Junior problems

J481. Find all triples (p, q, r) of primes such that

$$p^2 + 2q^2 + r^2 = 3pqr.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Polyahedra, Polk State College, USA
If neither p nor r is divisible by 3, then $p^2+2q^2+r^2 \equiv 1+2q^2+1 \equiv 2, 1 \pmod 3$, an impossibility. Suppose that $r = 3$. Then $p^2 + 2q^2 = 9(pq-1)$. If q is odd, then $p^2 + 2q^2$ and $9(pq-1)$ have opposite parity. So $q = 2$. Then $0 = p^2 - 18p + 17 = (p-1)(p-17)$, thus $p = 17$. By symmetry in p and r, the solutions are $(p, q, r) = (17, 2, 3)$ and $(3, 2, 17)$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Joe Simons, Utah Valley University, UT, USA; Joel Schlosberg, Bayside, NY, USA; Ioannis D. Sfikas, Athens, Greece; Joonsoo Lee, Dwight Englewood School, NJ, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nick Iliopoulos, 8th General Highschool of Trikala, Greece; Sushanth Sathish Kumar, PHS, SoCal, USA; Titu Zvonaru, Comănești, Romania; Jeffrey Roh, St. Andrew's School, DE, USA; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Sebastian Foulger, Charters School, Sunningdale, England, UK; Jeewoo Lee, Townsend Harris HS, NY, USA; Jenna Park, Blair Academy, Blairstown, NJ, USA; Min Jung Kim, Tabor Academy, MA, USA; Kelvin Kim, Bergen Catholic High School, NJ, USA; Chirita Andrei-Giovani, The Greek Catholic Timotei Cipariu High School, Bucharest, Romania; Bryant Hwang, Korea International School; Jiho Lee, Canterbury School, New Milford, CT, USA.

J482. Find all positive integers less than $10,000$ which are palindromic both in base 10 and base 11.

Proposed by Mircea Becheanu, Montreal, Canada

Solution by the author

When we mention the number of digits of a number, we consider this in base 10. The digits in base 11 are $1, 2, \ldots, 9, \zeta$, where $\zeta = 10$.. The one digit numbers $1, 2, \ldots, 9$ are obviously palindromic in both bases. A palindromic four digits numbers is divisible by 11, so it can not be palindromic in base 11. Hence we have to consider only three digits numbers.

Let N, $100 < N < 999$, be a number which is palindromic in base 11 and let

$$N = 11^2 a + 11b + a$$

be its representation in base 11. It is clear that $a \le 8$, as N is a three digits number. We try to obtain its representation in base 10.

$$N = (10+1)^2 a + (10+1)b + a = 10^2 a + 10(2a+b) + (2a+b).$$

If $2a + b < 10$, the number N can not be palindromic in base 10, because we can not have $a = 2a + b$. Also, $2a + b$ can not be divisible by 10.

We consider first the case $10 < 2a + b < 20$. Put $2a + b = 10 + c$ where $1 \le c \le 9$. Then

$$N = 10^2(a+1) + 10(c+1) + c \tag{1}$$

If $1 \le c \le 8$, the formula (1) gives a representation of N in base 10. In order to have N palindromic we put the condition $c = a + 1$, which gives $a + b = 11$ and $a \le 7$ So, we obtain the possibilities $(a = 1, b = \zeta)$; $(a = 2, b = 9)$, ...,$(a = 7, b = 4)$. They give numbers $1\zeta1$, 292, 383, 474, 565, 656, 747 in base 11, which have in base 10 representation $232, 343, 454, 565, 676, 787, 898$, respectively.

If $c = 9$, by plugging this in (1) we obtain

$$N = 10^2(a+2) + 9. \tag{2}$$

Hence, $a = 7, b = 5$ and we have in base 11 the number 757, which is 909 in base 10.

Let consider now the case $2a + b > 20$. Let denote $2a + b = 20 + c$. Then

$$N = 10^2(a+2) + 10(c+2) + c.$$

If $c < 8$ we obtain $a + 2 = c \Longrightarrow a + b = 22$ and this is impossible, as a, b are digits. If $c = 8$ or $c = 9$ we obtain $c = a + 3$ and this is again impossible.

So, the required integers are (in base 10):

1; 2; 3; 4; 5; 6; 7; 8; 9; 232; 343; 454; 565; 676; 787; 898 and 909.

Also solved by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Spain; Jeffrey Roh, St. Andrew's School, DE, USA; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Polyahedra, Polk State College, USA; Sebastian Foulger, Charters School, Sunningdale, England, UK; Ioannis D. Sfikas, Athens, Greece; Titu Zvonaru, Comănești, Romania.

J483. Let a, b, c be real numbers such that $13a + 41b + 13c = 2019$ and

$$\max\left(\left|\frac{41}{13}a - b\right|, \left|\frac{13}{41}b - c\right|, |c - a|\right) \leq 1.$$

Prove that $2019 \leq a^2 + b^2 + c^2 \leq 2020$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by the author

From Cauchy-Schwarz's Inequality,

$$2019^2 = (13a + 41b + 13c)^2 \leq (13^2 + 41^2 + 13^2)(a^2 + b^2 + c^2),$$

implying $2019 \leq a^2 + b^2 + c^2$.

We have

$$|41a - 13b| \leq 13, \ |13b - 41c| \leq 41, \ |13c - 13a| \leq 13,$$

so

$$(41a - 13b)^2 + (13b - 41c)^2 + (13c - 13a)^2 \leq 13^2 + 41^2 + 13^2.$$

Also,

$$(13a + 41b + 13c)^2 = 2019^2.$$

Adding the last two relations yields

$$(13^2 + 41^2 + 13^2)(a^2 + b^2 + c^2) \leq (13^2 + 41^2 + 13^2) + 2019^2.$$

Because $13^2 + 41^2 + 13^2 = 2019$, it follows that

$$a^2 + b^2 + c^2 \leq 1 + 2019 = 2020,$$

as desired.

Second solution by Daniel Lasaosa, Pamplona, Spain
Denote $a = 13 + 41u$ and $c = 13 + 41v$. Note that

$$b = \frac{2019 - 13(a+c)}{41} = 41 - 13(u+v).$$

Therefore,

$$a^2 + b^2 + c^2 = 2019 + 1681\left(u^2 + v^2\right) + 169(u+v)^2 \geq 2019,$$

with equality if $u = v = 0$, ie iff $a = c = 13$ and $b = 41$. Moreover,

$$|1850u + 169v| \leq 13, \qquad |169u + 1850v| \leq 41, \qquad |41u - 41v| \leq 1,$$

or since $1850^2 + 169^2 + 169 \cdot 41^2 = 2019 \cdot 1850$, then

$$1 = \frac{13^2 + 41^2 + 13^2}{2019} \geq \frac{|1850u + 169v|^2 + |169u + 1850v|^2 + 169|41u - 41v|^2}{2019} =$$

$$= 1850\left(u^2 + v^2\right) + 338uv = 1681\left(u^2 + v^2\right) + 169(u+v)^2.$$

Now, equality simultaneously requires

$$1850u + 169v = \pm 13, \qquad 169u + 1850v = \pm 41, \qquad 41u - 41v = \pm 1.$$

But from the first two relations we obtain $41u - 41v = \pm 1 \pm \frac{13}{41}$, clearly not equal to ± 1, or equality cannot hold. We conclude that

$$2019 \leq a^2 + b^2 + c^2 < 2020,$$

with equality in the lower bound iff $a = c = 13$ and $b = 41$.

Also solved by Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Jeffrey Roh, St. Andrew's School, DE, USA; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Polyahedra, Polk State College, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Jeewoo Lee, Townsend Harris HS, NY, USA; Jenna Park, Blair Academy, Blairstown, NJ, USA; Min Jung Kim, Tabor Academy, MA, USA; Kelvin Kim, Bergen Catholic High School, NJ, USA; Chirita Andrei-Giovani, The Greek Catholic Timotei Cipariu High School, Bucharest, Romania; Dumitru Barac, Sibiu, Romania; Joonsoo Lee, Dwight Englewood School, NJ, USA; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania; Jiho Lee, Canterbury School, New Milford, CT, USA.

