Junior problems

J493. In triangle ABC, $R = 4r$. Prove that $\angle A - \angle B = 90°$ if and only if

$$a - b = \sqrt{c^2 - \frac{ab}{2}}.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note that the second condition is equivalent to

$$\frac{3ab}{2} = a^2 + b^2 - c^2 = 2ab\cos C, \qquad \cos C = \frac{3}{4}, \qquad \sin\frac{C}{2} = \frac{1}{2\sqrt{2}}.$$

Since it is well known that $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, the second condition in a triangle such that $R = 4r$ also gives $\sin\frac{A}{2}\sin\frac{B}{2} = \frac{1}{4\sqrt{2}}$, hence

$$\cos\frac{A-B}{2} = \cos\frac{A+B}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2} = \sin\frac{C}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2} = \frac{1}{\sqrt{2}} = \cos 45°,$$

which implies to $\angle A - \angle B = 90°$. Conversely, if $\angle A - \angle B = 90°$, then using $R = 4r$ so $1 = 16\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ we find

$$\frac{1}{\sqrt{2}} = \cos\frac{A-B}{2} = \sin\frac{C}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2} = \sin\frac{C}{2} + \frac{1}{8\sin\frac{C}{2}}.$$

This quadratic equation in $\sin\frac{C}{2}$ has a double root and hence implies $\sin\frac{C}{2} = \frac{1}{2\sqrt{2}}$, which we saw above was equivalent to the second equation. The conclusion follows.

Also solved by Ioannis D. Sfikas, Athens, Greece; Polyahedra, Polk State College, USA; Arkady Alt, San Jose, CA, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Dumitru Barac, Sibiu, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Titu Zvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, India; Vicente Vicario García, Sevilla. Spain.

J494. Let a, b, c be positive real numbers. Prove that

$$\frac{ab+bc+ca+a+b+c}{(a+b)(b+c)(c+a)} \le \frac{3}{8}\left(1+\frac{1}{abc}\right).$$

Proposed by Florin Rotaru, Focşani, România

Solution by Polyahedra, Polk State College, USA

Let $p = (a+b+c)/3$, $q = (ab+bc+ca)/3$, and $r = abc$.

Then $pq \ge r$, so $(a+b)(b+c)(c+a) = 9pq - r \ge 8pq$, and thus it suffices to prove $pq(r+1) \ge (p+q)r$.

Note that $p^2 \ge q$. If either $p \le 1$ or $q \ge 1$, then $(p-1)(q-1) \ge 0$ and $pq(r+1)-(p+q)r = (p-1)(q-1)r+pq-r \ge 0$. It remains to consider $p > 1 > q$. Then $r \le q^{3/2} < q$, and therefore $pq(r+1)-(p+q)r = (p-1)qr+p(q-r) > 0$, completing the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ioannis D. Sfikas, Athens, Greece; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Titu Zvonaru, Comănești, Romania.

J495. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{a^2+b} + \frac{1}{b^2+c} + \frac{1}{c^2+a} \geq \frac{9}{2}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by the author

Let $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$ then the equation rewrites as

$$\frac{y}{x} + \frac{z}{y} + \frac{x}{z} + \frac{y^2z}{x^2z+y^3} + \frac{z^2x}{y^2x+z^3} + \frac{x^2y}{z^2y+x^3} \geq \frac{9}{2}.$$

LHS =

$$\frac{y^2(y+z)^2}{xy(y+z)^2} + \frac{z^2(z+x)^2}{zy(z+x)^2} + \frac{x^2(x+y)^2}{xz(x+y)^2} + \frac{y^2z^2}{x^2z^2+y^3z} + \frac{z^2x^2}{y^2x^2+z^3x} + \frac{x^2y^2}{z^2y^2+x^3y} \geq$$

$$\frac{[(xy+yz+zx)+x(x+y)+y(y+z)+z(z+x)]^2}{xy(y+z)^2+zy(z+x)^2+xz(x+y)^2+x^2y^2+y^2z^2+z^2x^2+x^3y+y^3z+z^3x} =$$

$$\frac{(x+y+z)^4}{(xy+yz+zx)(x^2+y^2+z^2+xy+yz+zx)} = \frac{(x+y+z)^4}{(xy+yz+zx)[(x+y+z)^2-(xy+yz+zx)]}$$

With $x+y+z = m, xy+yz+zx = k$ and $m^2 \geq 3k$ get

$$\frac{m^4}{k(m^2-k)} \geq \frac{9}{2} \Leftrightarrow 2m^4 - 9m^2k + 9k^2 \geq 0 \Leftrightarrow (2m^2-3k)(m^2-3k) \geq 0$$

and we are done.

