Joining the Incenter and Orthocenter Configurations: Properties Associated with a Tangential Quadrilateral

Andrew Wu

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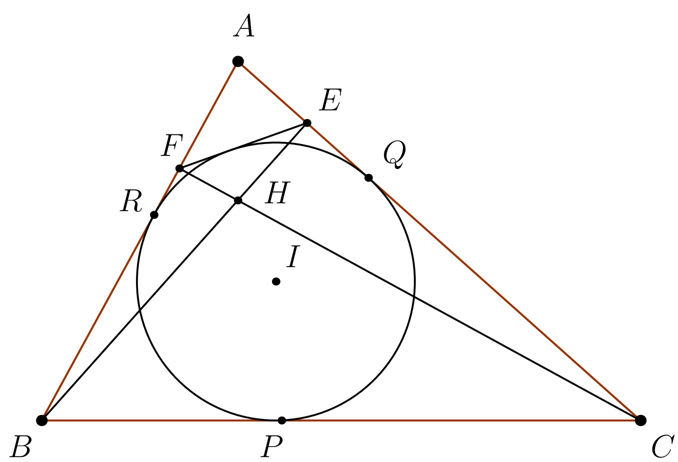
Abstract

Normally, it is quite rare to see olympiad geometry problems pertaining to both the incenter and orthocenter configurations. We discuss properties pertaining to a configuration related to both these centers with the condition that a certain quadrilateral is tangential.

1 The Configuration

Displayed below is a triangle ABC with orthocenter H such that E and F are the feet of the altitudes from B and C, respectively; furthermore, I is the incenter, and the incircle meets sides $\overline{BC}, \overline{CA}$, and $\overline{AB}$ at P, Q, and R, respectively. This triangle has the special property that quadrilateral $BFEC$ is tangential—that is, it has an inscribed circle, the incircle of $\triangle ABC$.



In general, it is not true that $BFEC$ will have an inscribed circle, but when it does we can find many interesting properties.

2 Properties Associated with This Configuration

You may have already observed that it seems as if R, H, and Q are collinear. Indeed, this follows immediately from a degenerate case of Brianchon's Theorem. However, there are many more nontrivial properties we can show, beginning with the following:

Property 1. *$AH = PI$, and in particular, $AHPI$ is a parallelogram.*

Proof. Of course, as $\overline{AH} \parallel \overline{IP}$, showing that $AH = PI$ will immediately imply that $AHPI$ is a parallelogram. We begin by applying Pitot's Theorem to quadrilateral $BFEC$; we have $BC + EF = BF + EC$. As triangles AEF and ABC are similar with scale factor $\cos\angle A$, we know that $EF = BC\cos\angle A$; this motivates us to express BF and EF in terms of BC, yielding the following equation:

