BEST POLYNOMIAL ESTIMATES IN A TRIANGLE

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Abstract. We prove the best polynomial estimates for some fourth degree polynomials on the side lengths of a triangle in terms of its inradius and circumradius.

1. Introduction

Let s, r, and R be the semi-perimeter, the inradius and the circumradius of a triangle, respectively. Then the Fundamental Inequality [3] says that

$$2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R(R-2r)} \leq s^2 \leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)},$$

where the equalities hold only for isosceles triangles. This double inequality was first proved by E. Rouche in 1851(c.f. [3]) and more than 100 years later Blundon [2] noticed that it is in fact the best inequality of the form

$$f(R,r) \leq s^2 \leq F(R,r),$$

where $f(x,y)$ and $F(x,y)$ are homogeneous real functions. Moreover, using the Fundamental Inequality, Blundon [2] proved that the best linear estimates for s in terms of r and R are the following:

$$3\sqrt{3}r \leq s \leq 2R + (3\sqrt{3}-4)r. \tag{1}$$

Blundon also found the best quadratic estimate for s^2 in terms of r and R, but this takes a little more explanation. First, it is natural to look only at inequalities that are equalities for the case of an equilateral triangle, that is, for $R = 2r$. For quadratic polynomials this means we are looking for upper and lower bounds of the form

$$2R^2 + 10Rr - r^2 \pm 2(R-2r)L(r,R)$$

for some linear polynomial $L(r,R)$. Comparing this to the Fundamental Inequality, we see that we need an upper bound $L(r,R) \geq \sqrt{R(R-2r)}$. Letting $x = R/r \geq 2$, we note that the graph $y = \sqrt{x^2-2x}$ is concave downward and hence any tangent line to this graph provides such an upper bound. A first reaction might be to look at $x = 2$ to get an upper bound that is optimal when the triangle is close to equilateral, but the tangent line at this point is vertical and we do not get an upper bound from it. Thus instead we ask for optimality at the other extreme, $x \to \infty$ or, equivalently, R much larger than r. From the asymptote (the tangent line at infinity) we get $\sqrt{x^2-2x} \leq x-1$ and hence $\sqrt{R(R-2r)} \leq R-r$. This gives Blundon's best quadratic inequalities

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2. \tag{2}$$

So, it is natural to ask what are the optimal inequalities of the form

$$p_n(R,r) \leq s^n \leq P_n(R,r), \tag{3}$$

where $p_n(x,y)$ and $P_n(x,y)$ are homogeneous polynomials of degree n, equality occurs for equilateral triangles, and the inequality is as strong as possible for R much bigger than r. A general approach for solving this problem for even n was developed in [4, 1] and in particular the following best polynomial inequalities were obtained:

$$24Rr - 12r^2 \leq a^2 + b^2 + c^2 \leq 8R^2 + 4r^2 \tag{4}$$

$$20Rr - 4r^2 \leq ab + bc + ca \leq 4R^2 + 8Rr + 4r^2 \tag{5}$$

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$$12\sqrt{3}Rr^2 \le abc \le 4Rr(2R + (3\sqrt{3} - 4)r) \tag{6}$$

$$288R^2r^2 - 368Rr^3 + 16r^4 \le a^4 + b^4 + c^4 \le 32R^4 - 16R^2r^2 - 16Rr^3 + 16r^4 \tag{7}$$

$$256R^2r^2 - 128Rr^3 - 39r^4 \le s^4 \le 16R^4 + 32R^3r + 32R^2r^2 + 24Rr^3 + 41r^4 \tag{8}$$

$$4096R^3r^3 - 3072R^2r^4 - 797r^6 \le s^6 \le 64R^6 + 192R^5r + 288R^4r^2 \tag{9}$$

$$+304R^3r^3 + 276R^2r^4 + 252Rr^5 + 795r^6.$$

In this paper we will use the above approach to obtain the best polynomial estimates for some fourth degree polynomials on the sides of a triangle in terms of its inradius and circumradius.

2. Preliminaries

The approach mentioned above is based on an estimate of the function $\sqrt{x^2 - 2x}$ by means of rational functions. Let

$$P(x) = a_0 + a_1x + \cdots + a_nx^n$$

be a polynomial of degree n with real coefficients. For every $0 \le k \le n$ denote by $P_k(x)$ the polynomial

$$P_k(x) = a_0 + a_1x + \cdots + a_kx^k$$

and by $P^*(x)$ the polynomial of degree $n+1$, defined by

$$P^*(x) = (x-1)P(x) - \sum_{k=1}^{n} \frac{(2k)!}{2^k k!(k+1)!} \cdot \frac{P(x) - P_{k-1}(x)}{x^k}. \tag{10}$$

Set

$$p^* = \inf_{(2,\infty)} (P^*(x) - P(x)\sqrt{x^2 - 2x}). \tag{11}$$

Then we have the following theorem [1].