Also solved by Ioannis D. Sfikas, Athens, Greece; Michael - Athanasios Peppas, Evangeliki Model High School of Smyrna, Athens, Greece.

J496. Let a_1, a_2, a_3, a_4, a_5 be positive real numbers. Prove that

$$\sum_{\text{cyc}} \frac{a_1}{2(a_1+a_2)+a_3} \cdot \sum_{\text{cyc}} \frac{a_2}{2(a_1+a_2)+a_3} \leq 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Polyahedra, Polk State College, USA

Note that $\sum_{cyc} a_3(2a_1+2a_2+a_3) = \left(\sum_{cyc} a_3\right)^2$. By the weighted AM-HM inequality,

$$\sum_{cyc} \frac{a_3}{2(a_1+a_2)+a_3} \geq \left(\sum_{cyc} a_3\right)^2 \cdot \frac{1}{\sum_{cyc} a_3(2a_1+2a_2+a_3)} = 1.$$

Hence, by the AM-GM inequality,

$$\sum_{cyc} \frac{a_1}{2(a_1+a_2)+a_3} \cdot \sum_{cyc} \frac{a_2}{2(a_1+a_2)+a_3} \leq \frac{1}{4}\left(\sum_{cyc} \frac{a_1+a_2}{2(a_1+a_2)+a_3}\right)^2$$

$$= \frac{1}{4}\left(\frac{5}{2} - \frac{1}{2}\sum_{cyc} \frac{a_3}{2(a_1+a_2)+a_3}\right)^2 \leq \frac{1}{4}\left(\frac{5}{2} - \frac{1}{2}\right)^2 = 1.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ioannis D. Sfikas, Athens, Greece; Titu Zvonaru, Comănești, Romania.

J497. Prove that for any positive real numbers a, b, c

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + \sqrt{bc} + \sqrt{ca} + \sqrt{ab} \geq 2(a + b + c).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain

By the weighted AM-GM inequality, and since $\frac{1}{14} + \frac{11}{14} + \frac{1}{7} + 1 = 2$, we have

$$\frac{1}{14}\frac{a^2}{b} + \frac{11}{14}\frac{b^2}{c} + \frac{1}{7}\frac{c^2}{a} + \sqrt{bc} \geq 2\sqrt{a^{\frac{1}{7}-\frac{1}{7}} \cdot b^{-\frac{1}{14}+\frac{11}{7}+\frac{1}{2}} \cdot c^{-\frac{11}{14}+\frac{2}{7}+\frac{1}{2}}} = 2b,$$

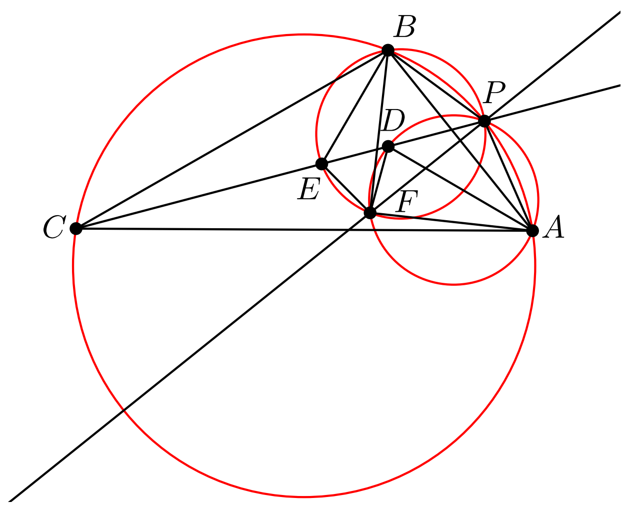
with equality iff $\frac{a^2}{b} = \frac{b^2}{c} = \frac{c^2}{a} = \sqrt{bc}$, ie iff $a = b = c$. Adding the cyclic permutations of this inequality produces the proposed result, where equality holds iff $a = b = c$.

Also solved by Albert Stadler, Herrliberg, Switzerland; Polyahedra, Polk State College, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Dumitru Barac, Sibiu, Romania; Taes Padhihary, Disha Delphi Public School, India; Ioannis D. Sfikas, Athens, Greece; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA.