$$BC + BC\cos\angle A = BC\cos\angle B + BC\cos\angle C.$$

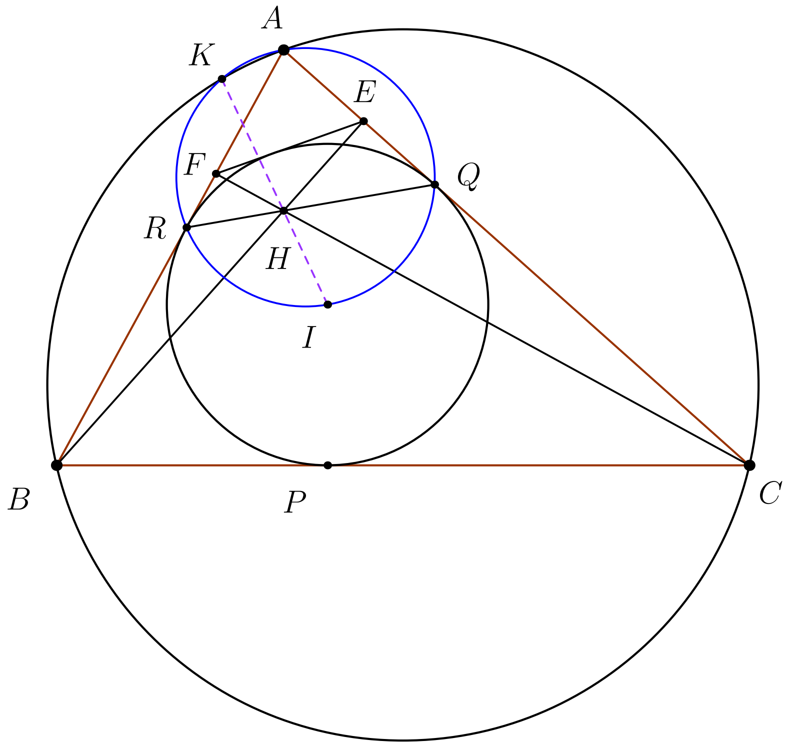
Of course, we divide through by BC and get $1 + \cos\angle A = \cos\angle B + \cos\angle C$. Now we will apply the identity $\cos\angle A + \cos\angle B + \cos\angle C = 1 + \frac{r}{R}$, where r is the length of the inradius and R is the length of the circumradius. (This identity can be proven through some mechanical expansion with identities, or by a clever application of Carnot's theorem.) Simplification with the identity yields

$$2\cos\angle A = \frac{r}{R}$$

which yields $2R\cos\angle A = r$, and as $AH = 2R\cos\angle A$, it follows that $AH = r = PI$. We are done.

Property 2. *The circumcircles of triangles AEF, APQ, and ABC are coaxial, and consequently $\overleftrightarrow{HI}$ passes through the midpoint M of $\overline{BC}$.*

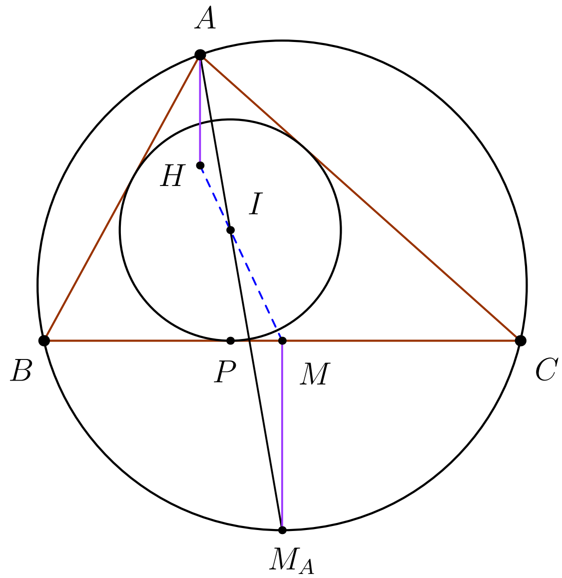
Proof. Using **Property 1**, we deduce that $\overline{PH} \perp \overline{QR}$. Let K be the intersection of the circumcircles of $\triangle AQR$ and $\triangle ABC$.



We claim that I, H, and K are collinear; as H is the foot of the P-altitude in $\triangle PQR$, it lies on the nine-point circle of $\triangle PQR$. Furthermore, upon an inversion about the incircle, the circumcircle of $\triangle ABC$ swaps with the nine-point circle of $\triangle PQR$, as A, B, and C map to the midpoints of intouch chords $\overline{QR}, \overline{RP}$, and $\overline{PQ}$, respectively. Thus under this inversion, H maps to a point on the circumcircle of $\triangle ABC$, and as H lies on $\overline{QR}$, it must also map to a point on the circumcircle of $\triangle IQR$. Thus H maps to K and this claim is proven.

The rest is easy: we know that $\angle AKI = 90°$, so therefore $\angle AKH = 90°$ and K lies on the circle with diameter $\overline{AH}$, as do E and F. As a consequence of this property, it follows that $\overleftrightarrow{HI}$ passes through the midpoint M of $\overline{BC}$ because it is well-known that K, H, and M are collinear (say, by orthocenter reflections.)

