Junior problems

J487. Let $ABCD$ be a cyclic kite. Prove that $3\angle A = \angle C$ or $\angle A = 3\angle C$ if and only if

$$\frac{AC}{BD} - \frac{BD}{AC} = \frac{1}{\sqrt{2}}.$$

Proposed by Titu Andreescu, University of Texas at Austin, USA

Solution by Daniel Lasaosa, Pamplona, Spain
Denoting $x = \frac{AC}{BD}$, the condition is equivalent to

$$0 = x^2 - \frac{x}{\sqrt{2}} - 1 = \left(x - \sqrt{2}\right)\left(x + \frac{1}{\sqrt{2}}\right),$$

or the condition is equivalent to $AC = \sqrt{2}BD$. Now, since the kite is cyclic, $\angle A + \angle C = 180°$ and $\angle B = \angle D = 90°$, or triangle ABC is rectangle at B, and the altitude from B is well known to have length equal to

$$\frac{BD}{2} = AC \sin\frac{A}{2}\sin\frac{C}{2} = AC\sin\frac{A}{2}\cos\frac{A}{2} = \frac{AC\sin A}{2}.$$

Now, this holds for every cyclic kite, or $AC = BD\sqrt{2}$ is equivalent to $\sin A = \frac{1}{\sqrt{2}}$, which in turn is equivalent to either $\angle A = 45° = \frac{135°}{3} = \frac{\angle C}{3}$, or to $\angle A = 135° = 3 \cdot 45° = 3\angle C$. The conclusion follows.

Also solved by Ioannis D. Sfikas, Athens, Greece; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Daniel Văcaru, Pitești, Romania; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Joonsoo Lee, Dwight Englewood School, NJ, USA; Polyahedra, Polk State College, USA; Taes Padhihary, Disha Delphi Public School, India; Jeffrey Roh, St. Andrew's School, Middletown, DE, USA; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Min Jung Kim, Tabor Academy; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Jenna Park, Blair Academy, Blairstown, NJ, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA.

J488. Let a and b be positive real numbers such that $ab = a + b$. Prove that

$$\sqrt{1+a^2} + \sqrt{1+b^2} \geq \sqrt{20 + (a-b)^2}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Polyahedra, Polk State College, USA

First, $ab = a + b \geq 2\sqrt{ab}$, so $ab \geq 4$. Next,

$$\left(\sqrt{1+a^2} + \sqrt{1+b^2}\right)^2 - 20 - (a-b)^2 = 2\left(\sqrt{1+a^2+b^2+a^2b^2} - 9 + ab\right).$$

Finally,

$$\begin{aligned} &1 + a^2 + b^2 + a^2b^2 - (9-ab)^2 = 1 + (a+b)^2 - 2ab + a^2b^2 - 81 + 18ab - a^2b^2 \\ &= a^2b^2 + 16ab - 80 = (ab-4)(ab+20) \geq 0. \end{aligned}$$

Also solved by Ioannis D. Sfikas, Athens, Greece; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Taes Padhihary, Disha Delphi Public School, India; Jeffrey Roh, St. Andrew's School, Middletown, DE, USA; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Min Jung Kim, Tabor Academy; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Jenna Park, Blair Academy, Blairstown, NJ, USA; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Joel Schlosberg, Bayside, NY, USA; Joonsoo Lee, Dwight Englewood School, NJ, USA; Daniel Lasaosa, Pamplona, Spain; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paul Revenant, École Normale Supérieure, Lyon, France; Sarah B. Seales, Prescott, AZ, USA; Sergio Esteban Muñoz, Universidad de Buenos Aires, Argentina; Sophie Bekerman, Los Gatos High School, CA, USA; Albert Stadler, Herrliberg, Switzeland; Sumanth Ravipati, George Mason University, Fairfax, VA, USA; Thomas Rainow, High School, Los Altos, California, USA; Titu Zvonaru, Comănești, Romania.

J489. Prove that in any triangle ABC,

$$8r(R-2r)\sqrt{r(16R-5r)} \le a^3+b^3+c^3-3abc \le 8R(R-2r)\sqrt{(2R+r)^2+2r^2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, USA

Using

$$r^2 s = (s-a)(s-b)(s-c) = s^3-(a+b+c)s^2+(ab+bc+ca)s-abc = -s^3+(ab+bc+ca)s-4Rrs,$$

we get $ab+bc+ca = s^2+4Rr+r^2$. Hence,

$$a^3+b^3+c^3-3abc = (a+b+c)\left((a+b+c)^2-3(ab+bc+ca)\right) = 2s\left(s^2-12Rr-3r^2\right).$$

From Gerretsen's inequalities: $r(16R-5r) \le s^2 \le (2R+r)^2+2r^2$, we obtain

$$4r(R-2r) \le s^2-12Rr-3r^2 \le 4R(R-2r).$$

Therefore,

$$8r(R-2r)\sqrt{r(16R-5r)} \le 2s\left(s^2-12Rr-3r^2\right) \le 8R(R-2r)\sqrt{(2R+r)^2+2r^2}.$$

Also solved by Ioannis D. Sfikas, Athens, Greece; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Arkady Alt, San Jose, CA, USA; Daniel Văcaru, Pitești, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Titu Zvonaru, Comănești, Romania.

J490. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{1+ab^2}+\frac{b^3}{1+bc^2}+\frac{c^3}{1+ca^2}\geq\frac{3abc}{1+abc}.$$

Proposed by An Zhenping, Xianyang and Li Xin, Wugong, China

First solution by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan

By Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\sum_{cyc}\frac{a^3}{1+ab^2}&=\sum_{cyc}\frac{a^4}{a+a^2b^2}\\&\geq\frac{(a^2+b^2+c^2)^2}{a+b+c+a^2b^2+b^2c^2+c^2a^2}.\end{aligned}$$

Therefore it suffices to show the inequality,

$$\begin{aligned}&\frac{(a^2+b^2+c^2)^2}{a+b+c+a^2b^2+b^2c^2+c^2a^2}\geq\frac{3abc}{1+abc}\\&\Leftrightarrow(a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2-3a^2bc-3ab^2c-3abc^2)\\&+abc(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)\geq 0\end{aligned}$$

By Muirhead's inequality,

$$\begin{aligned}&a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2-3a^2bc-3ab^2c-3abc^2\geq 0,\\&a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2\geq 0\end{aligned}$$

and we are done.

Second solution by Polyahedra, Polk State College, USA

By the Cauchy-Schwarz inequality,

$$\begin{aligned}&\left(a(1+ab^2)+b(1+bc^2)+c(1+ca^2)\right)\left(\frac{a^3}{1+ab^2}+\frac{b^3}{1+bc^2}=\frac{c^3}{1+ca^2}\right)\\&\geq\left(a^2+b^2+c^2\right)^2\geq 3\left(a^2b^2+b^2c^2+c^2a^2\right).\end{aligned}$$

Since $3x/(1+x)$ is an increasing function of $x>0$, and $a^2b^2+b^2c^2+c^2a^2\geq abc(a+b+c)$,

$$\begin{aligned}\frac{3abc}{1+abc}&\leq\frac{\frac{3(a^2b^2+b^2c^2+c^2a^2)}{a+b+c}}{1+\frac{a^2b^2+b^2c^2+c^2a^2}{a+b+c}}=\frac{3(a^2b^2+b^2c^2+c^2a^2)}{a(1+ab^2)+b(1+bc^2)+c(1+ca^2)}\\&\leq\frac{a^3}{1+ab^2}+\frac{b^3}{1+bc^2}+\frac{c^3}{1+ca^2}.\end{aligned}$$

Also solved by Ioannis D. Sfikas, Athens, Greece; Taes Padhihary, Disha Delphi Public School, India; Arkady Alt, San Jose, CA, USA; Arkady Alt, San Jose, CA, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Jamal Gadirov, Istanbul University, Turkey; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Evgenidis Nikolaos, Aristotle University of Thessaloniki, Thessaloniki, Greece; Albert Stadler, Herrliberg, Switzeland; Titu Zvonaru, Comănești, Romania.