J498. Let ABC be a triangle with $\angle A \neq \angle B$ and $\angle C = 30^\circ$. On the internal angle bisector of $\angle BCA$ consider the points D and E such that $\angle CAD = \angle CBE = 30^\circ$ and on the perpendicular bisector of AB, on the same side as C related to AB, consider the point F such that $\angle AFB = 90^\circ$. Prove that DEF is an equilateral triangle.

Proposed by Titu Andreescu, USA, and Marius Stănean, România

Solution by Polyahedra, Polk State College, USA



By the property of an angle bisector, CD intersects the circumcircle of $\triangle ABC$ again at P, the midpoint of the arc of AB not containing C. Hence, $\angle PBA = \angle PAB = 15^\circ$, and thus $\angle PBF = \angle PAF = 60^\circ$. Since FP is the perpendicular bisector of AB, $\angle BFP = 45^\circ = \angle BEP$ and $\angle AFP = 45^\circ = \angle ADP$. Therefore, $PBEF$ and $PAFD$ are cyclic quadrilaterals. Consequently, $\angle PEF = \angle PBF = 60^\circ$ and $\angle CDF = \angle PAF = 60^\circ$, completing the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Taes Padhihary, Disha Delphi Public School, India; Titu Zvonaru, Comănești, Romania.

Senior problems

S493. In triangle ABC, $R = 4r$. Prove that

$$\frac{19}{2} \le (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \le \frac{25}{2}$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution by Arkady Alt, San Jose, CA, USA

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = \frac{2s\,(ab+bc+ca)}{abc} = \frac{2s\left(s^2+4Rr+r^2\right)}{4Rrs} =$$

$$\frac{s^2+4Rr+r^2}{2Rr} = \frac{s^2+4\cdot 4r\cdot r+r^2}{2\cdot 4rr} = \frac{s^2+17r^2}{8r^2}$$

Since

$$s^2 \le 4R^2+4Rr+3r^2 = 4\cdot(4r)^2+4\cdot 4r\cdot r+3r^2 = 83r^2$$

and

$$s^2 \ge 16Rr-5r^2 = 16\cdot 4r\cdot r-5r^2 \ = 59r^2 \text{(Gerretsen's Inequalities)}$$

then

$$\frac{s^2+17r^2}{8r^2} \le \frac{83r^2+17r^2}{8r^2} = \frac{25}{2}$$

and

$$\frac{s^2+17r^2}{8r^2} \ge \frac{59r^2+17r^2}{8r^2} = \frac{19}{2}.$$

Also solved by Titu Zvonaru, Comănești, Romania; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Corneliu Mănescu-Avram, Ploiești, Romania; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Telemachus Baltsavias, Keramies Junior High School Kefalonia, Greece.

S494. Let $n > 1$ be an integer. Solve the equation

$$x^n - \lfloor x \rfloor = n.$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Daniel Lasaosa, Pamplona, Spain

If $|x| < 1$ we have $|x^n - \lfloor x \rfloor| < |1| + |1| = 2$, or no solution exists in this case. If $x = 1$ we have $x^n - \lfloor x \rfloor = 0$ for any $n \geq 2$, whereas if $x = -1$, we have $x^n - \lfloor x \rfloor = 0$ when n is odd and $x^n - \lfloor x \rfloor = 2$ when n is even, or the only possible solution with $|x| \leq 1$ is $x = -1$ and only for $n = 2$.

If $x > 1$, let $m = \lfloor x \rfloor$, or $m \geq 1$ and $x = m + \delta$, where $0 \leq \delta < 1$, or the equation rewrites as $m + n = (m+\delta)^n \geq m^n$. Or,

$$n \geq m^n - m = m\left(m^{n-1} - 1\right) = m(m-1)\left(m^{n-2} + m^{n-3} + \cdots + 1\right) \geq m(n-1)(m-1).$$

If $m \geq 2$, we have $n \geq 2(n-1)$, or $2 \geq n$, whereas if $m \geq 3$, we have $n \geq 6(n-1)$, or $n \leq \frac{6}{5} < 2$. It then follows that solutions with $x > 1$ exist only for $m = n = 2$ or for $m = 1$. If $m = n = 2$, we have $x^2 = n + m = 4$, or $x = 2$. If $m = 1$, we have $x^n = n + 1$, with solution $x = \sqrt[n]{n+1}$ for each positive integer $n \geq 2$, in which case clearly $\lfloor x \rfloor = 1$, as expected and needed. There can be no other solution for $x > 1$.