There is an alternative method of proving that H, I, and M are collinear without involving K; all we need is **Property 1**. For simplicity, in the diagram below, we have included only the essential components:



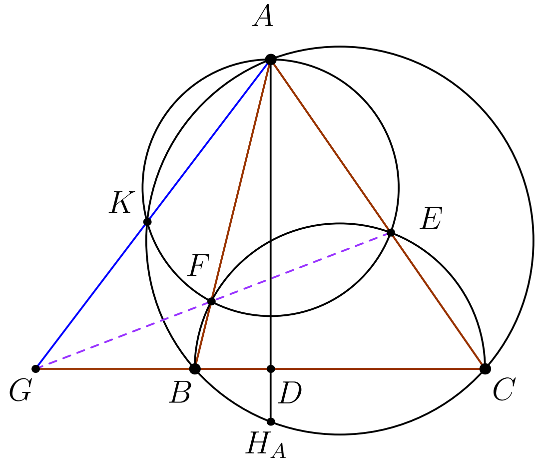
Here we let M_A be the midpoint of arc $\widehat{BC}$ not containing A, so that A, I, and M_A are collinear; to show that H, I, and M are collinear, it suffices to show that $\frac{IA}{AH} = \frac{IM_A}{M_AM}$.

By the Incenter-Excenter Lemma, we know that $M_AI = M_AB$, so $\frac{IM_A}{M_AM} = \frac{M_AB}{M_AM} = \csc\left(\frac{1}{2}\angle A\right)$. But $\frac{IA}{AH} = \frac{IA}{r}$ by **Property 1**, and it is clear that $\frac{IA}{r} = \csc\left(\frac{1}{2}\angle A\right)$, so therefore H, I, and M are collinear. From here we may let K be the intersection of the circle with diameter $\overline{AH}$ and the circumcircle of $\triangle ABC$, such that $\angle AKH = \angle AKI = 90^\circ$, once again showing the desired coaxiality.

Property 3. *If H_A is the reflection of the orthocenter H in $\overline{BC}$ and if L is the midpoint of arc $\widehat{BAC}$, then L, P, and H_A are collinear.*

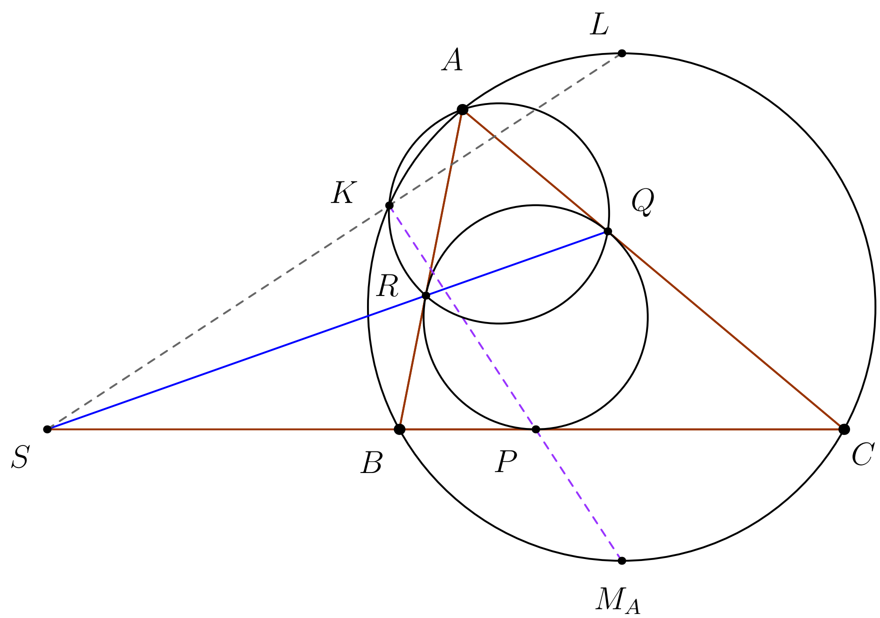
To prove this property, we will use two separate lemmas, one relying on the fact that K is the Miquel point of $BFEC$, and the other relying on the fact that K is the Miquel point of $BRQC$. (Of course, this is true because K is the intersection of the circumcircles of $\triangle AEF, \triangle AQR$, and $\triangle ABC$.)