J491. Find all triples (x, y, z) of positive integers such that

$$5\left(x^2+2y^2+z^2\right)=2(5xy-yz+4zx)$$

and at least one of x, y, z is a prime.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Daniel Lasaosa, Pamplona, Spain
Note first that

$$(x+y-2z)^2+(2x-3y-z)^2=5x^2+10y^2+5z^2-10xy+2yz-8zx=0,$$

or all such triples satisfy $x+y=2z$ and $2x=3y+z$, resulting in $5y=3z$, or since y, z are positive integers, a positive integer t exists such that $y=3t$, $z=5t$, for $x=7t$. Clearly t must be 1, otherwise none out of x, y, z would be prime. It follows that the only possible such solution is $(x, y, z)=(7,3,5)$, with both sides of the proposed equation taking value 460.

Also solved by Ioannis D. Sfikas, Athens, Greece; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Polyahedra, Polk State College, USA; Jeffrey Roh, St. Andrew's School, Middletown, DE, USA; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Min Jung Kim, Tabor Academy; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Jenna Park, Blair Academy, Blairstown, NJ, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Joonsoo Lee, Dwight Englewood School, NJ, USA; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Titu Zvonaru, Comănești, Romania.

J492. Let $n > 1$ be an integer and let a, b, c be positive real numbers such that $a^n + b^n + c^n = 3$. Prove that

$$\frac{1}{a^{n+1}+n} + \frac{1}{b^{n+1}+n} + \frac{1}{c^{n+1}+n} \geq \frac{3}{n+1}.$$

Proposed by Dragoljub Miloević, Gornji Milanovac, Serbia

Solution by Titu Zvonaru, Comănești, Romania

We will prove that for $x \geq 0$, we get

$$\frac{1}{x^{n+1}+n} \geq -\frac{x^n}{n(n+1)} + \frac{1}{n} \tag{1}$$

The inequality (1) is equivalent to $x^{n+1} - (n+1)x + n \geq 0$, which is true by the AM-GM and we obtain:

$$x^{n+1} + n = x^{n+1} + 1 + 1 + \ldots + 1 \geq (n+1)\sqrt[n+1]{x^{n+1} \cdot 1 \cdot 1 \cdot \ldots \cdot 1} = (n+1)x.$$

Using (1) we get

$$\frac{1}{a^{n+1}+n} + \frac{1}{b^{n+1}+n} + \frac{1}{c^{n+1}+n} \geq -\frac{a^n+b^n+c^n}{n(n+1)} + \frac{3}{n} = -\frac{3}{n(n+1)} + \frac{3}{n} = \frac{3}{n+1}.$$

The equality holds if and only if $a = b = c = 1$.

Also solved by Nicusor Zlota, Traian Vuia Technical College, Romania; Ioannis D. Sfikas, Athens, Greece; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Polyahedra, Polk State College, USA; Taes Padhihary, Disha Delphi Public School, India; Arkady Alt, San Jose, CA, USA; Daniel Văcaru, Pitești, Romania; Sergio Esteban Muñoz, Universidad de Buenos Aires, Argentina.

Senior problems

S487. Find all primes $a \geq b \geq c \geq d$ such that

$$a^2 + 2b^2 + c^2 + 2d^2 = 2(ab + bc - cd + da)$$

Proposed by Titu Andreescu, University of Texas at Austin, USA

Solution by the author

The condition in the display rewrites

$$(a - b - d)^2 + (b - c - d)^2 = 0,$$

implying $a = b + d$ and $b = c + d$.

It follows that $d = 2, a = 7, b = 5$, and $c = 3$.

Also solved by Ioannis D. Sfikas, Athens, Greece; Evgenidis Nikolaos, Aristotle University of Thessaloniki, Thessaloniki, Greece; Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Taes Padhihary, Disha Delphi Public School, India; Jeffrey Roh, St. Andrew's School, Middletown, DE, USA; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Min Jung Kim, Tabor Academy; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Jenna Park, Blair Academy, Blairstown, NJ, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Joonsoo Lee, Dwight Englewood School, NJ, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Titu Zvonaru, Comănești, Romania.

S488. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}+\frac{1}{3}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)\geq 2.$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

Solution by Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania

We denote by $p=a+b+c=3, q=ab+bc+ca, r=abc$. From Cauchy-Schwarz's Inequality,

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\geq\frac{9}{2(a^3+b^3+c^3)+3abc}=\frac{9}{54-18q+9r}$$

Therefore, it is sufficient to show that

$$\frac{9}{54-18q+9r}\geq\frac{2r-1}{r}$$

or $2r^2-2r(3q-5)+3q-6\leq 0$. Using the Schur's inequality, we have

$$pq\leq 9r\leq 4pq-p^3$$

Then, the inequality becomes

$$\frac{2}{9}q^2-\frac{2}{3}(4q-9)(3q-5)+3q-6\leq 0$$

or

$$70q^2-309q+324\geq 0$$

So, $\left(q-\frac{12}{7}\right)\left(q-\frac{27}{10}\right)\geq 0$, which is obvious.

Also solved by Ioannis D. Sfikas, Athens, Greece.

S489. Find all pairs (m,n) of positive integers with $m+n=2019$ for which there is a prime p such that

$$\frac{4}{m+3}+\frac{4}{n+3}=\frac{1}{p}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Denote $u=m+3$, $v=n+3$, or $u+v=2025$ and a prime p exists such that $uv=4p(u+v)=4\cdot 2025p$.

Therefore, u,v are the roots of the quadratic equation $z^2-2025z+4\cdot 2025p=0$, with solutions

$$z=\frac{2015\pm 45\sqrt{2025-16p}}{2},$$

and a positive integer w exists such that

$$16p=45^2-w^2=(45+w)(45-w).$$

Now, $(45+w)+(45-w)=90$ which is even but not divisible by 4, or $45+w,45-w$ are both even, but not both multiples of 4. Moreover, $45+w>8>2$, yielding either $45+w=2p$ or $45+w=8p$, respectively for $w=45-8=37$ and $w=45-2=43$. Substitution respectively yields $45+w=2p=82$ for $p=41$ and $45+w=8p=88$ for $p=11$, indeed both primes. In the first case the solutions of the quadratic equation are $(4,41)$, and in the second case $(1,44)$, resulting respectively in (m,n) being a permutation of $(4\cdot 45-3,41\cdot 45-3)=(177,1842)$, and (m,n) being a permutation of $(45-3,44\cdot 45-3)=(42,1977)$, producing respectively $p=41$ and $p=11$.

Also solved by Ioannis D. Sfikas, Athens, Greece; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Taes Padhihary, Disha Delphi Public School, India; Andrew Yang, Hotchkiss School, Lakeville, CT, USA; Min Jung Kim, Tabor Academy; Corneliu Mănescu-Avram, Ploiești, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Joonsoo Lee, Dwight Englewood School, NJ, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Sumanth Ravipati, George Mason University, Fairfax, VA, USA; Titu Zvonaru, Comănești, Romania; Hei Chun Leung, Hong Kong.

S490. Prove that there is a real function f for which there is no a real function g such that $f(x) = g(g(x))$ for all $x \in \mathbb{R}$.

