Junior Problems

**J487.** Let $ABCD$ be a cyclic kite. Prove that $3\angle A = \angle C$ or $\angle A = \angle 3C$ if and only if

$$\frac{AC}{BD} - \frac{BD}{AC} = \frac{1}{\sqrt{2}}.$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

**J488.** Let $a$ and $b$ be positive real numbers such that $ab = a + b$. Prove that

$$\sqrt{1 + a^2} + \sqrt{1 + b^2} \geq \sqrt{20 + (a - b)^2}.$$

*Proposed by An Zhenping, Xianyang Normal University, China*

**J489.** Prove that in any triangle $ABC$

$$8r(R - 2r)\sqrt{r(16R - 5r)} \leq a^3 + b^3 + c^3 - 3abc \leq 8R(R - 2r)\sqrt{(2R + r)^2 + 2r^2}.$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

**J490.** Let $a, b, c$ be positive real numbers. Prove that

$$\frac{a^3}{1 + ab^2} + \frac{b^3}{1 + bc^2} + \frac{c^3}{1 + ca^2} \geq \frac{3abc}{1 + abc}.$$

*Proposed by An Zhenping, Xianyang and Li Xin, Wugong, China*

**J491.** Find all triples $(x, y, z)$ of positive integers such that

$$5(x^2 + 2y^2 + z^2) = 2(5xy - yz + 4zx)$$

and at least one of $x, y, z$ is a prime.

*Proposed by Adrian Andreescu, University of Texas at Austin, USA*

**J492.** Let $n > 1$ be an integer and let $a, b, c$ be positive real numbers such that $a^n + b^n + c^n = 3$. Prove that

$$\frac{1}{a^{n+1} + n} + \frac{1}{b^{n+1} + n} + \frac{1}{c^{n+1} + n} \geq \frac{3}{n + 1}.$$  

*Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia*
Senior Problems

S487. Find all primes \( a \geq b \geq c \geq d \) such that

\[
a^2 + 2b^2 + c^2 + 2d^2 = 2(ab + bc - cd + da).
\]

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

S488. Let \( a, b, c \) be positive real numbers such that \( a + b + c = 3 \). Prove that

\[
\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} + \frac{1}{3 \left( \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \right)} \geq 2.
\]

*Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam*

S489. Find all pairs \((m, n)\) of positive integers with \( m + n = 2019 \) for which there is a prime \( p \) such that

\[
\frac{4}{m+3} + \frac{4}{n+3} = \frac{1}{p}.
\]

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

S490. Prove that there is a real function \( f \) for which there is no a real function \( g \) such that \( f(x) = g(g(x)) \) for all \( x \in \mathbb{R} \).

*Proposed by Pavel Gadzinski, Bielsko-Biała, Poland*

S491. Prove that in any acute triangle \( ABC \) the following inequality holds:

\[
\frac{1}{(\cos \frac{A}{2} + \cos \frac{B}{2})^2} + \frac{1}{(\cos \frac{B}{2} + \cos \frac{C}{2})^2} + \frac{1}{(\cos \frac{C}{2} + \cos \frac{A}{2})^2} \geq 1.
\]

*Proposed by Florin Rotaru, Focşani, România*

S492. Find the greatest real constant \( C \) such that the inequality

\[
(a^2 + 2)(b^2 + 2)(c^2 + 2) - (abc - 1)^2 \geq C(a + b + c)^2
\]

holds for all positive real numbers \( a, b, c \).

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*
Undergraduate Problems

**U487.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the following conditions hold simultaneously:

(a) $f(f(x)) = x$ for all $x \in \mathbb{R}$,
(b) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$,
(c) $\lim_{x \to \infty} f(x) = -\infty$.

*Proposed by Vincelot Ravoson, Lycée Henry IV, Paris, France*

**U488.** Let $a$ and $b$ be positive real numbers. Evaluate

$$\int_{a-b}^{a+b} \frac{\arctan x}{x} \, dx.$$

*Proposed by Michele Caselli, University of Modena, Italy*

**U489.** Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(f(x))) - 3f(f(x)) + 3f(x) - x = 0$$

for all $x \in \mathbb{R}$.

*Proposed by Titu Andreescu, USA, and Marian Tetiva, România*

**U490.** Find the greatest real number $k$ such that the inequality

$$\frac{a^2}{b} + \frac{b^2}{a} \geq \frac{2(a^{k+1} + b^{k+1})}{a^k + b^k}$$

holds for all positive real numbers $a$ and $b$.

*Proposed by Nguyen Viet Hung and Vo Quoc Ba Can, Hanoi, Vietnam*

**U491.** Find all polynomials $P$ with complex coefficients such that

$$P(a) + P(b) = 2P(a + b),$$

whenever $a$ and $b$ are complex numbers satisfying $a^2 + 5ab + b^2 = 0$.

*Proposed by Titu Andreescu, USA and Mircea Becheanu, Canada*

**U492.** Let $C$ be an arbitrary positive real number and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1^2 + x_2^2 + \ldots + x_n^2 = n$. Prove that

$$\sum_{i=1}^{n} \frac{x_i}{x_i + C} \leq \frac{n}{C + 1}.$$

*Proposed by Angel Plaza, University of Las Palmas de Gran Canaria. Spain*
Olympiad Problems

O487. Find all $n$ and all distinct positive integers $a_1, a_2, \ldots, a_n$ such that

\[
\left( \frac{a_1}{3} \right) + \cdots + \left( \frac{a_n}{3} \right) = \frac{1}{3} \left( \frac{a_1 + \cdots + a_n - n}{2} \right)
\]

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

O488. Let $m, n > 1$ be integers with $m$ even. Find the number of ordered systems $(a_1, a_2, \ldots, a_m)$ of integers such that:

(i) $0 \leq a_1 \leq a_2 \leq \cdots \leq a_m \leq n$,

(ii) $a_1 + a_3 + \cdots \equiv a_2 + a_4 + \cdots \text{mod}(n + 1)$.

*Proposed by Virgil Domocos, Montreal, Canada*

O489. In triangle $ABC$, $\angle A \geq \angle B \geq 60^\circ$. Prove that

\[
\frac{a}{b} + \frac{b}{a} \leq \frac{1}{3} \left( \frac{2R}{r} + \frac{2r}{R} + 1 \right)
\]

and

\[
\frac{a}{c} + \frac{c}{a} \geq \frac{1}{3} \left( 7 - \frac{2r}{R} \right)
\]

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

O490. Let $ABC$ be a triangle with incenter $I$ and $A$–excenter $I_A$. The line through $I$ perpendicular to $BI$ meets $AC$ at $X$, while the line through $I$ perpendicular to $CI$ meets $AB$ at $Y$. Prove that $X, I_A, Y$ are collinear if and only if $AB + AC = 3BC$.

*Proposed by Tovi Wen, USA*

O491. If $a, b, c$ are real numbers greater than $-1$ such that $a + b + c + abc = 4$, prove that

\[
\sqrt[3]{(a + 3)(b + 3)(c + 3)} + \sqrt[3]{(a^2 + 3)(b^2 + 3)(c^2 + 3)} \geq 2 \sqrt{ab + bc + ca + 13}.
\]

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

O492. Let $a, b, c, x, y, z$ be positive real numbers such that

\[
(a + b + c)(x + y + z) = (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = 4.
\]

Prove that

\[
\sqrt[4]{abcxyz} \leq \frac{4}{27}.
\]

*Proposed by Marius Stâncean, Zalău, România*