J484. Let a and b positive real numbers such that $a^2 + b^2 = 1$. Find the minimum value of

$$\frac{a+b}{1+ab}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam
First, we will show that

$$\frac{(a+b)^2(a^2+b^2)}{(a^2+ab+b^2)^2} \geq \frac{8}{9}.$$

Indeed, this is equivalent to

$$9(a^2+b^2+2ab)(a^2+b^2) \geq 8(a^2+b^2+ab)^2,$$

$$9[(a^2+b^2)^2+2ab(a^2+b^2)] \geq 8[(a^2+b^2)^2+2ab(a^2+b^2)+a^2b^2],$$

$$(a^2+b^2)^2+2ab(a^2+b^2) \geq 8a^2b^2,$$

$$(a^2-b^2)^2+2ab(a-b)^2 \geq 0$$

which is obvious. Hence our statement has been poven. From this statement and $a^2+b^2 = 1$ we get

$$\frac{a+b}{1+ab} \geq \frac{2\sqrt{2}}{3}.$$

The equality happens if and only if $a = b = \frac{1}{\sqrt{2}}$. So the minimum value of the given expression is $\frac{2\sqrt{2}}{3}$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Jeffrey Roh, St. Andrew's School, DE, USA; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Polyahedra, Polk State College, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Sebastian Foulger, Charters School, Sunningdale, England, UK; Jeewoo Lee, Townsend Harris HS, NY, USA; Jenna Park, Blair Academy, Blairstown, NJ, USA; Min Jung Kim, Tabor Academy, MA, USA; Chirita Andrei-Giovani, The Greek Catholic Timotei Cipariu High School, Bucharest, Romania; Bryant Hwang, Korea International School; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; Albert Stadler, Herrliberg, Switzerland; Arkady Alt, San Jose, CA, USA; Arleo Robles, Charters Sixth Form, Sunningdale, England, UK; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Henry Ricardo, Westchester Area Math Circle, NY, USA; Joel Schlosberg, Bayside, NY, USA; Ioannis D. Sfikas, Athens, Greece; Joonsoo Lee, Dwight Englewood School, NJ, USA; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Sumanth Ravipati, George Mason University, VA, USA; Sushanth Sathish Kumar, PHS, SoCal, USA; Titu Zvonaru, Comănești, Romania; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Nikos Kalapodis, Patras, Greece; Jiho Lee, Canterbury School, New Milford, CT, USA.

J485. Find the maximum and minimum of

$$\frac{1}{\sin^4 x + \cos^2 x} + \frac{1}{\sin^2 x + \cos^4 x}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA
Note

$$\begin{aligned}\sin^4 x + \cos^2 x &= \sin^4 x - \sin^2 x + 1 = \sin^2 x(\sin^2 x - 1) + 1\\ &= 1 - \sin^2 x\cos^2 x = 1 - \frac{1}{4}\sin^2 2x\end{aligned}$$

and

$$\begin{aligned}\sin^2 x + \cos^4 x &= 1 + \cos^4 x - \cos^2 x = 1 + \cos^2 x(\cos^2 x - 1)\\ &= 1 - \sin^2 x\cos^2 x = 1 - \frac{1}{4}\sin^2 2x,\end{aligned}$$

so

$$\frac{1}{\sin^4 x + \cos^2 x} + \frac{1}{\sin^2 x + \cos^4 x} = \frac{2}{1 - \frac{1}{4}\sin^2 2x}.$$

Now,

$$\frac{3}{4} \le 1 - \frac{1}{4}\sin^2 2x \le 1$$

implies

$$2 \le \frac{2}{1 - \frac{1}{4}\sin^2 2x} \le \frac{8}{3}.$$

Thus, the maximum value of

$$\frac{1}{\sin^4 x + \cos^2 x} + \frac{1}{\sin^2 x + \cos^4 x}$$

is 8/3, which occurs when $x = (2n+1)\pi/4$ for any integer n; the minimum value of

$$\frac{1}{\sin^4 x + \cos^2 x} + \frac{1}{\sin^2 x + \cos^4 x}$$

is 2, which occurs when $x = n\pi/2$ for any integer n.

Also solved by Daniel Lasaosa, Pamplona, Spain; Jeffrey Roh, St. Andrew's School, DE, USA; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Polyahedra, Polk State College, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Sebastian Foulger, Charters School, Sunningdale, England, UK; Jeewoo Lee, Townsend Harris HS, NY, USA; Jenna Park, Blair Academy, Blairstown, NJ, USA; Min Jung Kim, Tabor Academy, MA, USA; Chirita Andrei-Giovani, The Greek Catholic Timotei Cipariu High School, Bucharest, Romania; Nikos Kalapodis, Patras, Greece; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; Lukas Seier, Charters Sixth Form, Sunningdale, England, UK; Albert Stadler, Herrliberg, Switzerland; Arkady Alt, San Jose, CA, USA; Arleo Robles, Charters Sixth Form, Sunningdale, England, UK; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Henry Ricardo, Westchester Area Math Circle, NY, USA; Joel Schlosberg, Bayside, NY, USA; Ioannis D. Sfikas, Athens, Greece; Joonsoo Lee, Dwight Englewood School, NJ, USA; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Oscar Brown, Charters School, Sunningdale, UK; Sumanth Ravipati, George Mason University, VA, USA; Titu Zvonaru, Comănești, Romania; Jiho Lee, Canterbury School, New Milford, CT, USA.

J486. Let a, b, c be positive numbers. Prove that

$$\frac{bc}{(2a+b)(2a+c)} + \frac{ca}{(2b+c)(2b+a)} + \frac{ab}{(2c+a)(2c+b)} \geq \frac{1}{3}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Henry Ricardo, Westchester Area Math Circle, NY, USA
The Cauchy-Schwarz inequality gives us

$$\begin{aligned}
\sum_{cyclic} \frac{bc}{(2a+b)(2a+c)} &= \sum_{cyclic} \frac{(bc)^2}{bc\,(2a+b)(2a+c)} \\
&\geq \frac{\left(\sum_{cyclic} bc\right)^2}{\sum_{cyclic} bc\,(2a+b)(2a+c)} \\
&= \frac{\sum_{cyclic} b^2c^2 + 2abc \sum_{cyclic} a}{\sum_{cyclic} b^2c^2 + 8abc \sum_{cyclic} a} \\
&= \frac{X+2Y}{X+8Y} \geq \frac{1}{3} \iff X \geq Y.
\end{aligned}$$

But $X = \sum_{cyclic}(bc)^2 \geq (bc)(ca) + (ca)(ab) + (ab)(bc) = abc(a+b+c) = Y$ is a consequence of the AGM inequality. Equality holds if and only if $a = b = c$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Chirita Andrei-Giovani, The Greek Catholic Timotei Cipariu High School, Bucharest, Romania; Brian Bradie, Christopher Newport University, Newport News, VA , USA; Nikos Kalapodis, Patras, Greece; Albert Stadler, Herrliberg, Switzerland; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Ioan Viorel Codreanu, Satulung, Maramureş, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nick Iliopoulos, 8th General Highschool of Trikala, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania.