Proposed by Pavel Gadzinski, Bielsko-Biala, Poland

Solution by Dumitru Barac, Sibiu, Romania

We consider $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = \begin{cases} \sqrt{2}, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

First, suppose that there is a real function g such that $f(x) = g(g(x))$ for all $x \in \mathbb{R}$.

Next, assume that $g(\sqrt{2}) \in \mathbb{Q}$.

Setting $x = g(\sqrt{2}) \in \mathbb{Q}$, results that $g(g(g(\sqrt{2}))) = f(g(\sqrt{2})) = \sqrt{2}$. On the other hand, since $g(g(\sqrt{2})) = f(\sqrt{2}) = 1$, one deduces that $g(g(g(\sqrt{2}))) = g(f(\sqrt{2})) = g(1)$. It follows that $g(1) = \sqrt{2}$, hence, $g(g(1)) = g(\sqrt{2})$, or $g(\sqrt{2}) = \sqrt{2}$, contradiction.

Now, suppose that $g(\sqrt{2}) \notin \mathbb{Q}$. It results that $g(g(g(\sqrt{2}))) = f(g(\sqrt{2})) = 1$. On the other hand, because $g(g(\sqrt{2})) = f(\sqrt{2}) = 1$, one deduces that $g(g(g(\sqrt{2}))) = g(1)$ or $g(1) = 1$.

Therefore, we get that $g(g(1)) = g(1)$, and $g(1) = f(1) = \sqrt{2}$, contradiction.

Also solved by Ioannis D. Sfikas, Athens, Greece.

S491. Prove that in any acute triangle ABC the following inequality holds:

$$\frac{1}{\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)^2}+\frac{1}{\left(\cos\frac{B}{2}+\cos\frac{C}{2}\right)^2}+\frac{1}{\left(\cos\frac{C}{2}+\cos\frac{A}{2}\right)^2}\geq 1.$$

Proposed by Florin Rotaru, Focşani, România

Solution by Arkady Alt, San Jose, CA, USA

Let

$$\alpha:=\frac{\pi-A}{2},\beta:=\frac{\pi-B}{2},\gamma:=\frac{\pi-C}{2}.$$

Then $\alpha,\beta,\gamma>0,\alpha+\beta+\gamma=\pi,$

$$\sum\frac{1}{\left(\cos\dfrac{A}{2}+\cos\dfrac{B}{2}\right)^2}=\sum\frac{1}{(\sin\alpha+\sin\beta)^2}$$

and original inequality becomes

$$\sum\frac{1}{(\sin\alpha+\sin\beta)^2}\geq 1. \tag{1}$$

Let ABC be some triangle with angles α,β,γ and correspondent side lengths a,b,c. (Don't mixe this triangle with original acute triangle). Also, let R,r and s be circumradius, inradius and semiperimeter of this triangle.
Then from (1) it wollows that

$$\sum\frac{1}{(a+b)^2}\geq\frac{1}{4R^2}.$$

Since by Cauchy Inequality

$$\sum\frac{1}{(a+b)^2}\geq\frac{9}{\sum(a+b)^2}=\frac{9}{2\left(a^2+b^2+c^2+ab+bc+ca\right)}$$

then remains to prove inequality

$$\frac{9}{2\left(a^2+b^2+c^2+ab+bc+ca\right)}\geq\frac{1}{4R^2}\iff\quad a^2+b^2+c^2+ab+bc+ca\leq 18R^2.$$

The latter inequality holds because $a^2+b^2+c^2\leq 9R^2$ and $ab+bc+ca\leq a^2+b^2+c^2$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Min Jung Kim, Tabor Academy; Jeewoo Lee, Townsend Harris High School, Flushing, NY, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Daniel Văcaru, Pitești, Romania; Jamal Gadirov, Istanbul University, Turkey; Joonsoo Lee, Dwight Englewood School, NJ, USA; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Sergio Esteban Muñoz, Universidad de Buenos Aires, Argentina; Titu Zvonaru, Comănești, Romania; Hei Chun Leung, Hong Kong; Ioannis D. Sfikas, Athens, Greece.

S492. Find the greatest real constant C such that the inequality

$$(a^2+2)(b^2+2)(c^2+2)-(abc-1)^2 \geq C(a+b+c)^2$$

holds for all positive real numbers a, b, c.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Albert Stadler, Herrliberg, Switzerland

Setting $a=b=c=1$ results in $C \leq 3$. On the other side,

$$(a^2+2)(b^2+2)(c^2+2)-(abc-1)^2-3(a+b+c)^2 =$$

$$7+a^2+b^2+c^2-6ab-6bc-6ca+2abc+2a^2b^2+2b^2c^2+2c^2a^2 =$$

$$(a-1)^2+(b-1)^2+(c-1)^2+2(a-1)(b-1)(c-1)+2(ab-1)^2+2(bc-1)^2+2(ca-1)^2 \geq 0,$$

for if one of the variables lies between 0 and 1, let's assume that $0<a<1$, and by the AM-GM inequality:

$$(b-1)^2+(c-1)^2+2(a-1)(b-1)(c-1) \geq (b-1)^2+(c-1)^2-2|(b-1)(c-1)| \geq 0.$$

So $C \geq 3$, and thus $C=3$. Equality holds if and only if $a=b=c=1$.

Also solved by Ioannis D. Sfikas, Athens, Greece; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece.

Undergraduate problems

U487. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that the following conditions hold simultaneously:

(a) $f(f(x)) = x$, for all $x \in \mathbb{R}$,

(b) $f(x+y) = f(x) + f(y)$, for all $x \in \mathbb{R}$,

(c) $\lim_{x\to\infty} f(x) = -\infty$.

Proposed by Vincelot Ravoson, Lycée Henri IV, Paris, France

Solution by Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan

We show that if $f(x) = x$, then $x = 0$. Contrary assume that for some $x \neq 0$, $f(x) = x$. When $x > 0$, from (b), for $n \in \mathbb{N}$, $f(nx) = nf(x) = nx \to +\infty$ as $n \to \infty$, which contradicts (c). When $x < 0$, for $n \in \mathbb{N}$, $f(-nx) = -nf(x) = -nx \to +\infty$ as $n \to \infty$, which contradicts (c). On the other hand, for any $x \in \mathbb{R}$,

$$f(f(x)+x) = f \circ f(x) + f(x) = x + f(x).$$

Therefore the desired f is only $f(x) = -x$.

Also solved by Ioannis D. Sfikas, Athens, Greece; Sarah B. Seales, Prescott, AZ, USA; Joe Simons, Utah Valley University, Orem, UT.

U488. Let a and b be positive real numbers. Evaluate

$$\int_{a^{-b}}^{a^b} \frac{\arctan x}{x} dx.$$

Proposed by Michele Caselli, University of Modena, Italy

Solution by Alexandru Daniel Pîrvuceanu, National "Traian" College, Romania

We denote the given integral by I.

After we make the variable change $x = \frac{1}{t}$, we get that

$$I = -\int_{a^b}^{a^{-b}} \frac{\arctan \frac{1}{t}}{\frac{1}{t}} \cdot \frac{1}{t^2} dt = \int_{a^{-b}}^{a^b} \frac{\arctan \frac{1}{t}}{t} dt = \int_{a^{-b}}^{a^b} \frac{\frac{\pi}{2} - \arctan t}{t} = \frac{\pi}{2} \ln t \Big|_{a^{-b}}^{a^b} - I.$$

From here it follows that

$$I = \frac{\pi \cdot \ln a^b}{2}.$$

Also solved by Ioannis D. Sfikas, Athens, Greece; Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Daniel Văcaru, Pitești, Romania; Ivko Dimitrić, Pennsylvania State University Fayette, Lemont Furnace, PA, USA; Dumitru Barac, Sibiu, Romania; Jordan Goff, Utah Valley University, Orem, UT, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzeland; Sumanth Ravipati, George Mason University, Fairfax, VA, USA.