Lemma 1. In any arbitrary triangle ABC with D, E, and F as the feet of the A, B, and C-altitudes, respectively, if K is the Miquel Point of quadrilateral $BFEC$ and H_A is the intersection of $\overleftrightarrow{AD}$ and the circumcircle of $\triangle ABC$, then KBH_AC is a harmonic quadrilateral.



Proof of Lemma 1. Letting G be the intersection of $\overleftrightarrow{BC}$ and $\overleftrightarrow{AK}$, we firstly observe that G lies on $\overleftrightarrow{EF}$ by an application of the radical axis theorem. Next, it follows that $(G, D; B, C) = -1$, and thus $(G, D; B, C) \overset{A}{=} (K, H_A; B, C)$, so KBH_AC is a harmonic quadrilateral.

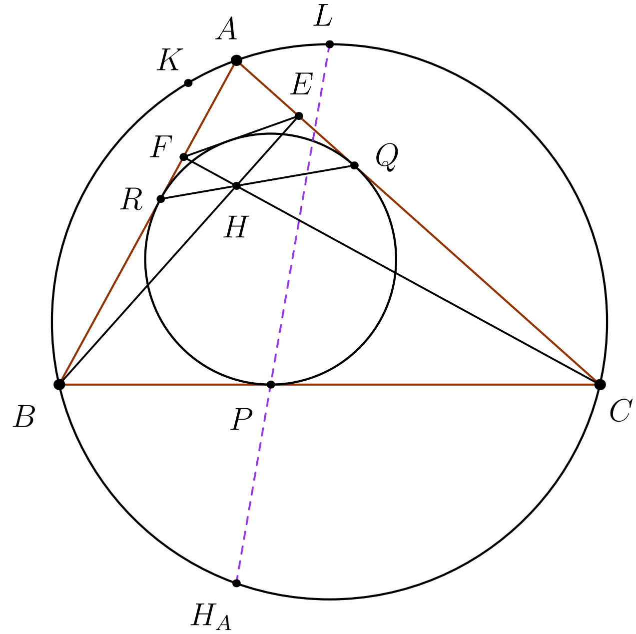
Lemma 2. In any arbitrary triangle ABC such that its incircle meets sides $\overline{BC}, \overline{CA}$, and $\overline{AB}$ at points P, Q, and R, respectively, if K is the Miquel Point of quadrilateral $BRQC$ and L is the midpoint of arc $\widehat{BAC}$, then $\overleftrightarrow{LK}, \overleftrightarrow{QR}$, and $\overleftrightarrow{BC}$ are concurrent. Furthermore, if $\overleftrightarrow{LP}$ meets the circumcircle of $\triangle ABC$ again at J, then $KBJC$ is a harmonic quadrilateral.



Proof of Lemma 2. We first show that $\overleftrightarrow{KP}$ bisects $\angle BKC$, which will show that it passes through M_A, the midpoint of arc $\widehat{BC}$ not containing A. As K is the center of the spiral similarity mapping $\overline{BR}$ to $\overline{CQ}$, it follows that $\frac{KB}{BR} = \frac{KC}{CQ}$; by equal tangents, $BR = BP$ and $CQ = CP$, so $\frac{KB}{BP} = \frac{KC}{CP}$, implying that K, P, and M_A are collinear by the Angle Bisector theorem.