Senior problems

S481. Let n be a positive integer. Evaluate

$$\sum_{k=1}^{n} \frac{(n+k)^4}{n^3+k^3}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, USA

By synthetic division,

$$\frac{(n+k)^4}{n^3+k^3} = \frac{k^3+3nk^2+3n^2k+n^3}{k^2-nk+n^2} = k+4n+3n^2 r_k,$$

where $r_k = \frac{2k-n}{k^2-nk+n^2}$. Notice that

$$r_{n-i} = \frac{2(n-i)-n}{(n-i)^2-n(n-i)+n^2} = \frac{n-2i}{i^2-ni+n^2} = -r_i.$$

Thus

$$\sum_{k=1}^{n} r_k \overset{k=n-i}{=\!=\!=} \sum_{i=0}^{n-1} r_{n-i} = r_n - r_0 + \sum_{i=1}^{n} r_{n-i} = \frac{2}{n} - \sum_{i=1}^{n} r_i,$$

so $\sum_{k=1}^{n} r_k = \frac{1}{n}$. Therefore,

$$\sum_{k=1}^{n} \frac{(n+k)^4}{n^3+k^3} = \sum_{k=1}^{n} \left(k+4n+3n^2 r_k\right) = \frac{n(n+1)}{2} + 4n^2 + 3n = \frac{n(9n+7)}{2}.$$

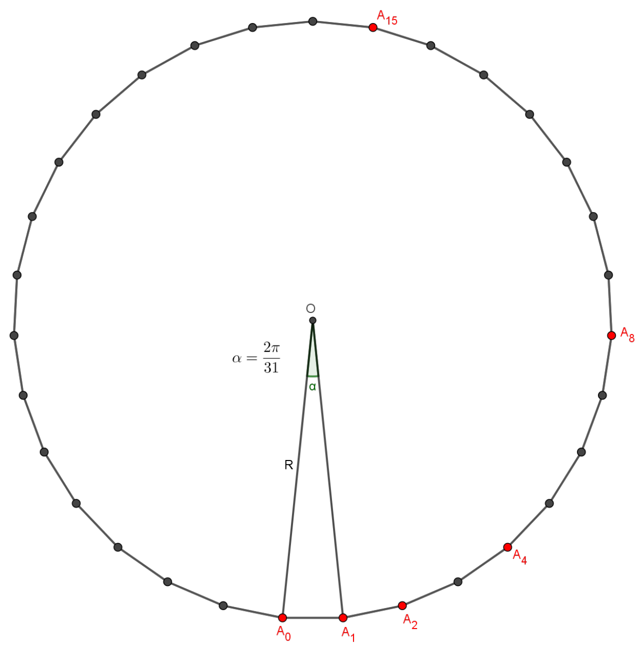
Also solved by Daniel Lasaosa, Pamplona, Spain; Jeffrey Roh, St. Andrew's School, DE, USA; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Sebastian Foulger, Charters School, Sunningdale, England, UK; Jeewoo Lee, Townsend Harris HS, NY, USA; Jenna Park, Blair Academy, Blairstown, NJ, USA; Kelvin Kim, Bergen Catholic High School, NJ, USA; Chirita Andrei-Giovani, The Greek Catholic Timotei Cipariu High School, Bucharest, Romania; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Lukas Seier, Charters Sixth Form, Sunningdale, England, UK; Albert Stadler, Herrliberg, Switzerland; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Dimoulios Ioannis, Thessaloniki, Greece; Dumitru Barac, Sibiu, Romania; Joonsoo Lee, Dwight Englewood School, NJ, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Sushanth Sathish Kumar, PHS, SoCal, USA; Jiho Lee, Canterbury School, New Milford, CT, USA.

S482. Prove that in any regular 31-gon $A_0A_1\ldots A_{30}$ the following equality holds:

$$\frac{1}{A_0A_1} = \frac{1}{A_0A_2} + \frac{1}{A_0A_4} + \frac{1}{A_0A_8} + \frac{1}{A_0A_{15}}.$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

Solution by Andrea Fanchini, Cantù, Italy



We denote with $\alpha = \frac{2\pi}{31}$ and with R the circumradius from the center of the regular 31-gon to one of the vertices.
We apply the cosine theorem to the triangle OA_0A_1

$$A_0A_1^2 = 2R^2(1-\cos\alpha) = 4R^2\sin^2\frac{\alpha}{2} \quad\Rightarrow\quad A_0A_1 = 2R\sin\frac{\alpha}{2}$$

in the same way from the triangles OA_0A_2, OA_0A_4, OA_0A_8 we obtain

$$A_0A_2 = 2R\sin\alpha, \qquad A_0A_4 = 2R\sin 2\alpha, \qquad A_0A_8 = 2R\sin 4\alpha$$

and from the triangle OA_0A_{15} we have

$$A_0A_{15}^2 = 2R^2\left(1-\cos\left(15\cdot\frac{2\pi}{31}\right)\right) = 2R^2\left(1-\cos\left(16\cdot\frac{2\pi}{31}\right)\right) = 4R^2\sin^2 8\alpha \quad\Rightarrow\quad 2R\sin 8\alpha$$

therefore we have to prove that

$$\csc\frac{\alpha}{2} = \csc\alpha + \csc 2\alpha + \csc 4\alpha + \csc 8\alpha \qquad (1)$$

Now we have

$$\csc\alpha + \cot\alpha = \frac{1}{\sin\alpha} + \frac{\cos\alpha}{\sin\alpha} = \frac{1+\cos\alpha}{\sin\alpha}$$

but $\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$ then

$$\csc\alpha + \cot\alpha = \frac{2\cos^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \cot\frac{\alpha}{2} \quad \Rightarrow \quad \csc\alpha = \cot\frac{\alpha}{2} - \cot\alpha$$

and in the same way we have that

$$\csc\frac{\alpha}{2} = \cot\frac{\alpha}{4} - \cot\frac{\alpha}{2}, \qquad \csc 2\alpha = \cot\alpha - \cot 2\alpha, \qquad \csc 4\alpha = \cot 2\alpha - \cot 4\alpha, \qquad \csc 8\alpha = \cot 4\alpha - \cot 8\alpha$$

so the (1) is also equal to

$$2\cot\frac{\alpha}{2} = \cot\frac{\alpha}{4} + \cot 8\alpha \quad \Rightarrow \quad 2\cot\frac{\pi}{31} = \cot\frac{\frac{\pi}{31}}{2} + \cot\frac{16\pi}{31} =$$

and finally the RHS is

$$= \cot\frac{\frac{\pi}{31}}{2} - \tan\frac{\frac{\pi}{31}}{2} = \frac{1+\cos\frac{\pi}{31}}{\sin\frac{\pi}{31}} - \frac{1-\cos\frac{\pi}{31}}{\sin\frac{\pi}{31}} = 2\cot\frac{\pi}{31}$$

Q.e.d.

Also solved by Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Li Zhou, Polk State College, USA; Albert Stadler, Herrliberg, Switzerland; Dimoulios Ioannis, Thessaloniki, Greece; Ioannis D. Sfikas, Athens, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania.

S483. For any real number a let $\lfloor a \rfloor$ and $\{a\}$ be the greatest integer less than or equal to a and the fractional part of a, respectively. Solve the equation

$$16x\lfloor x \rfloor - 10\{x\} = 2019.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan
Let $n = \lfloor x \rfloor$ and $a = \{x\}$.

$$\begin{aligned} &16x\lfloor x \rfloor - 10\{x\} = 2019 \\ &\Leftrightarrow 16n(n+a) - 10a = 2019 \\ &\Leftrightarrow a = \frac{2019 - 16n^2}{16n - 10}. \end{aligned}$$

We obtain $0 \le a = \frac{2019-16n^2}{16n-10} < 1$. If $n \ge 1$, then $2019 - 16n^2 \ge 0$ and $2019 - 16n^2 < 16n - 10$. Therefore we obtain $n = 11$, $a = 0.5$ and $x = 11.5$. Similarly, if $n \le 0$, then $16n^2 - 2019 \ge 0$ and $16n^2 - 2019 < 10 - 16n$, no solution satisfying these inequalities. The only solution is $x = 11.5$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Jeffrey Roh, St. Andrew's School, DE, USA; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Sebastian Foulger, Charters School, Sunningdale, England, UK; Jeewoo Lee, Townsend Harris HS, NY, USA; Min Jung Kim, Tabor Academy, MA, USA; Kelvin Kim, Bergen Catholic High School, NJ, USA; Chirita Andrei-Giovani, The Greek Catholic Timotei Cipariu High School, Bucharest, Romania; Bryant Hwang, Korea International School; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Li Zhou, Polk State College, USA; Albert Stadler, Herrliberg, Switzerland; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Joe Simons, Utah Valley University, UT, USA; Ioannis D. Sfikas, Athens, Greece; Joonsoo Lee, Dwight Englewood School, NJ, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Sushanth Sathish Kumar, PHS, SoCal, USA; Sachin Kumar, Bhabha Atomic Research Center, Mumbai, India; Sumanth Ravipati, George Mason University, VA, USA; Titu Zvonaru, Comănești, Romania; Jiho Lee, Canterbury School, New Milford, CT, USA.