If $x < -1$, let $m = -\lfloor x \rfloor$, where $m \geq 2$, and $x = -m + \delta$, where $0 \leq \delta < 1$, and the equation rewrites as $(-m+\delta)^n + m = n$. If n is odd, then $m = n + (m-\delta)^n > n + (m-1)^n > n + m - 1 \geq m + 1$, absurd, or solutions may exist only for even n. If n is even, then using that $m \geq 2$, we have

$$n > (m-1)^n + m = (m-2)\left((m-1)^{n-1} + (m-1)^{n-2} + \cdots + 1\right) + m + 1 \geq$$

$$\geq (m-2)n + m + 1.$$

Note therefore that for $m \geq 3$ there are no solutions of we must have $m = 2$, and consequently $x^n = n - 2$, or $x = -\sqrt[n]{n-2}$, with $\lfloor x \rfloor = -2$, as expected and needed.

It follows that all solutions are

- $x = -1$ and $x = 2$ for $n = 2$.
- $x = \sqrt[n]{n+1}$ for any integer $n \geq 2$.
- $x = -\sqrt[n]{n-2}$ for any even integer $n \geq 4$.

Also solved by Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Taes Padhihary, Disha Delphi Public School, India; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Ioannis D. Sfikas, Athens, Greece.

S495. Let a, b, c be real numbers not less than $\frac{1}{2}$ such that $a+b+c=3$. Prove that

$$\sqrt{a^3+3ab+b^3-1}+\sqrt{b^3+3bc+c^3-1}+\sqrt{c^3+3ca+a^3-1}+\frac{1}{4}(a+5)(b+5)(c+5) \le 60.$$

When does equality hold?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam
Using the well-known identity

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

and the AM-GM inequality we have

$$\begin{aligned}\sqrt{a^3+3ab+b^3-1} &= \sqrt{(a+b-1)(a^2+b^2+1-ab+a+b)} \\ &= \frac{1}{2}\sqrt{4(a+b-1)(a^2+b^2+1-ab+a+b)} \\ &\le \frac{4(a+b-1)+(a^2+b^2+1-ab+a+b)}{4} \\ &= \frac{a^2+b^2-ab+5a+5b-3}{4}.\end{aligned}$$

Writing two similar inequalities and adding them up we get

$$\begin{aligned}\sum_{\text{cyc}}\sqrt{a^3+3ab+b^3-1} &\le \frac{a^2+b^2+c^2}{2}-\frac{ab+bc+ca}{4}+\frac{5(a+b+c)}{2}-\frac{9}{4} \\ &= \frac{a^2+b^2+c^2}{2}-\frac{ab+bc+ca}{4}+\frac{21}{4}.\end{aligned}$$

On the other hand

$$\begin{aligned}(a+5)(b+5)(c+5) &= abc+5(ab+bc+ca)+25(a+b+c)+125 \\ &\le \frac{(a+b+c)^3}{27}+5(ab+bc+ca)+25(a+b+c)+125 \\ &= 5(ab+bc+ca)+201.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{\text{cyc}}\sqrt{a^3+3ab+b^3-1}+\frac{1}{4}(a+5)(b+5)(c+5) &\le \frac{a^2+b^2+c^2}{2}+ab+bc+ca+\frac{111}{2} \\ &= \frac{(a+b+c)^2}{2}+\frac{111}{2} \\ &= 60\end{aligned}$$

as desired. Equality holds if and only if $a=b=c=1$.

Also solved by Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Taes Padhihary, Disha Delphi Public School, India; Ioannis D. Sfikas, Athens, Greece; Titu Zvonaru, Comănești, Romania.

S496. Let ABC be a triangle and let a, b, c be the lengths of its sides. Prove that the centroid of the triangle lies on the incircle if and only if

$$(a-b)^2+(b-c)^2+(c-a)^2=\frac{1}{8}(a+b+c)^2.$$

Proposed by Albert Stadler, Herrliberg, Switzerland

Solution by Ivko Dimitric, Penn State University Fayette, Lemont Furnace, PA, USA

In homogeneous barycentric coordinates where $A=(1:0:0)$, $B=(0:1:0)$ and $C=(0:0:1)$, the centroid of ABC has the coordinates $G=(1:1:1)$ and the incircle is given by the equation

$$-a^2yz-b^2zx-c^2xy+(x+y+z)[(s-a)^2x+(s-b)^2y+(s-c)^2z]=0,$$

where $s=\frac{1}{2}(a+b+c)$ is the semiperimeter. The centroid of the triangle lies on the incircle if and only if the coordinates of G satisfy the equation of the incircle, i. e.