Theorem 1. Let $P(x)$ be a polynomial of degree n with real coefficients. The best inequality of the form

$$P(x)\sqrt{x^2 - 2x} \le Q(x),\ x \in (2,\infty), \tag{12}$$

where $Q(x)$ is a polynomial of degree $n+1$ is obtained when $Q(x) = P^*(x) - p^*$.
Note that if $P^*(x) - P(x)\sqrt{x^2 - 2x}$ is a decreasing function in the interval $(2,\infty)$, then

$$p^* = \inf_{(2,\infty)} (P^*(x) - P(x)\sqrt{x^2 - 2x}) = \lim_{x\to\infty} (P^*(x) - P(x)\sqrt{x^2 - 2x}) = 0.$$

Similarly, if $P^*(x) - P(x)\sqrt{x^2 - 2x}$ is an increasing function in the interval $(2,\infty)$, then

$$p^* = \inf_{(2,\infty)} (P^*(x) - P(x)\sqrt{x^2 - 2x}) = \lim_{x\to 2} (P^*(x) - P(x)\sqrt{x^2 - 2x}) = P^*(2).$$

These observations imply the following

Corollary 1. Let $P(x)$ be a polynomial such that $P^*(x) - P(x)\sqrt{x^2 - 2x}$ is either a decreasing or an increasing function in the interval $(2,\infty)$. Then the best inequality of the form (12) is obtained in the first case for the polynomial $Q(x) = P^*(x)$ whereas in the second case it is reached for the polynomial $Q(x) = P^*(x) - P^*(2)$.

3. Best polynomial estimates for $(a-b)^4+(b-c)^4+(c-a)^4$

To obtain the best polynomial estimates for $(a-b)^4+(b-c)^4+(c-a)^4$ we first compute this expression in terms of s, r, R by using the following well known formulas for the symmetric functions of the sides a, b, c of a triangle:

$$ab+bc+ca = s^2+r^2+4Rr \tag{13}$$

$$a^2+b^2+c^2 = 2(s^2-r^2-4R^2) \tag{14}$$

$$abc = 4srR. \tag{15}$$

It is also easy to check that if $x+y+z=0$, then

$$x^4+y^4+z^4 = \frac{1}{2}(x^2+y^2+z^2)^2.$$

Hence

$$\begin{aligned}(a-b)^4+(b-c)^4+(c-a)^4 &= \frac{1}{2}(a-b)^2+(b-c)^2+(c-a)^2)^2 \\ &= 2(a^2+b^2+c^2-ab-bc-ca)^2\end{aligned}$$

and formulas (13) and (14) imply the identity

$$(a-b)^4+(b-c)^4+(c-a)^4 = 2s^4-12s^2(r^2+4Rr)+18(r^2+4Rr)^2. \tag{16}$$

Consider the following region in the plane:

$$\begin{aligned}\mathcal{B} = \{(x,y)\in\mathbb{R}^2 \mid 2x^2+10x-1-2(x-2)\sqrt{x^2-2x} \\ \le y^2 \le 2x^2+10x-1+2(x-2)\sqrt{x^2-2x}\}.\end{aligned}$$

Its boundary is defined by the curves

$$\beta_1 : y^2 = 2x^2+10x-1-2(x-2)\sqrt{x^2-2x},\ x\ge 2, \tag{17}$$

and

$$\beta_2 : y^2 = 2x^2+10x-1+2(x-2)\sqrt{x^2-2x},\ x\ge 2. \tag{18}$$

It is well-known [2] that up to similarity, the map $\left(\frac{R}{r}, \frac{s}{r}\right) \mapsto (x,y)$ is a one-to-one correspondence between the set of all triangles in the plane and the set $\mathcal{B}$. In this way, the boundary points of $\mathcal{B}$ correspond to the classes of similar isosceles triangles. Hence we have to find the best inequalities of the form

$$q(x) \le y^4-6y^2(4x+1)+9(4x+1)^2 \le Q(x) \tag{19}$$

for all points $(x,y)\in\mathcal{B}$, where $q(x)$ and $Q(x)$ are polynomials of degree 4. Having in mind (17) and (18) we see that both inequalities in (19) are reduced to the problem of finding the best inequality of the form

$$(x^2-x-2)\sqrt{x^2-2x} \le R(x), \tag{20}$$

where $R(x)$ is a cubic polynomial. Using the same notations as in the previous section we have $P(x) = x^2-x-2$ and $P^*(x) = x^3-2x^2-\frac{3}{2}x+2$. Set

$$f(x) = P^*(x)-P(x)\sqrt{x^2-2x} = x^3-2x^2-\frac{3}{2}x+2-(x^2-x-2)\sqrt{x^2-2x}.$$