S484. Let a, b, c be positive real numbers such that $a + b + c = 2$. Prove that

$$a^2\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)+b^2\left(\frac{1}{c}-1\right)\left(\frac{1}{a}-1\right)+c^2\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\geq\frac{1}{3}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Dimoulios Ioannis, Thessaloniki, Greece

Note that all $\sum$ notations refer to $\sum_{cyclic}$, unless otherwise mentioned.
Let $x = a/2, y = b/2, z = c/2$. Then $x + y + z = 1$. Plugging in it suffices to prove that

$$\begin{aligned}
\sum a^2\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)\geq\frac{1}{3} &\iff\\
\sum 4x^2\left(\frac{1}{2y}-1\right)\left(\frac{1}{2z}-1\right)\geq\frac{1}{3} &\iff\\
3\sum x^3(x-y+z)(x+y-z)\geq xyz &\iff\\
3\sum(x^5+2x^3yz-x^3y^2-x^3z^2)\geq xyz(x+y+z)^2 &\iff\\
3\sum x^5+6\sum x^3yz\geq\sum x^3yz+2\sum x^2y^2z+3\sum x^3y^2+3\sum x^3z^2 &\iff\\
3\sum x^5+5\sum x^3yz\geq 2\sum x^2y^2z+3\sum_{sym} x^3y^2 &
\end{aligned}$$

By Muirhead's Inequality

$$\sum_{sym} x^3yz\geq\sum_{sym} x^2y^2z$$

and by Shur's Inequality of 5^{th} degree and Muirhead's Inequality

$$\sum x^5+\sum x^3yz\geq\sum_{sym} x^4y\geq\sum_{sym} x^3y^2$$

Adding the second to the first inequality three times gives the desired result.

Also solved by Daniel Lasaosa, Pamplona, Spain; Nikos Kalapodis, Patras, Greece; Jeffrey Roh, St. Andrew's School, DE, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Chirita Andrei-Giovani, The Greek Catholic Timotei Cipariu High School, Bucharest, Romania; Li Zhou, Polk State College, USA; Albert Stadler, Herrliberg, Switzerland; Arkady Alt, San Jose, CA, USA; Dumitru Barac, Sibiu, Romania; Ioannis D. Sfikas, Athens, Greece; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Titu Zvonaru, Comănești, Romania.

S485. Find all positive integers n for which there is a real constant c such that

$$(c+1)\left(\sin^{2n}x+\cos^{2n}x\right)-c\left(\sin^{2(n+1)}x+\cos^{2(n+1)}x\right)=1,$$

for all real numbers x.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, USA

Let $f_n(c,x)$ be the left side of the given equation. Then $f_1(0,x)=\sin^2 x+\cos^2 x=1$ for all real x. Also,

$$\begin{aligned} f_2(2,x) &= 3\left(\sin^4 x+\cos^4 x\right)-2\left(\sin^6 x+\cos^6 x\right)\\ &= 3\left(\sin^2 x+\cos^2 x\right)^2-2\left(\sin^2 x+\cos^2 x\right)^3=1 \end{aligned}$$

for all real x. Now suppose that $n\geq 3$. We have $f_n\left(c,\frac{\pi}{4}\right)=\frac{c+2}{2^n}=1$, so $c=2^n-2$. Meanwhile,

$$f_n\left(c,\frac{\pi}{3}\right)=\frac{(c+1)\left(3^n+1\right)}{4^n}-\frac{c\left(3^{n+1}+1\right)}{4^{n+1}}=1,$$

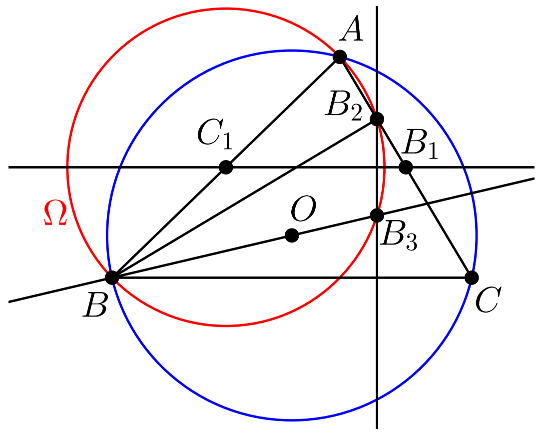
so $c=\frac{4^{n+1}-4(3^n+1)}{3^n+3}$. Equating these two expressions of c we get $4^{n+1}+2=6^n+2\cdot 3^n+3\cdot 2^n$, which is false for $n=3$. Moreover, since $4^5<6^4$, $4^{n+1}<6^n$ for all $n\geq 4$. Therefore, $n=1$ and 2 are the only such integers.

Also solved by Daniel Lasaosa, Pamplona, Spain; Jeffrey Roh, St. Andrew's School, DE, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Sebastian Foulger, Charters School, Sunningdale, England, UK; Jeewoo Lee, Townsend Harris HS, NY, USA; Chirita Andrei-Giovani, The Greek Catholic Timotei Cipariu High School, Bucharest, Romania; Jiho Lee, Canterbury School, New Milford, CT, USA; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Ploiești, Romania; Ioannis D. Sfikas, Athens, Greece; Joonsoo Lee, Dwight Englewood School, NJ, USA; Dumitru Barac, Sibiu, Romania.

S486. Let ABC be an acute triangle. Let B_1, C_1 be the midpoint of AC and AB, respectively and B_2, C_2 be the foot of altitude from B, C, respectively. Let B_3, C_3 be the reflection of B_2, C_2 across the line B_1C_1. The lines BB_3 and CC_3 intersect in X. Prove that $XB = XC$.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution by Li Zhou, Polk State College, USA



Since C_1 is the center of the circumcircle Ω of ABB_2, B_3 is on Ω. Since $C_1B_1 \parallel BC$, $B_2B_3 \perp BC$. Hence, $\angle CBB_3 = 90^\circ - A$, which implies that the circumcenter O of ABC is on BB_3. Likewise, O is on CC_3 as well. Therefore, $X = O$, completing the proof.

Also solved by Andrea Fanchini, Cantù, Italy; Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Chirita Andrei-Giovani, The Greek Catholic Timotei Cipariu High School, Bucharest, Romania; Corneliu Mănescu-Avram, Ploiești, Romania; Dimoulios Ioannis, Thessaloniki, Greece; Albert Stadler, Herrliberg, Switzerland; Sushanth Sathish Kumar, PHS, SoCal, USA; Titu Zvonaru, Comănești, Romania.

Undergraduate problems

U481. Evaluate

$$\lim_{n\to\infty} \frac{1}{n}\left(\left\lfloor e^{\frac{1}{n}}\right\rfloor + \left\lfloor e^{\frac{2}{n}}\right\rfloor + \cdots + \left\lfloor e^{\frac{n}{n}}\right\rfloor\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain

Since for all $1 \leq k \leq n$ we have $0 < \frac{k}{n} \leq 1$, then $3 > e \geq e^{\frac{k}{n}} > 1$, or $\left\lfloor e^{\frac{k}{n}}\right\rfloor$ takes only values 1 or 2, taking value 2 iff $k \geq n\ln 2$. For each n, denote $K_n = \lfloor n\ln 2\rfloor$, where since $\ln 2$ is irrational, the value is 1 for $k = 1, 2, \ldots, K_n$ and 2 for $k = K_n + 1, K_n + 2, \ldots, n$, resulting in

$$\left\lfloor e^{\frac{1}{n}}\right\rfloor + \left\lfloor e^{\frac{2}{n}}\right\rfloor + \cdots + \left\lfloor e^{\frac{n}{n}}\right\rfloor = K_n + 2(n - K_n) = 2n - K_n.$$

Now, clearly a real $0 < \delta_n < 1$ exists such that $K_n = n\ln 2 - \delta_n$, or the proposed limit equals

$$\lim_{n\to\infty} \frac{2n - K_n}{n} = 2 - \ln 2 + \lim_{n\to\infty} \frac{\delta_n}{n} = 2 - \ln 2.$$

The conclusion follows.