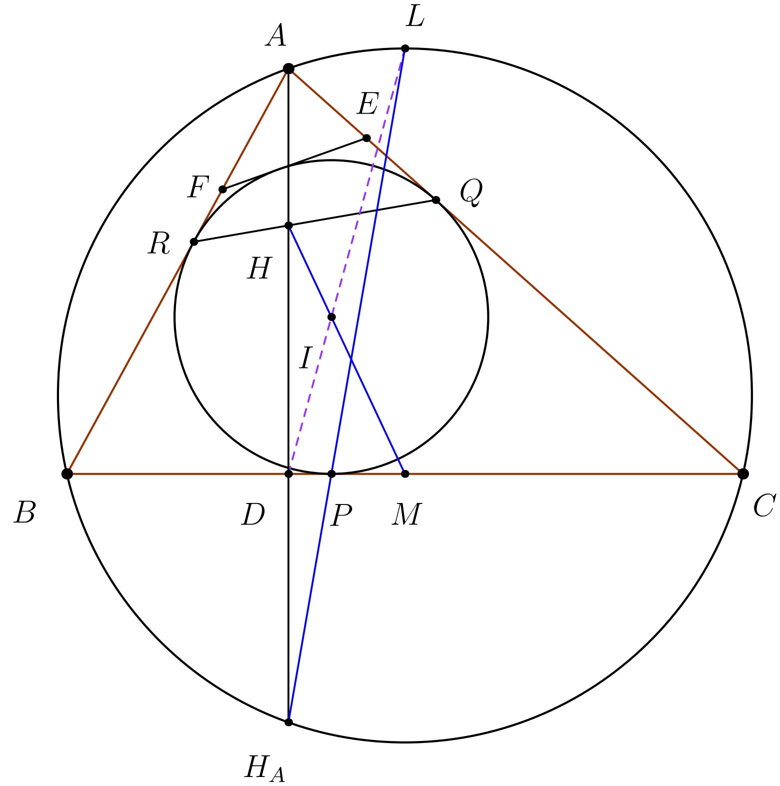
Next, we know that $(L, M_A; B, C) = -1$ and $(S, P; B, C) = -1$, where S denotes the intersection of $\overleftrightarrow{QR}$ and $\overleftrightarrow{BC}$. It follows from projecting through K that L, K, and S are collinear. Furthermore, $KBJC$ is a harmonic quadrilateral from projecting $(S, P; B, C)$ through L. The lemma is proven.

Having proven these two lemmas, we may return to the main proof, which is now quite easy. Refer to the diagram below:



Proof of Property 3. Let H'_A be the intersection of $\overleftrightarrow{LP}$ and the circumcircle of $\triangle ABC$. As K is the Miquel Point of quadrilateral $BRQC$, it follows that $(K, H'_A; B, C)$ is harmonic. But as K is the Miquel Point of quadrilateral $BFEC$, we know that $(K, H_A; B, C)$ is harmonic. Thus $H'_A = H_A$ and L, P, and H_A are indeed collinear.

Property 4. *L, I, and D are collinear, where L is the midpoint of arc $\widehat{BAC}$ and D is the foot of the A-altitude.*