Then

$$f'(x) = 3x^2-4x-\frac{3}{2}-\frac{3x^3-7x^2+x+2}{\sqrt{x^2-2x}}$$

and a long but straightforward computation shows that $f'(x)>0$ for $x>2$. Hence the function $f(x)$ is increasing and from Corollary 1 it follows that

$$R(x) = P^*(x)-P^*(2) = x^3-2x^2-\frac{3}{2}+3.$$

This together with (17) and (18) implies that the best inequalities of the form (18) are obtained for

$$Q(x) = (2x^2+10x-1)^2 + 4(x-2)^2(x^2-2x) - 6(4x+1)(2x^2+10x-1)$$
$$+9(4x+1)^2 + 8(x-2)(x^3-2x^2-\frac{3}{2}x+3) = 8(x-2)^2(2x^2-1)$$

and

$$q(x) = (2x^2+10x-1)^2 + 4(x-2)^2(x^2-2x) - 6(4x+1)(2x^2+10x-1)$$
$$+9(4x+1)^2 - 8(x-2)(x^3-2x^2-\frac{3}{2}x+3) = 16(x-2)^2.$$

Hence the best polynomial estimates for $(a-b)^4+(b-c)^4+(c-a)^4$ are:

(21) $$32r^2(R-2r)^2 \le (a-b)^4+(b-c)^4+(c-a)^4 \le 16(R-2r)^2(2R^2-r^2).$$

In both inequalities the equality is attained only for equilateral triangles.

4. Best polynomial estimates for $a^2b^2+b^2c^2+c^2a^2$

To obtain the best polynomial estimates for $a^2b^2+b^2c^2+c^2a^2$ we proceed as in the previous section. Formulas (13) and (15) give

$$a^2b^2+b^2c^2+c^2a^2 = (ab+bc+ca)^2 - 2abc(a+b+c)$$

(22) $$= (s^2+r^2+4Rr)^2 - 16s^2rR = s^4 - 2s^2(4Rr-r^2) + (4Rr+r^2)^2.$$

Hence we have to find the best inequalities of the form

(23) $$q(x) \le y^4 - 2y^2(4x-1) + (4x+1)^2 \le Q(x),$$

where $q(x)$ and $Q(x)$ are polynomials of degree 4. In this case the problem is reduced to finding the best inequality of the form

(24) $$(x^2+3x)\sqrt{x^2-x} \le R(x),$$

where $R(x)$ is a cubic polynomial. Now $P(x) = x^2+3x$ and we obtain by (10) that

$$P^*(x) = x^3+2x^2-\frac{7}{2}x-2.$$

Set

$$f(x) = x^3+2x^2-\frac{7}{2}x-2-(x^2+3x)\sqrt{x^2-2x}.$$

Then

$$f'(x) = 3x^2+4x-\frac{7}{2}-\frac{x(3x^2+x-9)}{\sqrt{x^2-2x}}$$

and it is easy to check that $f'(x)<0$ for $x>2$. Hence the function $f(x)$ is decreasing for $x>2$ and by Corollary 1 we obtain

$$R(x) = P^*(x) = x^3+2x^2-\frac{7}{2}x-2.$$

This together with (17) and (18) implies that the best inequalities of the form (22) are obtained for

$$Q(x) = (2x^2+10x-1)^2 + 4(x-2)^2(x^2-2x) - 2(4x-1)(2x^2+10x-1)$$
$$+(4x+1)^2 + 8(x-2)((x^3+2x^2-\frac{7}{2}x-2) = 16x^4+24x^2+24x+32$$

and

$$q(x) = (2x^2+10x-1)^2 + 4(x-2)^2(x^2-2x) - 2(4x-1)(2x^2+10x-1)$$
$$+(4x+1)^2 - 8(x-2)((x^3+2x^2-\frac{7}{2}x-2) = 144x^2-56x-32$$

Hence the best polynomial estimates for $a^2b^2+b^2c^2+c^2a^2$ are:

(25) $$144R^2r^2-56Rr^3-32r^4 \le a^2b^2+b^2c^2+c^2a^2 \le 16R^4+24R^2r^2$$
$$+24Rr^3+32r^4.$$

In both inequalities the equality is attained only for equilateral triangles.

References

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