Also solved by Jeffrey Roh, St. Andrew's School, DE, USA; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Sebastian Foulger, Charters School, Sunningdale, England, UK; Jeewoo Lee, Townsend Harris HS, NY, USA; Jenna Park, Blair Academy, Blairstown, NJ, USA; Min Jung Kim, Tabor Academy, MA, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Li Zhou, Polk State College, USA; Jiho Lee, Canterbury School, New Milford, CT, USA; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Henry Ricardo, Westchester Area Math Circle, NY, USA; Joel Schlosberg, Bayside, NY, USA; Ioannis D. Sfikas, Athens, Greece; Joonsoo Lee, Dwight Englewood School, NJ, USA; Khakimboy Egamberganov, ICTP; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Sumanth Ravipati, George Mason University, VA, USA; Sushanth Sathish Kumar, PHS, SoCal, USA; Tamoghno Kandar, IIT Bombay, India; Thiago Landim de Souza Leão, Federal University of Pernambuco, Recife, Brazil; Akash Singha Roy, Chennai Mathematical Institute, India; Prajnanaswaroopa S, Amrita University, Coimbatore, India; Arkady Alt, San Jose, CA, USA.

U482. For any positive integer n consider the polynomial $f_n = x^{2n} + x^n + 1$. Prove that for any positive integer m there is a positive integer n such that f_n has exactly m irreducible factors in $\mathbb{Z}[X]$.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

First solution by Daniel Lasaosa, Pamplona, Spain
Note first that all $n \geq 1$, we have

$$x^{3\cdot 2^{n+1}} - 1 = \left(x^{3\cdot 2^{n-1}} - 1\right)\left(x^{3\cdot 2^{n-1}} + 1\right)\left(x^{2^n} + 1\right)\left(x^{2^{n+1}} - x^{2^n} + 1\right),$$

or since the cyclotomic polynomial $\Phi_{3\cdot 2^{n+1}}(x)$ has degree $\varphi\left(3 \cdot 2^{n+1}\right) = 2 \cdot 2^n = 2^{n+1}$, where φ is the Euler totient function, and it is irreducible in $\mathbb{Z}[X]$, then

$$\Phi_{3\cdot 2^{n+1}}(x) = x^{2^{n+1}} - x^{2^n} + 1,$$

and it is irreducible.

Lemma: For every non negative integer k, the polynomial

$$p_k(x) = x^{2^{k+1}} + x^{2^k} + 1$$

is the product of exactly $k+1$ irreducible factors in $\mathbb{Z}[X]$.

Proof: For $k = 0$, we have $p_0(x) = x^2 + x + 1$ is irreducible in $\mathbb{Z}[X]$, since otherwise either -1 or $+1$ would be a root, but $p_0(-1) = 1$ and $p_0(1) = 3$. If the Lemma holds for k, note that for $k+1$ we have

$$p_{k+1}(x) = \left(x^{2^{k+1}} + x^{2^k} + 1\right)\left(x^{2^{k+1}} - x^{2^k} + 1\right) = p_k(x) \cdot \Phi_{3\cdot 2^{n+1}}(x)$$

or since p_k has exactly $k+1$ irreducible factors and $\Phi_{3\cdot 2^{n+1}}(x)$ is irreducible, then p_{k+1} has exactly $k+2$ irreducible factors in $\mathbb{Z}[X]$. The Lemma follows.

By the Lemma, for every positive integer m, $p_{m-1}(x) = f_{2^{m-1}}(x)$ has exactly m irreducible factors in $\mathbb{Z}[X]$. The conclusion follows.

Second solution by Li Zhou, Polk State College, USA

Actually, for each m, there are infinitely many such n. Indeed, take any prime p different from 3, and let $n = p^{m-1}$. Denote by $\Phi_d(x)$ the d-th cyclotomic polynomial.

Then it is well known (`http://mathworld.wolfram.com/CyclotomicPolynomial.html`) that

$$f_n(x) = \frac{x^{3n}-1}{x^n-1} = \frac{\prod_{d|3n}\Phi_d(x)}{\prod_{d|n}\Phi_d(x)} = \prod_{k=1}^{m}\Phi_{3p^{k-1}}(x)$$

and each $\Phi_{3p^{k-1}}(x)$ is irreducible in $\mathbb{Z}[X]$.

Also solved by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Akash Singha Roy, Chennai Mathematical Institute, India; Joel Schlosberg, Bayside, NY, USA; Ioannis D. Sfikas, Athens, Greece; Moubinool Omarjee, Lycée Henri IV, Paris, France; Albert Stadler, Herrliberg, Switzerland.

U483. Evaluate

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j<k\le n}\cot^{-1}\left(\frac{i}{n}\right)\cot^{-1}\left(\frac{j}{n}\right)\cot^{-1}\left(\frac{k}{n}\right)$$

Proposed by Nicuşor Zlota, Focşani, Romania

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA
Recognize that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j<k\le n}&\cot^{-1}\left(\frac{i}{n}\right)\cot^{-1}\left(\frac{j}{n}\right)\cot^{-1}\left(\frac{k}{n}\right)\\&=\int_0^1\int_0^x\int_0^y\cot^{-1}z\cot^{-1}y\cot^{-1}x\,dz\,dy\,dx.\end{aligned}$$

By integration by parts,

$$\int\cot^{-1}z\,dz=z\cot^{-1}z+\int\frac{z}{1+z^2}\,dz=z\cot^{-1}z+\frac{1}{2}\ln(1+z^2)+C,$$

so

$$\int_0^y\cot^{-1}z\,dz=y\cot^{-1}y+\frac{1}{2}\ln(1+y^2).$$

Next,

$$\begin{aligned}\int_0^x\cot^{-1}y\left(y\cot^{-1}y+\frac{1}{2}\ln(1+y^2)\right)dy&=\frac{1}{2}\left(y\cot^{-1}y+\frac{1}{2}\ln(1+y^2)\right)^2\Bigg|_0^x\\&=\frac{1}{2}\left(x\cot^{-1}x+\frac{1}{2}\ln(1+x^2)\right)^2\end{aligned}$$

and

$$\begin{aligned}\int_0^1\cot^{-1}x\cdot\frac{1}{2}\left(x\cot^{-1}x+\frac{1}{2}\ln(1+x^2)\right)^2dx&=\frac{1}{6}\left(x\cot^{-1}x+\frac{1}{2}\ln(1+x^2)\right)^3\Bigg|_0^1\\&=\frac{1}{6}\left(\frac{\pi}{4}+\frac{1}{2}\ln 2\right)^3=\frac{1}{384}(\pi+2\ln 2)^3.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j<k\le n}\cot^{-1}\left(\frac{i}{n}\right)\cot^{-1}\left(\frac{j}{n}\right)\cot^{-1}\left(\frac{k}{n}\right)=\frac{1}{384}(\pi+2\ln 2)^3.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Li Zhou, Polk State College, USA; Akash Singha Roy, Chennai Mathematical Institute, India; Albert Stadler, Herrliberg, Switzerland; Dumitru Barac, Sibiu, Romania; Ioannis D. Sfikas, Athens, Greece.

U484. Find all polynomials $P(x)$ for which:

$$P(a+b)=6\left(P(a)+P(b)\right)+15a^2b^2(a+b),$$

for all complex numbers a and b such that $a^2+b^2=ab$.

Proposed by Titu Andreescu, USA and Mircea Becheanu, Canada

Solution by Li Zhou, Polk State College, USA

Letting $a=b=0$ yields $P(0)=12P(0)$, so $P(0)=0$. Clearly, $P\not\equiv 0$. Thus $P(x)=c_nx^n+\cdots+c_1x$, with $n\geq 1$ and $c_n\neq 0$.