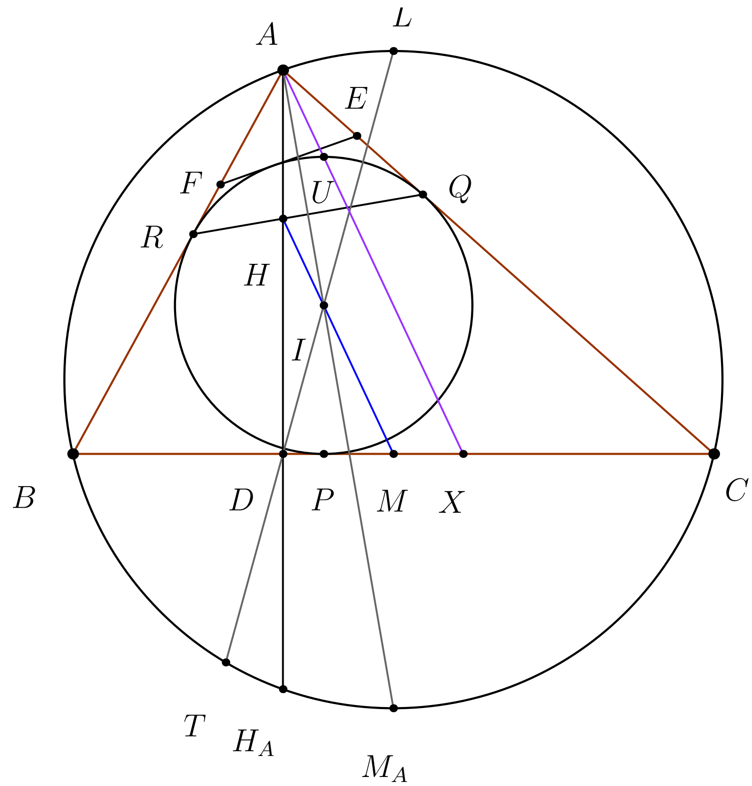
Proof. As H, I, and M are collinear, it follows by similar triangles that $\frac{IP}{HD} = \frac{MP}{MD}$. Orthocenter reflections yield that $HD = H_AD$, and as triangles MPL and DPH_A are similar, we know that $\frac{LP}{LH_A} = \frac{MP}{MD}$. It follows that

$$\frac{LP}{LH_A} = \frac{MP}{MD} = \frac{IP}{HD} = \frac{IP}{H_AD}.$$

Therefore L, I, and D are collinear by similar triangles.

Property 5. *$\overline{AI}$ is tangent to the circumcircle of $\triangle HID$.*

Proof. It suffices to show that $\angle AIH = \angle ADI$. Let U be the antipode of P with respect to the incircle. Our first claim is that $\overline{HI}$ is parallel to the A-extouch cevian; let X be the reflection of P in M, such that A, U, and X are collinear and lie on the A-extouch cevian. (This is true by homothety at A mapping the incircle to the A-excircle.) Then $\overline{MI}$ is the P-midline of $\triangle PUX$, so $\overline{MI} \parallel \overline{AX}$ and $\angle AIH = \angle IAX$.

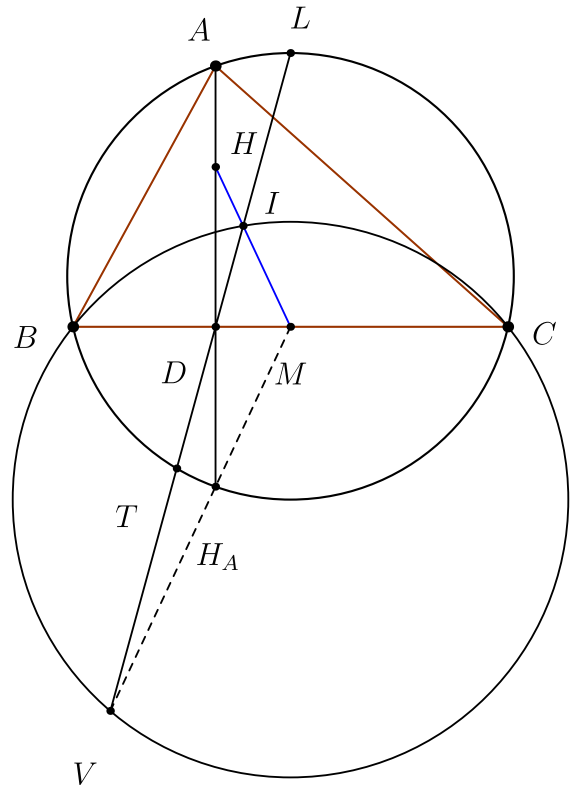


Now let T be the A-mixtilinear touchpoint, such that by the well-known isogonality of $\overleftrightarrow{AT}$ and $\overleftrightarrow{AX}$ we have $\angle AIH = \angle IAX = \angle IAT$. But it is well-known that L, I, and T are collinear, so $\angle IAT = \frac{1}{2}\widehat{TM_A}$. As LAH_AM_A is an isosceles trapezoid, we have $\frac{1}{2}\widehat{TM_A} = \frac{1}{2}\widehat{TH_A} + \frac{1}{2}\widehat{AL} = \angle ADI$, and the conclusion follows.