U489. Find all continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(f(f(x))) - 3f(f(x)) + 3f(x) - x = 0$$

for all $x \in \mathbb{R}$

Proposed by Titu Andreescu, USA, and Marian Tetiva, România

Solution by the authors

We will prove that the only solutions are the functions $f : \mathbb{R} \to \mathbb{R}$ defined by a formula of the type $f(x) = x + c$ for every $x \in \mathbb{R}$ (c being some real number).

We start by noting that any continuous solution of the given functional equation is injective (since, if we have $f(x) = f(y)$, then we get $f(f(x)) = f(f(y))$ and $f(f(f(x))) = f(f(f(y)))$, too, thus, by the given condition $x = y$ follows; actually, continuity is not needed for proving injectivity) and, also, surjective. Indeed, being continuous and injective, f is strictly monotone. Let $m = \inf_{x\in\mathbb{R}} f(x)$ (which exists, by the monotony of f) and let (x_n) be a sequence of real numbers such that $\lim_{n\to\infty} f(x_n) = m$. Assume that m is a real number. By the continuity of f, we have $f(m) = \lim_{n\to\infty} f(f(x_n))$ and $f(f(m)) = \lim_{n\to\infty} f(f(f(x_n)))$, therefore, because

$$x_n = f(f(f(x_n))) - 3f(f(x_n)) + 3f(x_n), \quad \forall n \in \mathbb{N}^*,$$

we infer that the sequence (x_n) is convergent. Now, if $l = \lim_{n\to\infty} x_n$ we get (again by the continuity of f) $m = \lim_{n\to\infty} f(x_n) = f(l)$. This, obviously, is not possible for a function $f : \mathbb{R} \to \mathbb{R}$ which is strictly monotone. (For instance, if f was strictly increasing, we would get $f(t) < f(l) = m = \inf_{x\in\mathbb{R}} f(x)$ when $t < l$.) Thus the assumption $m \in \mathbb{R}$ is wrong, hence $\inf_{x\in\mathbb{R}} f(x)$ must be either ∞ or $-\infty$. Similarly, we see that $\sup_{x\in\mathbb{R}} f(x)$ is either ∞ or $-\infty$. As f is continuous and strictly monotone, the fact that $f(\mathbb{R}) = \mathbb{R}$ follows, that is, f is surjective.

Thus any continuous solution f of the given functional equation has to be bijective and strictly monotone. But, if f is strictly decreasing, $f \circ f \circ f$ is also strictly decreasing, while $f \circ f$ is strictly increasing; also, $1_{\mathbb{R}}$ (the identical map of $\mathbb{R}$, defined by $1_{\mathbb{R}}(x) = x$ for every $x \in \mathbb{R}$) is strictly increasing. Thus the given equality $f \circ f \circ f - 3f \circ f + 3f - 1_{\mathbb{R}} = 0$ would be impossible, as in the left-hand side we would have a strictly decreasing function, which cannot be equal to the constant zero function from the right-hand side. The contradiction thus obtained shows that any continuous solution of our functional equation has to be strictly increasing.

Now, for positive integer $n \in \mathbb{N}^*$, let us denote by $f^{[n]} = f \circ \cdots \circ f$ (with f appearing n times) the nth iteration of f, and, also, let $f^{[-n]} = f^{-1} \circ \cdots \circ f^{-1}$ (where f^{-1}, the inverse of f, appears n times); finally, let $f^{[0]} = 1_{\mathbb{R}}$. Thus, the given functional equation reads $f^{[3]}(x) - 3f^{[2]}(x) + 3f^{[1]}(x) - f^{[0]}(x) = 0$, $\forall x \in \mathbb{R}$, and by replacing here x with $f^{[n]}(x)$ we see that we also have

$$f^{[n+3]}(x) - 3f^{[n+2]}(x) + 3f^{[n+1]}(x) - f^{[n]}(x) = 0,$$

for any real number x and any integer n. By induction (or by using the theory of linear recurrences) one easily sees that

$$f^{[n]}(x) = x - \frac{f^{[2]}(x) - 4f(x) + 3x}{2} n + \frac{f^{[2]}(x) - 2f(x) + x}{2} n^2$$

for any $n \in \mathbb{Z}$ and for any $x \in \mathbb{R}$.

Let us consider some $x \in \mathbb{R}$. Clearly, if we have $f(x) = x$, then $f(f(x)) = f(x) = x$, also, therefore $f(f(x)) - 2f(x) + x = 0$. On the other hand, if $x < f(x)$, we get (by repeatedly applying f and by using its monotony)

$$\cdots < f^{[-2]}(x) < f^{[-1]}(x) < f^{[0]}(x) < f^{[1]}(x) < f^{[2]}(x) < \cdots$$

and, analogously, the assumption $f(x) < x$ leads to the conclusion that the sequence $(f^{[n]}(x))_{n\in\mathbb{Z}}$ is strictly decreasing. On the other hand, if we look to the above formula for $f^{[n]}(x)$ we see that both limits $\lim_{n\to\infty} f^{[n]}(x)$ and $\lim_{n\to-\infty} f^{[n]}(x)$ are equal to either ∞ or $-\infty$ when $f^{[2]}(x) - 2f(x) + x$ is not 0 (the limits are ∞ when $f^{[2]}(x) - 2f(x) + x > 0$ and they equal $-\infty$ when $f^{[2]}(x) - 2f(x) + x < 0$). Since this is not acceptable for a strictly monotone sequence $(f^{[n]}(x))_{n\in\mathbb{Z}}$, the only possibility $f^{[2]}(x) - 2f(x) + x = 0$ remains.

Thus we have $f(f(x)) - 2f(x) + x = 0$ for any $x \in \mathbb{R}$, and the above formula gives us

$$f^{[n]}(x) = x + (f(x) - x)n$$

for each $n \in \mathbb{Z}$ and for any $x \in \mathbb{R}$.

Because f is strictly increasing on $\mathbb{R}$, so will be any iteration $f^{[n]}$, $n \in \mathbb{Z}$; consequently, for integer n and for any $x, y \in \mathbb{R}$ with $x < y$, we have $f^{[n]}(x) < f^{[n]}(y)$, that is

$$x + (f(x) - x)n < y + (f(y) - y)n,$$

leading to

$$\frac{x}{n} + f(x) - x < \frac{y}{n} + f(y) - y$$

for $n > 0$, and to

$$\frac{x}{n} + f(x) - x > \frac{y}{n} + f(y) - y$$

for $n < 0$. By passing to the limit for $n \to \infty$ in the first realtion we get $f(x) - x \le f(y) - y$ and, similarly, passing to the limit for $n \to -\infty$ in the second one, we get $f(x) - x \ge f(y) - y$. Thus we actually have $f(x) - x = f(y) - y$ for any $x, y \in \mathbb{R}$ with $x < y$, showing that the function $x \mapsto f(x) - x$ is a constant function, that is, there exists $c \in \mathbb{R}$ such that $f(x) = x + c$ for any real number x. As any function of the type $x \mapsto x + c$ satisfies all the conditions from the statement of the problem, we conclude that these functions are the only solutions, as announced from the beginning, and the proof ends here.

Also solved by Ioannis D. Sfikas, Athens, Greece; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.