Suppose that $a^2+b^2=ab$. We claim that for each $k\geq 1$, there is a constant f_k such that $a^k+b^k=f_k(a+b)^k$. Indeed, $f_1=1$, and since $a^2+b^2=(a+b)^2-2ab=(a+b)^2-2\left(a^2+b^2\right)$, $f_2=\frac{1}{3}$. Inductively, for $k\geq 2$,

$$a^{k+1}+b^{k+1}=\left(a^k+b^k\right)(a+b)-ab\left(a^{k-1}+b^{k-1}\right)=f_k(a+b)^{k+1}-\frac{1}{3}f_{k-1}(a+b)^{k+1},$$

that is, $f_{k+1}=f_k-\frac{1}{3}f_{k-1}$. In particular, $f_3=0$ and $f_4=-\frac{1}{9}=f_5$. Using a generating function we can also obtain $f_k=\left(\frac{\sqrt{3}+i}{2\sqrt{3}}\right)^k+\left(\frac{\sqrt{3}-i}{2\sqrt{3}}\right)^k$. Therefore, for $k\geq 6$,

$$|6f_k|\leq 12\left|\frac{\sqrt{3}+i}{2\sqrt{3}}\right|^k=12\left(\frac{1}{\sqrt{3}}\right)^k\leq\frac{12}{27}<1.$$

Equating coefficients of $(a+b)^k$ in $P(a+b)=6(P(a)+P(b))+\frac{5}{3}(a+b)^5$, we get $(1-6f_k)c_k=0$ for all $k\neq 5$, thus $c_k=0$ for all $k\neq 5$. Also, $c_5=6f_5c_5+\frac{5}{3}=-\frac{2}{3}c_5+\frac{5}{3}$, so $c_5=1$. In conclusion, $P(x)=x^5$ is the only solution.

Also solved by Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Jeffrey Roh, St. Andrew's School, DE, USA; Akash Singha Roy, Chennai Mathematical Institute, India; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Ploiești, Romania; Ioannis D. Sfikas, Athens, Greece.

U485. Let $f:[0,1]\to(0,\infty)$ be a continuous function and let A be the set of all positive integers n for which there is a real number x_n such that

$$\int_{x_n}^{1} f(t)dt = \frac{1}{n}.$$

Prove that the set $\{x_n\}_{n\in A}$ is an infinite sequence and find

$$\lim_{n\to\infty} n\left(x_n-1\right).$$

Proposed by Florin Rotaru, Focşani, Romania

Solution by Thiago Landim de Souza Leão, Federal University of Pernambuco, Recife, Brazil

Since $f(t)>0\ \forall t\in[0,1]$, then $\int_0^1 f(t)ṭ = \epsilon > 0$ and $\exists n_0$ such that $\epsilon > \frac{1}{n_0}$. Moreover,

$$F(x) = \int_x^1 f(t)ṭ$$

is a decreasing differentiable function $F\colon[0,1]\to[0,\epsilon]$ and satisfying $F'(x) = -f(x)$.

Hence, $\forall n\geq n_0\ \exists x_n$ such that $F(x_n)=\frac{1}{n}$.

Even more, by the Mean Value Theorem, $\exists\xi\in[x_n,1]$ such that

$$F(1)-F(x_n) = (1-x_n)F'(\xi) = (1-x_n)f(\xi)$$

and $n(1-x_n) = -\frac{1}{f(\xi)}$. Taking $n\to\infty$, we get

$$\lim_{n\to\infty} n(1-x_n) = -\frac{1}{f(1)}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Li Zhou, Polk State College, USA; Akash Singha Roy, Chennai Mathematical Institute, India; Prajnanaswaroopa S, Amrita University, Coimbatore, India; Khakimboy Egamberganov, ICTP; Moubinool Omarjee, Lycée Henri IV, Paris, France; Sergio Esteban Muñoz, Universidad de Buenos Aires, Argentina; Albert Stadler, Herrliberg, Switzerland.

U486. Let $\lfloor x \rfloor$ be the floor function and let $k \geq 3$ be a positive integer. Evaluate

$$\int_0^\infty \frac{\lfloor x \rfloor}{x^k} dx.$$

Proposed by Metin Can Aydemir, Ankara, Turkey

Solution by Joel Schlosberg, Bayside, NY, USA

$$\begin{aligned}
&\int_0^\infty \frac{\lfloor x \rfloor}{x^k}\, dx \\
&= \sum_{n=0}^{\infty} \int_n^{n+1} \frac{\lfloor x \rfloor}{x^k}\, dx \\
&= \sum_{n=0}^{\infty} \int_n^{n+1} \frac{n}{x^k}\, dx \\
&= 0 + \sum_{n=1}^{\infty} \frac{n}{(1-k)x^{k-1}} \Bigg|_n^{n+1} \\
&= \frac{1}{k-1} \sum_{n=1}^{\infty} \left(\frac{n}{n^{k-1}} - \frac{n}{(n+1)^{k-1}} \right) \\
&= \frac{1}{k-1} \sum_{n=1}^{\infty} \left(\frac{1}{n^{k-2}} - \frac{1}{(n+1)^{k-2}} + \frac{1}{(n+1)^{k-1}} \right) \\
&= \frac{1}{k-1} \left[\zeta(k-2) - (\zeta(k-2) - 1) + (\zeta(k-1) - 1) \right] \\
&= \frac{\zeta(k-1)}{k-1}.
\end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Jeffrey Roh, St. Andrew's School, DE, USA; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Sebastian Foulger, Charters School, Sunningdale, England, UK; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Li Zhou, Polk State College, USA; Akash Singha Roy, Chennai Mathematical Institute, India; Prajnanaswaroopa S, Amrita University, Coimbatore, India; Albert Stadler, Herrliberg, Switzerland; Corneliu Mănescu-Avram, Ploiești, Romania; Henry Ricardo, Westchester Area Math Circle, NY, USA; Ioannis D. Sfikas, Athens, Greece; Khakimboy Egamberganov, ICTP; Daniel López-Aguayo, Instituto Tecnológico y de Estudios Superiores de Monterrey, Mexico; Moubinool Omarjee, Lycée Henri IV, Paris, France; Sumanth Ravipati, George Mason University, VA, USA; Sushanth Sathish Kumar, PHS, SoCal, USA; Thiago Landim de Souza Leão, Federal University of Pernambuco, Recife, Brazil.

Olympiad problems

O481. Prove that

$$\prod_{k=1}^{n}\left(1-4\sin\frac{\pi}{5^k}\sin\frac{3\pi}{5^k}\right)=-\sec\frac{\pi}{5^n}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Li Zhou, Polk State College, USA

By the double-angle and product-to-sum formulas,

$$\cos\frac{\pi}{5^k}\left(1-4\sin\frac{\pi}{5^k}\sin\frac{3\pi}{5^k}\right)=\cos\frac{\pi}{5^k}-2\sin\frac{2\pi}{5^k}\sin\frac{3\pi}{5^k}=\cos\frac{\pi}{5^k}+\cos\frac{\pi}{5^{k-1}}-\cos\frac{\pi}{5^k}=\cos\frac{\pi}{5^{k-1}}.$$

By telescoping,

$$\prod_{k=1}^{n}\left(1-4\sin\frac{\pi}{5^k}\sin\frac{3\pi}{5^k}\right)=\prod_{k=1}^{n}\cos\frac{\pi}{5^{k-1}}\sec\frac{\pi}{5^k}=\cos\frac{\pi}{5^0}\sec\frac{\pi}{5^n}=-\sec\frac{\pi}{5^n}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Sebastian Foulger, Charters School, Sunningdale, England, UK; Jeewoo Lee, Townsend Harris HS, NY, USA; Min Jung Kim, Tabor Academy, MA, USA; Chirita Andrei-Giovani, The Greek Catholic Timotei Cipariu High School, Bucharest, Romania; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Albert Stadler, Herrliberg, Switzerland; Dumitru Barac, Sibiu, Romania; Joonsoo Lee, Dwight Englewood School, NJ, USA; M.A.Prasad, India; Moubinool Omarjee, Lycée Henri IV, Paris, France; Sumanth Ravipati, George Mason University, VA, USA; Titu Zvonaru, Comănești, Romania; Akash Singha Roy, Chennai Mathematical Institute, India; Jiho Lee, Canterbury School, New Milford, CT, USA.

