint; let $N$ be the tangency point between the incircle of $\triangle ABC$ and $EF$. Then quadrilateral $TPIN$ is cyclic.

**Proof.** The key idea is once again to note the isogonality of extouch and mixtilinear lines. Indeed, observe that $\triangle AEF$ and $\triangle ABC$ are inversely similar, and that $N$ is actually the $A$-extouch point in $\triangle AEF$. It follows that $\overrightarrow{AN}$ and $\overrightarrow{AX}$ are isogonal in $\angle A$, and thus that $A, N, T$ are collinear.
Now see that $\overrightarrow{TP}$ passes through the reflection of $A$ in the perpendicular bisector of $BC$ (to prove this, we may reflect $T$ and $P$ over $\overrightarrow{LM_A}$, such that $P$ maps to $X$ and $T$ maps to the intersection of $\overrightarrow{AX}$ and the circumcircle of $\triangle ABC$.) Then, as $T$, $I$, and $L$ are collinear, it follows that $\overrightarrow{TI}$ bisects $\angle NTP$. But $N$ and $P$ lie on the incircle, so $IP = IN$, and we have proven the desired concyclicity.

In particular, we may also observe that $K$ lies on the circle circumscribing this cyclic quadrilateral. We proved earlier that $K$, $P$, and $M_A$ are collinear; then $\angle PKT = \angle M_AKT = \angle M_ALT = \angle PIT$ by parallel $\overrightarrow{MA}$ and $\overrightarrow{PI}$, showing the desired concyclicity.

At this point, we’ve already proven many results related to this particular configuration; we leave it to the reader to solve the exercises in the following section.
3 Exercises

We'll use the same labels as before. Hints begin on the next page.

Property 8. The A-mixtilinear incircle is tangent to the circumcircle of $\triangle AEF$.

Property 9. The A-mixtilinear incircle is tangent to the circumcircle of $\triangle BHC$. Consequently, the center of the A-mixtilinear incircle is the foot of the A-angle bisector.

Property 10. $KT$ is perpendicular to $BC$.

Property 11. $\overrightarrow{MA}$, $\overrightarrow{EF}$, and $\overrightarrow{BC}$ are concurrent.

Property 12. If $\overrightarrow{HA}$ meets $BC$ at $Y$, then $KY$ and $\overrightarrow{MA}$ concur on the circumcircle of $\triangle ABC$. If $Z$ is that point of concurrency, then $ZH_A$ passes through the intersection of $KT$ and $BC$.

Property 13. $MZ$ passes through the reflection of $T$ in $BC$. 
4 Hints to Exercises

*Hint 8.* Invert using Property 5.

*Hint 9.* Same as *Hint 8.*

*Hint 10.* Just some angle-chasing. Show that $\widehat{KL} = \widehat{MA}$.

*Hint 11.* Project a harmonic quadrilateral from $T$.

*Hint 12.* Let $\overrightarrow{KY}$ meet the circumcircle of $\triangle ABC$ at $Z$ and show that $ZBTC$ is a harmonic quadrilateral, using Property 10. Then project from $M_A$ onto $\overrightarrow{BC}$.

*Hint 13.* $\overrightarrow{TM}$ and $\overrightarrow{AX}$ meet on the circumcircle of $\triangle ABC$ (why?) so project $(B, C; M, P_{\infty})$ from $Z$ onto the circumcircle. ($P_{\infty}$ denotes the infinity point on $\overrightarrow{BC}$.)