U490. Find the greatest real number k such that the inequality

$$\frac{a^2}{b}+\frac{b^2}{a}\geq\frac{2\left(a^{k+1}+b^{k+1}\right)}{a^k+b^k}$$

holds for all positive real numbers a and b.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Dumitru Barac, Sibiu, Romania

The inequality is $\dfrac{a^3+b^3}{ab}\geq\dfrac{2\left(a^{k+1}+b^{k+1}\right)}{a^k+b^k}$. Denoting $\dfrac{a}{b}=x>0$ yields to

$$\frac{x^3+1}{x}\geq\frac{2(x^{k+1}+1)}{x^k+1}\iff x^{k+3}+x^k+x^3+1\geq 2x^{k+2}+2x\iff$$

$$x^{k+3}-2x^{k+2}+x^k+x^3-2x+1\geq 0,\ \text{ for all } x>0.$$

The inequality holds for $k=4$:

$$x^7-2x^6+x^4+x^3-2x+1\geq 0\iff (x-1)^4(x+1)(x^2+x+1)\geq 0.$$

If $P_k(x)\geq 0$ for all $x>0$, then $k\leq 4$, where $P_k(x)=x^{k+3}-2x^{k+2}+x^k+x^3-2x+1$.
Indeed, if $k>4$, we have $\lim_{x\to 1}\dfrac{P_k(x)}{(x-1)^2}\geq 0$. However,

$$\begin{aligned}P_k''(1)=&(k+3)(k+2)-2(k+2)(k+1)+k(k-1)+6=\\&k^2+5k+6-2k^2-6k-4+k^2-k+6=-2k+8<0.\end{aligned}$$

Hence, the greatest value is $k=4$.

Also solved by Ioannis D. Sfikas, Athens, Greece; Arkady Alt, San Jose, CA, USA; Min Jung Kim, Tabor Academy; Joonsoo Lee, Dwight Englewood School, NJ, USA; Titu Zvonaru, Comănești, Romania.

U491. Find all polynomials P with complex coefficients such that

$$P(a) + P(b) = 2P(a+b),$$

whenever a and b are complex numbers satisfying $a^2 + 5ab + b^2 = 0$.

Proposed by Titu Andreescu, USA and Mircea Becheanu, Canada

First solution by the authors

First, we see that every constant polynomial is a solution. Let (a, b) be a pair of non zero numbers which satisfy the condition $a^2 + 5ab + b^2 = 0$. Then

$$\frac{a}{b} = \frac{-5 \pm \sqrt{21}}{2} \tag{1}$$

Let denote $\lambda = \frac{-5+\sqrt{21}}{2}$. Then, for every real number t, the pairs $(\lambda t, t)$ satisfy the given condition.

Let $P(x) = \sum_{i=0}^{n} a_i x^i$ be a non constant polynomial which is a solution of the problem. Then we have:

$$\sum_{i=0}^{n} a_i \lambda^i t^i + \sum_{i=0}^{n} a_i t^i = 2\sum_{i=0}^{n} a_i (\lambda+1)^i t^i,$$

for all numbers t. Writing this equality in the form

$$\sum_{i=0}^{n} a_i (1+\lambda^i) t^i = 2\sum_{i=0}^{n} a_i (\lambda+1)^i t^i \tag{2}$$

we obtain an equality of polynomials in one variable t, which shows that whenever $a_i \neq 0$ we have the condition:

$$1 + \lambda^i = 2(1+\lambda)^i \tag{3}$$

This is true for $i = 3$ and it can not happen for $i = 1$ or $i = 2$. We will show that it can not happen for $i > 3$.

To begin, we will show that i is necessarily odd. From (3) it follows that λ is a root of the rational polynomial $f(X) = 2(X+1)^i - X^i - 1$. But λ is a root of $g(X) = X^2 + 5X + 1$ which is its minimal polynomial. So $g(X)$ divides $f(X)$ in $\mathbb{Z}[X]$, that is

$$2(X+1)^i - X^i - 1 = (X^2 + 5X + 1)h(X).$$

For $x = -1$ in this equality we obtain $-(-1)^i - 1 = -3h(-1)$, showing that i is odd.

Now, we have $0 < \lambda + 1 < 1$, which shows that the function $p(i) = 2(1+\lambda)^i$ decreases as i increases and it goes to 0. The function $q(i) = 1 + t^i$ can be written, for i odd, as $q(i) = 1 + (-1)^i(-t)^i = 1 - (-t)^i$ and it increases as i increases. So, the equation (3) has only one root and this is 3. Now, it is easy to see that any polynomial $a_0 + a_3 x^3$ is a solution of the problem.

Second solution by Daniel Lasaosa, Pamplona, Spain

Let M be any complex number, and let $a = M\cdot\frac{\sqrt{3}+\sqrt{7}}{2}$ and $b = M\cdot\frac{\sqrt{3}-\sqrt{7}}{2}$. Then, $a^2+b^2 = 5M^2$, and $ab = -M^2$, or indeed $a^2+5ab+b^2 = 0$ for all such complex values of a, b. Note further that $a+b = M\cdot\sqrt{3}$. Let now n be the degree of $P(x)$, and consider

$$\frac{P(a)}{c_n M^n} + \frac{P(b)}{c_n M^n} = \frac{2P(a+b)}{c_n M^n},$$

where c_n is the leading coefficient in $P(x)$. Letting $M \to \infty$, the limit of the previous expression becomes $a^n + b^n = 2(a+b)^n$, or the degree n of P must satisfy

$$\left(\frac{\sqrt{3}+\sqrt{7}}{2\sqrt{3}}\right)^n + \left(\frac{\sqrt{3}-\sqrt{7}}{2\sqrt{3}}\right)^n = 2.$$

Note now that

$$\frac{\sqrt{3}+\sqrt{7}}{2\sqrt{3}} > \frac{5}{4}, \qquad\qquad 0 > \frac{\sqrt{3}-\sqrt{7}}{2\sqrt{3}} > -\frac{1}{3}.$$

Indeed, the first inequality is equivalent, after some algebra, to $2\sqrt{7} > 3\sqrt{3}$, which after squaring becomes $28 > 27$, clearly true, while for the second inequality, the upper bound is obvious, and the lower bound is equivalent to $5 > \sqrt{21}$, again clearly true. It follows that, for all $n \geq 4$ even, we have

$$\left(\frac{\sqrt{3}+\sqrt{7}}{2\sqrt{3}}\right)^n + \left(\frac{\sqrt{3}-\sqrt{7}}{2\sqrt{3}}\right)^n > \left(\frac{\sqrt{3}+\sqrt{7}}{2\sqrt{3}}\right)^4 > \frac{5^4}{4^4} = \frac{625}{256} > 2.$$

