Junior Problems

J493. In triangle ABC, $R = 4r$. Prove that $\angle A - \angle B = 90^\circ$ if and only if

$$a - b = \sqrt{c^2 - \frac{ab}{2}}.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J494. Let a, b, c be positive real numbers. Prove that

$$\frac{ab + bc + ca + a + b + c}{(a+b)(b+c)(c+a)} \le \frac{3}{8}\left(1 + \frac{1}{abc}\right)$$

Proposed by Florin Rotaru, Focşani, România

J495. Let a, b, c be postive numbers such that $abc = 1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{a^2+b} + \frac{1}{b^2+c} + \frac{1}{c^2+a} \ge \frac{9}{2}.$$

Proposed by An Zhenping, Xianyang Normal University, China

J496. Let a_1, a_2, a_3, a_4, a_5 be positive real numbers. Prove that

$$\sum_{\text{cyc}} \frac{a_1}{2(a_1+a_2)+a_3} \cdot \sum_{\text{cyc}} \frac{a_2}{2(a_1+a_2)+a_3} \le 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J497. Prove that for any positive real numbers a, b, c

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \ge 2(a+b+c).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J498. Let ABC be a triangle with $\angle A \neq \angle B$ and $\angle C = 30^\circ$. On the internal angle bisector of $\angle BCA$ consider the points D and E such that $\angle CAD = \angle CBE = 30^\circ$ and on the perpendicular bisector of AB, on the same side as C related to AB, consider the point F such that $\angle AFB = 90^\circ$. Prove that DEF is an equilateral triangle.

Proposed by Titu Andreescu, USA, and Marius Stănean, România

Senior Problems

S493. In triangle ABC, $R = 4r$. Prove that

$$\frac{19}{2} \leq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \leq \frac{25}{2}$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

S494. Let $n > 1$ be an integer. Solve the equation

$$x^n - \lfloor x \rfloor = n.$$

Proposed by Alessandro Ventullo, Milan, Italy

S495. Let a, b, c be real numbers not less than $\frac{1}{2}$ such that $a+b+c=3$. Prove that

$$\sqrt{a^3+3ab+b^3-1} + \sqrt{b^3+3bc+c^3-1} + \sqrt{c^3+3ca+a^3-1} +$$

$$+\frac{1}{4}(a+5)(b+5)(c+5) \leq 60.$$

When does equality hold?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S496. Let ABC be a triangle and let a, b, c be the lengths of its sides. Prove that the centroid of the triangle lies on the incircle if and only if

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = \frac{1}{8}(a+b+c)^2.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

S497. Let $a, b, c \geq \frac{6}{5}$ be real numbers such that

$$a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+8.$$

Prove that

$$ab+bc+ca \leq 27.$$

Proposed by Marius Stănean, Zalău, România

S498. Solve in integers the equation

$$(mn+8)^3 + (m+n+5)^3 = (m-1)^2(n-1)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Undergraduate Problems

U493. Let A, B, C be matrices of order n such that $ABC = BCA = A+B+C$. Prove that $A(B+C) = -BC$ if and only if $(B+C)A = -BC$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U494. Let m be a real number such that the roots a, b, c of the polynomial $X^3 + mX^2 + X + 1$ satisfy the condition:

$$a^3b + b^3c + c^3a + ab^3 + bc^3 + ca^3 = 0.$$

Prove that a, b, c cannot all be real numbers.

Proposed by Mircea Becheanu, Montreal, Canada

U495. Let $g : \mathbb{N} \longrightarrow \mathbb{N}$ be a one-to-one function such that $\mathbb{N} \setminus g(\mathbb{N})$ is infinite. Let $n \geq 2$ be an arbitrary positive integer. Prove that g admits a functional n^{th} root, that is there is a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that $f \circ \cdots \circ f = g$, where f appears n times.

Proposed by Titu Andreescu, USA, and Marian Tetiva, România

U496. Prove that the polynomial $X^7 - 4X^6 + 4$ is irreducible in $\mathbb{Z}[X]$.

Proposed by Mircea Becheanu, Montreal, Canada

U497. Evaluate

$$\int_0^1 (2x^3 - 3x^2 + x)^{2019} dx$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U498. Let $f : [0,1] \to \mathbb{R}$ be the function defined by

$$f(x) = x \arctan x - \ln\left(1 + x^2\right).$$

Prove that

$$\int_{\frac{1}{2}}^{1} f(x)\, \mathrm{d}x \geq 3 \int_{0}^{\frac{1}{2}} f(x)\, \mathrm{d}x.$$

Proposed by Mihaela Berindeanu, Bucharest, România

Olympiad Problems

O493. Let x, y, z be positive real numbers such that $xy + yz + zx = 3$. Prove that

$$\frac{1}{x^2+5} + \frac{1}{y^2+5} + \frac{1}{z^2+5} \le \frac{1}{2}.$$

Proposed by Titu Andreescu, USA, and Marius Stănean, România

O494. Positive real numbers a and b satisfy the following system of equations:

$$a^2 + b = 1$$

$$ab + b^2 = 1.$$

Prove that there is a triangle with side lengths a, a, b, and find the measures of the angles of that triangle.

Proposed by Waldemar Pompe, Warsaw, Poland

O495. Let ABC be an acute triangle. Prove that

$$\frac{h_b h_c}{a^2} + \frac{h_c h_a}{b^2} + \frac{h_a h_b}{c^2} \le 1 + \frac{r}{R} + \frac{1}{3}\left(1 + \frac{r}{R}\right)^2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O496. Let M be the set of points with integer coordinates in the plane. Every point (a, b) in M is connected by an edge to all points (ab, c) in M with $c > ab$. Prove that no matter how the points in M are colored with finitely many colors, there is an edge with its endpoints colored with the same color.

Proposed by Titu Andreescu, USA and Marian Tetiva, România

O497. Let $A_1A_2\dots A_{2n+1}$ be a regular $(2n{+}1)$-gon with center O. Line l passes through O and meets line A_iA_{i+1} at point X_i $(i = 1, 2, \dots, 2n{+}1,\ A_{2n+2} = A_1)$. Prove that

$$\sum_{i=1}^{2n+1} \overrightarrow{\frac{1}{OX_i}} = 0\,.$$

Here, $\overrightarrow{\frac{1}{OX_i}}$ is the vector having the orientation of $\overrightarrow{OX_i}$ and the size $\frac{1}{OX_i}$.

Proposed by Waldemar Pompe, Warsaw, Poland

O498. In triangle ABC, let D,E,F be the feet of the altitudes from A,B,C respectively. Let H be the orthocenter of triangle ABC, M be the midpoint of the segment AH, and N be the intersection point of lines AD and EF. The line through A and parallel to BM intersects BC at P. Prove that the midpoint of the segment NP lies on AB.

Proposed by Titu Andreescu, USA, and Marius Stănean, România