Alternatively, we can provide a proof using ratios. It suffices to show that $AI^2 = AH \cdot AD$, or that $\frac{AI}{AD} = \frac{AH}{AI}$. But we know that $AH = r$, the length of the inradius, so therefore $\frac{AH}{AI} = \sin\frac{1}{2}\angle A$. Similar triangles and **Property 4** yield that $\frac{AI}{AD} = \frac{M_AI}{M_AL}$. By the Incenter-Excenter lemma, we know that $M_AI = M_AB$, so we get $\frac{M_AI}{M_AL} = \frac{M_AB}{M_AL} = \sin\angle BLM_A = \sin\frac{1}{2}\angle A$, so we may conclude.

Property 6. *Lines $\overleftrightarrow{MH_A}$ and $\overleftrightarrow{LI}$ concur on the circumcircle of $\triangle BIC$.*

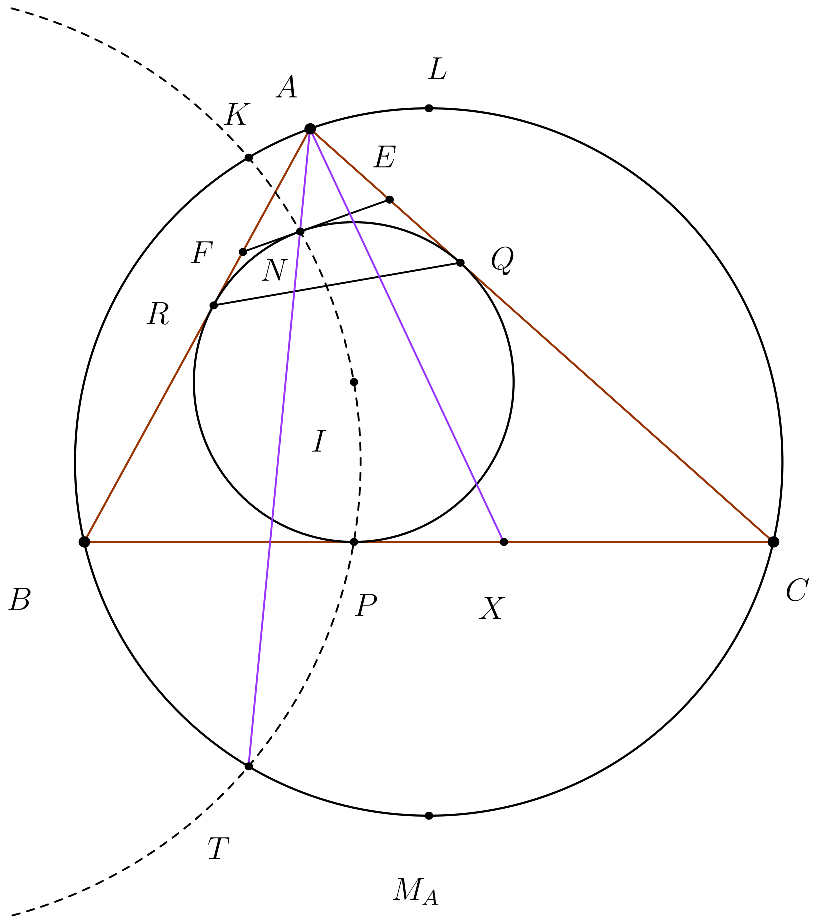
Proof. Suppose that $\overleftrightarrow{LI}$ meets the circumcircle of $\triangle BIC$ again at V. By the Incenter-Excenter lemma, $\overline{LB}$ and $\overline{LC}$ are tangent to the circumcircle of $\triangle BIC$.