Moreover,

$$\left(\frac{\sqrt{3}+\sqrt{7}}{2\sqrt{3}}\right)^2 + \left(\frac{\sqrt{3}-\sqrt{7}}{2\sqrt{3}}\right)^2 = \frac{2(3+7)}{12} = \frac{5}{3} \neq 2,$$

or if n is even, then $n = 0$. Note also that any constant polynomial $P(x) = k$ trivially satisfies $P(a) + P(b) = k + k = 2k = 2P(a+b)$. On the other hand, if $n \geq 5$ is odd, then

$$\left(\frac{\sqrt{3}+\sqrt{7}}{2\sqrt{3}}\right)^5 + \left(\frac{\sqrt{3}-\sqrt{7}}{2\sqrt{3}}\right)^5 > \frac{5^5}{4^5} - \frac{1}{3^5} = \frac{3125}{1024} - \frac{1}{3^5} > 3 - 1 = 2,$$

or if n is odd, it must be $n \leq 3$. Furthermore, $n = 1$ does not hold since $\frac{\sqrt{3}+\sqrt{7}}{2\sqrt{3}} + \frac{\sqrt{3}-\sqrt{7}}{2\sqrt{3}} = 1 < 2$, whereas for $n = 3$, we have

$$\left(\frac{\sqrt{3}+\sqrt{7}}{2\sqrt{3}}\right)^3 + \left(\frac{\sqrt{3}-\sqrt{7}}{2\sqrt{3}}\right)^3 = \frac{24\sqrt{3}+16\sqrt{7}}{24\sqrt{3}} + \frac{24\sqrt{3}-16\sqrt{7}}{24\sqrt{3}} = 2,$$

or $n = 3$ is the only possible odd degree for P. Or, the maximum possible degree of P is 3, and $Q(x) = P(x) - c_3x^3$ has a degree lower than 3, and by linearity must satisfy the same relation. It follows that all polynomials P which satisfy the condition given in the problem statement are

$$P(x) = Ax^3 + B,$$

where A, B are any complex constants. Indeed, for any such polynomial, and a, b such that $a^2 + 5ab + b^2 = 0$, we have

$$P(a) + P(b) = A(a+b)(a^2 - ab + b^2) + 2B = A(a+b)(a^2 - ab + b^2 + a^2 + 5ab + b^2) + 2B =$$

$$= A(a+b)(2a^2 + 4ab + 2b^2) + 2B = 2A(a+b)^3 + 2B = 2P(a+b).$$

Also solved by Ioannis D. Sfikas, Athens, Greece; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Min Jung Kim, Tabor Academy; Albert Stadler, Herrliberg, Switzeland.

U492. Let C be an arbitrary positive real number and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that ${x_1}^2 + {x_2}^2 + \cdots + {x_n}^2 = n$. Prove that

$$\sum_{i=1}^{n} \frac{x_i}{x_i + C} \le \frac{n}{C+1}.$$

Proposed by Angel Plaza, University of Las Palmas de Gran Canaria. Spain

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

By the arithmetic mean - quadratic mean inequality,

$$\frac{1}{n}\sum_{i=1}^{n} x_i \le \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^{1/2} = 1.$$

Because $C > 0$,

$$C \cdot \frac{1}{n}\sum_{i=1}^{n} x_i \le C,$$

from which it follows

$$\frac{1}{n}\sum_{i=1}^{n} x_i + C \cdot \frac{1}{n}\sum_{i=1}^{n} x_i \le \frac{1}{n}\sum_{i=1}^{n} x_i + C$$

and

$$\frac{\frac{1}{n}\sum_{i=1}^{n} x_i}{\frac{1}{n}\sum_{i=1}^{n} x_i + C} \le \frac{1}{C+1}.$$

The function

$$f(x) = \frac{x}{x+C}$$

is concave for $x > -C$. With $C > 0$, each $x_i > -C$. Therefore, by Jensen's inequality,

$$\sum_{i=1}^{n} \frac{x_i}{x_i + C} = \sum_{i=1}^{n} f(x_i) \le n f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = n \cdot \frac{\frac{1}{n}\sum_{i=1}^{n} x_i}{\frac{1}{n}\sum_{i=1}^{n} x_i + C} \le \frac{n}{C+1}.$$

Equality holds if and only if each $x_i = 1$.

Also solved by Ioannis D. Sfikas, Athens, Greece; Daniel Lasaosa, Pamplona, Spain; Takuji Grigorovich Imaiida, Fujisawa, Kanagawa, Japan; Min Jung Kim, Tabor Academy; Jenna Park, Blair Academy, Blairstown, NJ, USA; Alexandru Daniel Pîrvuceanu, National "Traian" College, Romania; Arkady Alt, San Jose, CA, USA; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Daniel Văcaru, Pitești, Romania; Dumitru Barac, Sibiu, Romania; Joonsoo Lee, Dwight Englewood School, NJ, USA; Sergio Esteban Muñoz, Universidad de Buenos Aires, Argentina; Titu Zvonaru, Comănești, Romania.

O487. Find all n and all distinct positive integers $a_1, a_2, \ldots, a_n$ such that

$$\binom{a_1}{3} + \cdots + \binom{a_n}{3} = \frac{1}{3}\binom{a_1 + a_2 + \cdots + a_n - n}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

For $1 \le i \le n$, let $b_i = a_i - 1$, then

$$\binom{a_i}{3} = \frac{(b_i+1)b_i(b_i-1)}{6} = \frac{b_i^3 - b_i}{6},$$

and

$$\frac{1}{3}\binom{a_1 + \cdots + a_n - n}{2} = \frac{1}{3}\binom{b_1 + \cdots + b_n}{2} = \frac{(b_1 + \cdots + b_n)^2 - (b_1 + \cdots + b_n)}{6}.$$

Therefore, the equation is equivalent to $b_1^3 + \cdots + b_n^3 = (b_1 + \cdots + b_n)^2$. We prove the following claim.

Claim: Let $0 \le b_1 < b_2 < \cdots < b_n$ be n distinct integers. Then $b_1^3 + b_2^3 + \cdots + b_n^3 \ge (b_1 + b_2 + \cdots + b_n)^2$, with equality iff either $(b_1, b_2, \ldots, b_n) = (0, 1, \ldots, n-1)$ or $(b_1, b_2, \ldots, b_n) = (1, 2, \ldots, n)$.

Proof: Since $b_1 = 0$ does not alter either side of the inequality, we may assume wlog that $b_1 \ge 1$. Therefore,

$$b_1 + b_2 + \cdots + b_{n-1} \le 1 + 2 + \cdots + (b_n - 2) + (b_n - 1) = \frac{b_n(b_n - 1)}{2},$$

with equality iff $(b_1, b_2 \cdots, b_{n-1}) = (1, 2, \ldots, b_n - 1)$, ie iff $b_i = i$ for $i = 1, 2, \ldots, n$. Then,

$$(b_1 + b_2 + \cdots + b_n)^2 - (b_1 + b_2 + \cdots + b_{n-1})^2 = b_n^2 + 2b_n(b_1 + b_2 + \cdots + b_{n-1}) \le$$

$$\le b_n^2 + 2b_n \cdot \frac{b_n(b_n - 1)}{2} = b_n^3,$$

or

$$b_1^3 + b_2^3 + \cdots + b_n^3 - (b_1 + b_2 + \cdots + b_n)^2 \ge b_1^3 + b_2^3 + \cdots + b_{n-1}^3 - (b_1 + b_2 + \cdots + b_{n-1})^2,$$

with equality iff $b_i = i$ for $i = 1, 2, \ldots, n$. Since the base case $b_1^3 - b_1^2 \ge 0$ holds for all positive integer, with equality iff $b_1 = 1$, by induction the proposed result is true for all n, with equality iff $b_i = i$ for $i = 1, 2, \ldots, n$. Restoring generality, a 0 may be added for $b_i = i - 1$ for $i = 1, 2, \ldots, n$. The Claim follows. We conclude that equality occurs for any positive integer n, iff $(a_1, a_2, \ldots, a_n)$ is any permutation of $(1, 2, \ldots, n)$ or of $(2, 3, \ldots, n+1)$.

Also solved by Ioannis D. Sfikas, Athens, Greece.

O488. Let $m, n > 1$ be integers with m even. Find the number of ordered systems $(a_1, a_2, \ldots, a_m)$ of integers such that:

(i) $0 \le a_1 \le a_2 \le \cdots \le a_m \le n$,

(ii) $a_1 + a_3 + \cdots \equiv a_2 + a_4 + \cdots \pmod{n+1}$.