O482. Let a,b,c be positive real numbers such that $a^2+b^2+c^2=1$. Prove that

$$\frac{a^2}{c^3}+\frac{b^2}{a^3}+\frac{c^2}{b^3}\geq(a+b+c)^3.$$

Proposed by Dragoljub Miloşević, Gornji Milanovac, Serbia

Solution by Adamopoulos Dionysios, 3rd High School, Pyrgos, Greece

By Hölder inequality we have that

$$\left(\frac{a^2}{c^3}+\frac{b^2}{a^3}+\frac{c^2}{b^3}\right)(a^2+b^2+c^2)(c^2+a^2+b^2)\geq\left(\sqrt[3]{\frac{a^4}{c}}+\sqrt[3]{\frac{b^4}{a}}+\sqrt[3]{\frac{c^4}{b}}\right)^3$$

Since $a^2+b^2+c^2=1$, we obtain that

$$\frac{a^2}{c^3}+\frac{b^2}{a^3}+\frac{c^2}{b^3}\geq\left(\sqrt[3]{\frac{a^4}{c}}+\sqrt[3]{\frac{b^4}{a}}+\sqrt[3]{\frac{c^4}{b}}\right)^3$$

As a result it suffices to prove that

$$\sqrt[3]{\frac{a^4}{c}}+\sqrt[3]{\frac{b^4}{a}}+\sqrt[3]{\frac{c^4}{b}}\geq a+b+c$$

Setting $a=x^3$, $b=y^3$, $c=z^3$ our inequality becomes

$$\frac{x^4}{z}+\frac{y^4}{x}+\frac{z^4}{y}\geq x^3+y^3+z^3$$

which is obvious due to the rearrangement inequality for triples (x^4,y^4,z^4) and $\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Nikos Kalapodis, Patras, Greece; Albert Stadler, Herrliberg, Switzerland; Ioan Viorel Codreanu, Satulung, Maramureş, Romania; Dimoulios Ioannis, Thessaloniki, Greece; Dumitru Barac, Sibiu, Romania; Erfan Amouzad Khalili, Iran; Jamal Gadirov, Istanbul University, Turkey; Ioannis D. Sfikas, Athens, Greece; Khakimboy Egamberganov, ICTP; Leung Hei Chun, SKH Tang Shiu Kin Secondary School, Hong Kong; M.A.Prasad, India; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nick Iliopoulos, 8th General Highschool of Trikala, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Sarah B. Seales, Prescott, AZ, USA; Sumanth Ravipati, George Mason University, VA, USA; Sushanth Sathish Kumar, PHS, SoCal, USA; Titu Zvonaru, Comănești, Romania; Li Zhou, Li Zhou, Polk State College, USA; Akash Singha Roy, Chennai Mathematical Institute, India; Chirita Andrei-Giovani, The Greek Catholic Timotei Cipariu High School, Bucharest, Romania.

O483. Find all integers n for which $\left(4n^2-1\right)\left(n^2+n\right)+2019$ is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan
Let $f(n)=\sqrt{(4n^2-1)(n^2-n)+2019}$, then

$$f(1)=45,\quad 45<f(2)<46,,\quad 49<f(3)<50,\quad 57<f(4)<58,\quad 70<f(5)<71,$$

$$89<f(6)<90,\quad 113<f(7)<114,\quad 142<f(8)<143,\quad 176<f(9)<177.$$

For $n\geq 10$ we obtain $2n^2+n-1<f(n)<2n^2+n+5$, that is,

$$f(n)=2n^2+n,\ 2n^2+n+1,\ 2n^2+n+2,\ 2n^2+n+3,\ 2n^2+n+4.$$

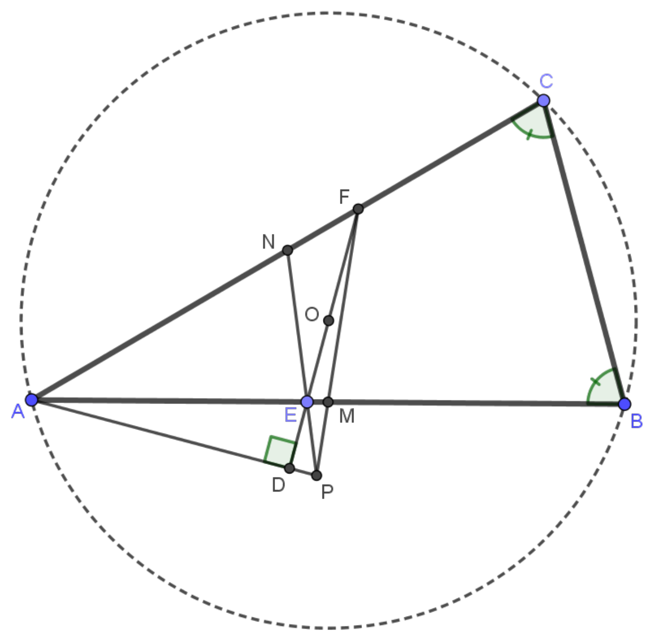
If $f(n)=2n^2+n$, no integer solution. If $f(n)=2n^2+n+1$, no integer solution. If $f(n)=2n^2+n+2$, no integer solution. If $f(n)=2n^2+n+3$, no integer solution. If $f(n)=2n^2+n+4$, no integer solution. Therefore the only solution is $n=1$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Jeffrey Roh, St. Andrew's School, DE, USA; Sebastian Foulger, Charters School, Sunningdale, England, UK; Jeewoo Lee, Townsend Harris HS, NY, USA; Chirita Andrei-Giovani, The Greek Catholic Timotei Cipariu High School, Bucharest, Romania; Albert Stadler, Herrliberg, Switzerland; Nick Iliopoulos, 8th General Highschool of Trikala, Greece; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Ioannis D. Sfikas, Athens, Greece; Khakimboy Egamberganov, ICTP; M.A.Prasad, India; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Sumanth Ravipati, George Mason University, VA, USA; Titu Zvonaru, Comănești, Romania; Akash Singha Roy, Chennai Mathematical Institute, India; Li Zhou, Polk State College, USA.

O484. Let ABC be a triangle with $AB = AC$. Points E and F lie on AB and AC, respectively so that EF passes through the circumcenter of ABC. Let M be the midpoint of AB, let N be the midpoint of AC and set $P = FM \cap EN$. Prove that the lines AP and EF are perpendicular.

Proposed by Tovi Wen, USA

First solution by by Andrea Fanchini, Cantù, Italy



We use barycentric coordinates with reference to the triangle ABC.

We consider a generic point $E(t, 1-t, 0)$ on AB where t is a parameter. Then the line that passes through the circumcenter of ABC is

$$EO : (1-t)b^2x - b^2ty + ((3S_A + S_B)t - 2S_A)z = 0$$

so the point F is

$$F = EO \cap AC = (2S_A - (3S_A + S_B)t : 0 : (1-t)b^2)$$

and the line EF

$$EF : (1-t)b^2x - b^2ty - (2S_A - (3S_A + S_B)t)z = 0$$

now we have $M(1:1:0)$ and $N(1:0:1)$ therefore

$$FM : -(1-t)b^2x + (1-t)b^2y + (2S_A - (3S_A + S_B)t)z = 0$$

$$EN : (1-t)x - ty - (1-t)z = 0$$

Then, the point P is

$$P = FM \cap EN = \left(-2(S_B + 2S_A)t^2 + 2(S_B + 2S_A)t - b^2 : 2S_At^2 + (S_B - 3S_A)t + (S_A - S_B) : (1-t)(2t-1)b^2\right)$$

and the line AP

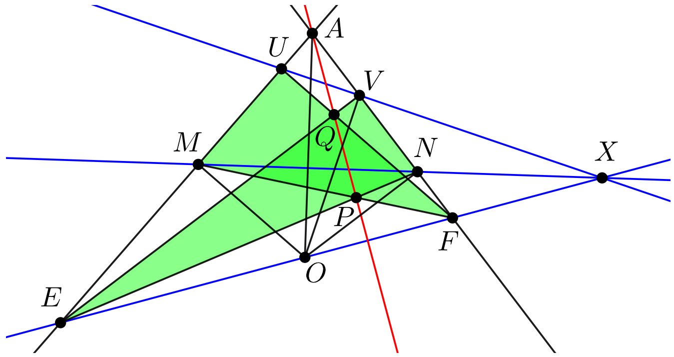
$$AP : (1-t)(2t-1)b^2y - (2S_At^2 + (S_B - 3S_A)t + (S_A - S_B))z = 0$$

Finally, we have

$$EF_\infty = AP_{\infty\perp} = (-2(2S_A + S_B)t + 2S_A : 2(2S_A + S_B)t - (3S_A + S_B) : S_A + S_B)$$

Q.e.d.