It follows that $\overleftrightarrow{BC}$ is the polar of L with respect to the circumcircle of $\triangle BIC$, and thus $(L, D; I, V) = -1$. But projecting this quadruplet from M onto the A-altitude yields that M, H_A, and V are collinear, so we are done.

Property 7. *Again, let T be the A-mixtilinear touchpoint; let N be the tangency point between the incircle of $\triangle ABC$ and $\overline{EF}$. Then quadrilateral $TPIN$ is cyclic.*

Proof. The key idea is once again to note the isogonality of extouch and mixtilinear lines. Indeed, observe that $\triangle AEF$ and $\triangle ABC$ are inversely similar, and that N is actually the A-extouch point in $\triangle AEF$. It follows that $\overleftrightarrow{AN}$ and $\overleftrightarrow{AX}$ are isogonal in $\angle A$, and thus that A, N, and T are collinear.



Now see that $\overleftrightarrow{TP}$ passes through the reflection of A in the perpendicular bisector of $\overline{BC}$ (to prove this, we may reflect T and P over $\overleftrightarrow{LM_A}$, such that P maps to X and T maps to the intersection of $\overleftrightarrow{AX}$ and the circumcircle of $\triangle ABC$.) Then, as T, I, and L are collinear, it follows that $\overleftrightarrow{TI}$ bisects $\angle NTP$. But N and P lie on the incircle, so $IP = IN$, and we have proven the desired concyclity.

In particular, we may also observe that K lies on the circle circumscribing this cyclic quadrilateral. We proved earlier that K, P, and M_A are collinear; then $\angle PKT = \angle M_AKT = \angle M_ALT = \angle PIT$ by parallel $\overline{M_AL}$ and $\overline{PI}$, showing the desired concyclity.

At this point, we've already proven many results related to this particular configuration; we leave it to the reader to solve the exercises in the following section.

3 Exercises

We'll use the same labels as before. Hints begin on the next page.

Property 8. *The A-mixtilinear incircle is tangent to the circumcircle of* $\triangle AEF$.

Property 9. *The A-mixtilinear incircle is tangent to the circumcircle of* $\triangle BHC$. *Consequently, the center of the A-mixtilinear incircle is the foot of the A-angle bisector.*

Property 10. $\overline{KT}$ *is perpendicular to* $\overline{BC}$.

Property 11. $\overleftrightarrow{M_AT}, \overleftrightarrow{EF}$, *and* $\overleftrightarrow{BC}$ *are concurrent.*

Property 12. *If* $\overleftrightarrow{H_AT}$ *meets* $\overleftrightarrow{BC}$ *at* Y, *then* $\overleftrightarrow{KY}$ *and* $\overleftrightarrow{M_AD}$ *concur on the circumcircle of* $\triangle ABC$. *If* Z *is that point of concurrency, then* $\overleftrightarrow{ZH_A}$ *passes through the intersection of* $\overline{KT}$ *and* $\overline{BC}$.

Property 13. $\overline{MZ}$ *passes through the reflection of* T *in* $\overline{BC}$.

4 Hints to Exercises

Hint 8. Invert using **Property 5**.

Hint 9. Same as *Hint 8*.

Hint 10. Just some angle-chasing. Show that $\widehat{KL} = \widehat{M_A T}$.

Hint 11. Project a harmonic quadrilateral from T.

Hint 12. Let $\overleftrightarrow{KY}$ meet the circumcircle of $\triangle ABC$ at Z and show that $ZBTC$ is a harmonic quadrilateral, using **Property 10**. Then project from M_A onto $\overleftrightarrow{BC}$.

Hint 13. $\overleftrightarrow{T'M}$ and $\overleftrightarrow{AX}$ meet on the circumcircle of $\triangle ABC$ (why?) so project $(B, C; M, P_\infty)$ from Z onto the circumcircle. (P_∞ denotes the infinity point on $\overleftrightarrow{BC}$.)