Proposed by Virgil Domocos, Montreal, Canada

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that

$$(a_2 + a_4 + \cdots + a_m) - (a_1 + a_3 + \cdots + a_{m-1}) = (a_2 - a_1) + \cdots + (a_m - a_{m-1}) \ge 0,$$

since $a_{2u} - a_{2u-1} \ge 0$ for $u = 1, 2, \ldots, \frac{m}{2}$, and with equality iff $a_{2u} = a_{2u-1}$ for $u = 1, 2, \ldots, \frac{m}{2}$. Moreover,

$$(a_2 + a_4 + \cdots + a_m) - (a_1 + a_3 + \cdots + a_{m-1}) =$$

$$= (a_2 - a_3) + \cdots + (a_{m-2} - a_{m-1}) + (a_m - a_1) \le a_m - a_1 \le n - 0 = n,$$

since $a_{2u+1} - a_{2u} \le 0$ for $u = 1, 2, \ldots, \frac{m-2}{2}$, $a_m \le n$ and $a_1 \ge 0$. Therefore, condition (ii) is satisfied iff $a_1 + a_3 + \cdots = a_2 + a_4 + \cdots$, or by the first bound, iff $a_{2u-1} = a_{2u}$ for all $u = 1, 2, \ldots, \frac{m}{2}$. Denote $b_u = a_{2u-1} = a_{2u}$ for $u = 1, 2, \ldots, \frac{m}{2}$, or the problem is equivalent to finding the number of $\frac{m}{2}$-tuples $0 \le b_1 \le b_2 \le \cdots \le b_{\frac{m}{2}} \le n$.

Consider a path from $(0,0)$ to $\left(\frac{m}{2} + 1, n\right)$ moving from (x, y) to either $(x+1, y)$ or to $(x, y+1)$ at each step. For each $u = 1, 2, \ldots, \frac{m}{2}$, let b_u be the largest value of y present in the path with $x = u$. Clearly the resulting sequence of b_u's satisfies the condition of the problem. Reciprocally, given a sequence which satisfies the conditions of the problem, the corresponding path may be constructed. Moreover, the path is unique, since the path up to $x = \frac{m}{2}$ is clearly determined univocally by the b_u's, whereas once $b_{\frac{m}{2}}$ is known, from $\left(\frac{m}{2}, b_{\frac{m}{2}}\right)$ the path must necessarily go to $\left(\frac{m}{2} + 1, b_{\frac{m}{2}}\right)$, and from there only the y coordinate may increase up to $\left(\frac{m}{2} + 1, n\right)$. Therefore, the number of systems is

$$\binom{n + \frac{m}{2} + 1}{n}.$$

Also solved by Ioannis D. Sfikas, Athens, Greece.

O489. In triangle ABC, $\angle A \geq \angle B \geq 60°$. Prove that

$$\frac{a}{b}+\frac{b}{a} \leq \frac{1}{3}\left(\frac{2R}{r}+\frac{2r}{R}+1\right)$$

and

$$\frac{a}{c}+\frac{c}{a} \geq \frac{1}{3}\left(7-\frac{2r}{R}\right)$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

It is known that

$$20Rr-4r^2 \leq ab+bc+ca \leq 4R^2+8Rr+4r^2,$$

Note: See for example page 1 of the article in the same issue (MR4).

Also,

$$\frac{a+b+c}{abc}=\frac{\frac{2K}{r}}{4RK}=\frac{1}{2Rr},$$

hence, by multiplication with the above inequalities,

$$10-\frac{2r}{R} \leq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \leq \frac{2R}{r}+4+\frac{2r}{R}.$$

Subtracting 3 from each of the three quantities yields

$$7-\frac{2r}{R} \leq \left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{c}{a}+\frac{a}{c}\right) \leq \frac{2R}{r}+\frac{2r}{R}+1.$$

It now suffice to prove that

$$\frac{a}{b}+\frac{b}{a} \leq \frac{b}{c}+\frac{c}{b} \leq \frac{c}{a}+\frac{a}{c}.$$

We have $60° \leq A < 120°$, $60° \leq B < 90°$, $0° < C < 60°$, and $A \geq B$ so, by the Law of Sines, $a \geq b > c$.

The first inequality rewrites $(c-a)(b^2-ac) \leq 0$ and is true because $c-a<0$ and $b^2=a^2+c^2-2ac\cos B \geq a^2+c^2-ac \geq ac$.

The second inequality rewrites $(b-a)(ab-c^2) \leq 0$ and is true because $b-a<0$ and $ab > c \cdot c$.

Also solved by Ioannis D. Sfikas, Athens, Greece.

O490. Let ABC be a triangle with incenter I and A-excenter I_A. The line through I perpendicular to BI meets AC at X, while the line through I perpendicular to CI meets AB at Y. Prove that X, I_A, Y are collinear if and only if $AB + AC = 3BC$.

Proposed by Tovi Wen, USA

First solution by Daniel Lasaosa, Pamplona, Spain

Note first that $\angle BIC = 180° - \frac{B}{2} - \frac{C}{2} = 90° + \frac{A}{2}$, or $\angle BIY = \angle CIX = \frac{A}{2}$. Now, $\angle IBY = 180° - \angle ABI = 180° - \frac{B}{2}$, or $\angle BYI = \frac{B-A}{2}$, and similarly $\angle CXI = \frac{C-A}{2}$. Or, by the Sine Law we have

$$\frac{BY}{BA} = \frac{BI}{BA} \cdot \frac{\sin\frac{A}{2}}{\sin\frac{B-A}{2}} = \frac{\sin^2\frac{A}{2}}{\sin\frac{B+A}{2}\sin\frac{B-A}{2}} = \frac{1-\cos A}{\cos A - \cos B},$$

$$AY = AB\frac{1-\cos B}{\cos A - \cos B}, \qquad AX = AC\frac{1-\cos C}{\cos A - \cos C}.$$

By the similarity between incircle and A-excircle, and using the Cosine Law to simplify $(b+c+a)(a+b-c)$ and $(b+c-a)(a+b-c)$, we have

$$\frac{AI_A}{AB} = \frac{AI_A}{AI} \cdot \frac{AI}{AB} = \frac{b+c+a}{b+c-a} \cdot \frac{\sin\frac{B}{2}}{\sin\frac{B+A}{2}} = \frac{\cos\frac{B}{2}}{\sin\frac{C}{2}},$$

$$\frac{AI_A}{AY} = \frac{AI_A}{AB} \cdot \frac{AB}{AY} = \frac{\cos\frac{B}{2}\sin\frac{B-A}{2}}{\sin^2\frac{B}{2}}$$

Or by the Cosine Law and after some algebra,

$$\frac{YI_A{}^2}{AY^2} = \frac{AY^2 + AI_A{}^2 - 2AY \cdot AI_A\cos\frac{A}{2}}{AY^2} =$$

$$= 1 + \frac{\cos^2\frac{B}{2}\sin^2\frac{B-A}{2}}{\sin^4\frac{B}{2}} - 2\frac{\cos\frac{B}{2}\sin\frac{B-A}{2}\cos\frac{A}{2}}{\sin^2\frac{B}{2}} =$$

$$= \left(\frac{\sin\frac{A}{2}}{\sin^2\frac{B}{2}} - \frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{B}{2}} + \cos\frac{A}{2} - \sin\frac{A}{2}\right)^2 - \sin^2\frac{A}{2}.$$

Now, the Apollonious circle defined by the points Z of the plane such that $\frac{XZ}{YZ} = \frac{AX}{AY}$ intersects the internal bisector of A at two points, one A itself, and the other one the point where this internal bisector meets segment XY. Now, this point is I_A iff I_A is on the Apollonious circle, or since by symmetry a similar expression may be obtained for $\frac{XI_A{}^2}{AY^2}$, we have that X, Y, I_A are collinear iff

$$\frac{\sin\frac{A}{2}}{\sin^2\frac{B}{2}} - \frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{B}{2}} = \frac{\sin\frac{A}{2}}{\sin^2\frac{C}{2}} - \frac{\cos\frac{A}{2}\cos\frac{C}{2}}{\sin\frac{C}{2}},$$

or after some algebra, iff

$$2\sin\frac{A}{2} = 2\sin\frac{B}{2}\sin\frac{C}{2} = \cos\frac{B-C}{2} - \cos\frac{B+C}{2} = \cos\frac{B-C}{2} - \sin\frac{A}{2},$$

$$3\sin A = 2\cos\frac{A}{2}\cos\frac{B-C}{2} = 2\sin\frac{B+C}{2}\cos\frac{B-C}{2} = \sin B + \sin C.$$

The conclusion follows.