Second solution by Li Zhou, Polk State College, USA



Let O be the circumcenter of ABC, Q be the orthocenter of AEF, and EV and FU be two altitudes of AEF. It suffices to show that A, Q, P are collinear. By Desargues' theorem, this will follow from the claim that triangles ENV and FMU are perspective from a point X. Indeed,

$$\frac{AU}{AV} = \frac{AF}{AE} = \frac{FO}{OE} = \frac{FO}{FE} \cdot \frac{FE}{OE} = \frac{FN}{FV} \cdot \frac{UE}{ME},$$

thus

$$\frac{AU}{UE} \cdot \frac{FV}{VA} = \frac{FN}{ME} = \frac{AM}{ME} \cdot \frac{FN}{NA}.$$

By Menelaus' theorem, the lines UV, MN, and EF are concurrent, completing the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Chirita Andrei-Giovani, The Greek Catholic Timotei Cipariu High School, Bucharest, Romania; Adamopoulos Dionysios, 3rd High School, Pyrgos, Greece; Corneliu Mănescu-Avram, Ploiești, Romania; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

O485. Prove that any infinite set of positive integers contains two numbers whose sum has a prime divisor greater than 10^{2020}.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by Daniel Lasaosa, Pamplona, Spain

We will prove a (slightly) more general statement, namely that for any infinite set A of positive integers, there are infinitely many primes p such that one number of the form $m+n$, where $m \neq n$ and $m,n \in A$, is divisible by p.

Assume the contrary, ie that given an infinite set A, there is a finite set of distinct primes $p_1, p_2, \dots, p_u$ such that any number of the form $m+n$, with $m \neq n$ and $m,n \in A$, can be written as a product of said primes. Define the sequence of sets $A_0 = A, A_1, A_2, \dots, A_u$ such that $A_u \subset A_{u-1} \subset \dots \subset A_1 \subset A_0 = A$ are all infinite, and such that A_i is defined starting from A_{i-1} for $i = 1, 2, \dots, u$, depending on whether p_i satisfies one of the following two conditions:

Case 1: If there is a (possibly zero) multiplicity α_i such that there are infinitely many elements $n \in A$ divisible by p_i with multiplicity α_i, then $c = \frac{n}{{p_i}^{\alpha_i}}$ is a positive integer which is not a multiple of p_i for infinitely many elements $n \in A$; if this occurs for more than one multiplicity α_i, choose one of them at random. There is therefore one (or possible more than one) of the $p_i - 1$ possible nonzero remainders modulo p_i, denoted by r_i, which repeats itself infinitely many times in the c's (if more than one, then choose one at random). If p_i is an odd prime, define A_i as the infinite set of such n's. In the particular case where $p_i = 2$, note that the c's are odd, or each one leaves a remainder either 1 or 3 when divided by 4, one of these two remainders appearing infinitely many times (if both, then choose one of them at random). Define in this case A_i as the set of the infinitely many n's such that the corresponding c's leave this remainder.

Case 2: If no nonnegative multiplicity α_i is such that infinitely many elements $n \in A$ are divisible by p_i with multiplicity α_i, then there must be infinitely many multiplicities α_i such that at least one $n \in A$ is divided by p_i with multiplicity α_i. For each such multiplicity, chose at random exactly one of the n's which are divisible by p_i with multiplicity α_i, and let A_i be the set formed by the infinitely many such n's.

Consider now any two distinct elements $m,n \in A_u$, and let D be their greatest common divider. Since $D \leq m,n$ and $m \neq n$, we have $2D < m+n$.

Now, if p_i is an odd prime which falls under Case 1, clearly p_i divides D with multiplicity α_i since it divides both m,n with said multiplicity, and since $\frac{m+n}{{p_i}^{\alpha_i}}$ leaves a remainder $2r_i$ when divided by p_i, where r_i is a nonzero remainder modulo p_i, then p_i also divides $m+n$ with multiplicity α_i. If $p_i = 2$ falls under Case 1, then 2 divides D again with multiplicity α_i, but $\frac{m+n}{{p_i}^{\alpha_i}}$ yields a remainder of 2 when divided by 4, regardless of whether $\frac{m}{{p_i}^{\alpha_i}}$ and $\frac{n}{{p_i}^{\alpha_i}}$ yield both a remainder of 1, or both a remainder of 3, when divided by 4. Hence $p_i = 2$ divides $m+n$ with multiplicity $\alpha_i + 1$. On the other hand, if p_i is any prime which falls under Case 2, then $m+n$ is divisible by p_i with multiplicity equal to the least of the two multiplicities with which p_i divides m and n, and likewise for D.

It follows that, except for a possible additional factor of 2 in $m+n$ if prime $p_i = 2$ falls under Case 1, $m+n$ and D are divisible by exactly the same primes and with the same multiplicity, or $m+n \in \{D, 2D\}$, for $m+n \leq 2D < m+n$. The conclusion follows from this contradiction.

Also solved by Chirita Andrei-Giovani, The Greek Catholic Timotei Cipariu High School, Bucharest, Romania; M.A.Prasad, India; Akash Singha Roy, Chennai Mathematical Institute, India; Albert Stadler, Herrliberg, Switzerland.

O486. Let a, b, c be positive real numbers. Prove that

$$a^2+b^2+c^2 \geq a\sqrt[3]{\frac{b^3+c^3}{2}}+b\sqrt[3]{\frac{c^3+a^3}{2}}+c\sqrt[3]{\frac{a^3+b^3}{2}}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Li Zhou, Polk State College, USA

First,

$$\begin{aligned}&\left(a^2+b^2+c^2\right)^2-(a+b+c)\left[a\left(b^2-bc+c^2\right)+b\left(c^2-ca+a^2\right)+c\left(a^2-ab+b^2\right)\right]\\&=a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b)\geq 0\end{aligned}$$

by Schur's inequality. Now we use Hölder's inequality to get

$$\begin{aligned}&\left(a^2+b^2+c^2\right)^3=\left(a^2+b^2+c^2\right)\left(a^2+b^2+c^2\right)^2\\&\geq\left(a^2+b^2+c^2\right)\left(\frac{b+c}{2}+\frac{c+a}{2}+\frac{a+b}{2}\right)\left[a\left(b^2-bc+c^2\right)+b\left(c^2-ca+a^2\right)+c\left(a^2-ab+b^2\right)\right]\\&\geq\left(\sqrt[3]{a^2\cdot\frac{b+c}{2}\cdot a\left(b^2-bc+c^2\right)}+\sqrt[3]{b^2\cdot\frac{c+a}{2}\cdot b\left(c^2-ca+a^2\right)}+\sqrt[3]{c^2\cdot\frac{a+b}{2}\cdot c\left(a^2-ab+b^2\right)}\right)^3\\&=\left(a\sqrt[3]{\frac{b^3+c^3}{2}}+b\sqrt[3]{\frac{c^3+a^3}{2}}+c\sqrt[3]{\frac{a^3+b^3}{2}}\right)^3.\end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Nikos Kalapodis, Patras, Greece; Albert Stadler, Herrliberg, Switzerland; Ioan Viorel Codreanu, Satulung, Maramureş, Romania; Ioannis D. Sfikas, Athens, Greece; Khakimboy Egamberganov, ICTP; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Sarah B. Seales, Prescott, AZ, USA; Titu Zvonaru, Comănești, Romania; Akash Singha Roy, Chennai Mathematical Institute, India.