Second solution by Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece

Let M be the midpoint of the small arc BC.

From Ptolemy's theorem we know that $AB \cdot CM + AC \cdot BM = AM \cdot BC \Leftrightarrow (AB + AC)IM = BC \cdot AM \Leftrightarrow AM = 3IM$ or $AI = 2IM$ (here we used the fact that $CM = BM = IM$)

We apply inversion such as the incircle as the circle of the inversion.

Here we consider the touching triangle to be $\triangle DEF$ (D is on BC, E is on AC and F is on AB). These points are invariant.

The conjugates of A, B, C are the midpoints of EF, DF, DE, denoted by A', B', C' respectively (Lemma).

It is clear that the circumcircle of $\triangle ABC$ and the Euler circle of $\triangle DEF$ are conjugates. Therefore, M's conjugate, M', is the second intersection of IA' and the Euler circle. It is well-known that $II_A = 2IM$, so now $I_A{}'$ is the midpoint of IM'.

Now we define X' and Y'.

X belongs to AC, which has conjugate the circle $\odot A'IC'$ (apparently this circle also passes through E) and the line IX' although preserved, can be defined as parallel to DF (since DF is also perpendicular to IC). So X' is on $\odot A'IC'$ and $IX' \parallel DF$ or $IX' \parallel A'C'$ (A', C' are midpoints of EF and DE respectively).

In a similar manner we show that Y' is on $\odot A'IB'$ such as $IY' \parallel A'B'$. Now line XY goes to the circle $\odot X'IY'$. Let N be the center of the Euler circle of $\triangle DEF$. N lies on the perpendicular bisector of $A'C'$, which is also the perpendicular bisector of IX'. In a similar manner we show that N lies also on the perpendicular bisector of IY'.

As a result, N is the center of circle $\odot X'IY'$.

Now, let S be the second intersection of the circle $X'IY'$ and IM'.

We want to examine the case $I_A{}' \equiv S$.

First observe that since N is the center of circle $\odot X'IY'$ and of the Euler circle, it is $SM = IA'$.

Combining with the fact that $I_A{}'$ is the midpoint of IM', it is easy to get that $I_A{}' \equiv S$ if and only if $2IA' = IM'$ that is to say if and only if $AI = 2IM$.

Also solved by Ioannis D. Sfikas, Athens, Greece; Corneliu Mănescu-Avram, Ploiești, Romania.

O491. If a, b, c are real numbers greater than -1 such that $a+b+c+abc=4$, prove that

$$\sqrt[3]{(a+3)(b+3)(c+3)}+\sqrt[3]{(a^2+3)(b^2+3)(c^2+3)} \geq 2\sqrt[3]{(ab+bc+ca+13)}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

Because $(x+3)(x^2+3)=(x+1)^3+8$, by the AM-GM Inequality the left hand side is greater than equal to

$$2\sqrt[6]{[(a+1)^3+8][(b+1)^3+8][(c+1)^3+8]}.$$

Hence, it suffices to prove that

$$[(a+1)^3+2^3][(b+1)^3+2^3][(c+1)^3+2^3] \geq (ab+bc+ca+13)^3.$$

But this follows from Holder's Inequality and the fact that

$$(a+1)(b+1)(c+1)+2\times 2\times 2=(abc+a+b+c)+(ab+bc+ca)+1+8=ab+bc+ca+13.$$

The equality holds if and only if $a+1=b+1=c+1=2$, that is if and only if $a=b=c=1$.

Also solved by Ioannis D. Sfikas, Athens, Greece.

O492. Let a, b, c, x, y, z be positive real numbers such that

$$(a+b+c)(x+y+z) = (a^2+b^2+c^2)(x^2+y^2+z^2) = 4.$$

Prove that

$$\sqrt{abcxyz} \le \frac{4}{27}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by Albert Stadler, Herrliberg, Switzeland

By replacing, if necessary, a, b, c, x, y, z by $a/\lambda, b/\lambda, c/\lambda, x\lambda, y\lambda, z\lambda$ for a suitable constant $\lambda > 0$ we may assume that $a+b+c = 2$ and $x+y+z = 2$. Then, by the Cauchy-Schwarz inequality, $a^2+b^2+c^2 \ge \frac{4}{3}$ and $x^2+y^2+z^2 \ge \frac{4}{3}$. Assume that $a^2+b^2+c^2 = r$ and $x^2+y^2+z^2 = \frac{4}{r}$,where $\frac{4}{3} \le r \le 3$. We claim that the maximum of abc under the constraints $a+b+c = 2$ and $a^2+b^2+c^2 = r$, $\frac{4}{3} \le r \le 3$ equals

$$\frac{40-18r+\sqrt{2}(3r-4)^{\frac{3}{2}}}{54} \tag{1}$$

The constraints imply that

$$a = \frac{1}{2}\left(2-c \pm \sqrt{-3c^2+4c+2r-4}\right),$$
$$b = \frac{1}{2}\left(2-c \mp \sqrt{-3c^2+4c+2r-4}\right),$$

The term under the square roots must be nonnegative. $-3c^2+4c+2r-4 \ge 0$ implies $c \le \frac{1}{3}(2+\sqrt{2}\sqrt{3r-4})$. So $abc = c^3-2c^2+\left(2-\frac{r}{2}\right)c$. The function $c \to c^3-2c^2+\left(2-\frac{r}{2}\right)c$ has a local maximum at $c = \frac{1}{6}(4-\sqrt{2}\sqrt{3r-4})$ that equals $\dfrac{40-18r+\sqrt{2}(3r-4)^{\frac{3}{2}}}{54}$

The value of $c^3-2c^2+\left(2-\frac{r}{2}\right)c$ at $c = \frac{1}{3}\left(2+\sqrt{2}\sqrt{3r-4}\right)$ equals $\dfrac{40-18r+\sqrt{2}(3r-4)^{\frac{3}{2}}}{54}$ as well and (1) follows.

It remains to show that $\dfrac{40-18r+\sqrt{2}(3r-4)^{\frac{3}{2}}}{54} \cdot \dfrac{40-18\frac{4}{r}+\sqrt{2}\left(3\frac{4}{r}-4\right)^{\frac{3}{2}}}{54} \le \left(\dfrac{4}{27}\right)^2$ which is equivalent to

$$\frac{(r-2)^2(-44+49r-11r^2)}{162r^2} \ge 0.$$

However, the last inequality is true for $\frac{4}{3} \le r \le 3$, since $-44+49r-11r^2 = 0$ has the roots $\frac{1}{22}(49 \pm \sqrt{465})$ which approximately equals to 1.247 and 3.207, and therefore $-44+49r-11r^2 > 0$ for $\frac{4}{3} \le r \le 3$.

Also solved by Ioannis D. Sfikas, Athens, Greece.