$$\begin{aligned} & -a^2-b^2-c^2+3\left[(s-a)^2+(s-b)^2+(s-c^2)\right]=0 \\ \iff\ & -(a^2+b^2+c^2)+3\left[3s^2+(a^2+b^2+c^2)-2s(a+b+c)\right]=0 \\ \iff\ & 2(a^2+b^2+c^2)=3s^2. \end{aligned}$$

On the other hand, the given condition is equivalent to

$$\begin{aligned} & (a-b)^2+(b-c)^2+(c-a)^2=\frac{1}{8}\cdot 4s^2 \\ \iff\ & 2(a^2+b^2+c^2)-2(ab+bc+ca)=\frac{1}{2}s^2 \\ \iff\ & 2(a^2+b^2+c^2)-\left[(a+b+c)^2-(a^2+b^2+c^2)\right]=\frac{1}{2}s^2 \\ \iff\ & 3(a^2+b^2+c^2)=\frac{9}{2}s^2 \\ \iff\ & 2(a^2+b^2+c^2)=3s^2, \end{aligned}$$

which, from the above, is equivalent to G belonging to the incircle.

Also solved by Daniel Lasaosa, Pamplona, Spain; Vicente Vicario García, Sevilla. Spain; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Dumitru Barac, Sibiu, Romania; Corneliu Mănescu-Avram, Ploiești, Romania; Ioannis D. Sfikas, Athens, Greece; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Nguyen Viet Hung, Hanoi University of Science, Vietnam; Titu Zvonaru, Comănești, Romania; Jenna Park, Blair Academy, Blairstown, NJ, USA.

S497. Let $a, b, c \geq \frac{6}{5}$ be real numbers such that

$$a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 8$$

Prove that

$$ab + bc + ca \leq 27.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author
Denote $t = a + b + c$. We have

$$\frac{5a-6}{a} + \frac{5b-6}{b} + \frac{5c-6}{c} = 63 - 6(a+b+c) = 63 - 6t.$$

It follows that $t < \frac{21}{2}$. Also

$$t - 8 \geq \frac{9}{t} \Longrightarrow t \geq 9.$$

By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} 63 - 6t &= \frac{(5a-6)^2}{a(5a-6)} + \frac{(5b-6)^2}{b(5b-6)} + \frac{(5c-6)^2}{c(5c-6)} \\ &\geq \frac{(5(a+b+c)-18)^2}{5(a^2+b^2+c^2)-6(a+b+c)} \\ &= \frac{(5t-18)^2}{5t^2-10(ab+bc+ca)-6t}. \end{aligned}$$

Therefore

$$10(ab+bc+ca) \leq 5t^2 - 6t - \frac{(5t-18)^2}{63-6t}$$

and

$$5t^2 - 6t - \frac{(5t-18)^2}{63-6t} - 270 = -\frac{2(t-9)^2(15t+107)}{63-6t} \leq 0.$$

S498. Solve in integers the equation

$$(mn+8)^3+(m+n+5)^3=(m-1)^2(n-1)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

With the substitutions $mn+8=x$ and $-(m+n+5)=y$ the equation becomes

$$x^3-y^3=(x+y-2)^2.$$

We have $x \geq y$. If $x=y$, then $x+y=2$, yielding the solution $(x,y)=(1,1)$. Let now consider $x-y=d>0$. Then

$$d[(y+d)^2+(y+d)y+y^2]=[2y+(d-2)]^2,$$

implying

$$(3d-4)y^2+(3d^2-4d+8)y+(d^3-d^2+4d-4)=0. \tag{1}$$

This is a quadratic equation in y with discriminant

$$\Delta=(3d^2-4d+8)^2-4(3d-4)(d^3-d^2+4d-4)=-3d^4+4d^3+48d.$$

Δ must be a perfect square, so first of all, $\Delta \geq 0$, implying

$$(3d-4)d^2 \leq 48.$$

It follows that $d<4$, so $d \in \{1,2,3\}$. If $d=1$, then $\Delta=49$ and (1) becomes $-y^2+7y=0$, yielding solutions $(x,y)=(1,0)$ and $(x,y)=(8,7)$. If $d=2$, then $\Delta=80$, not a perfect square. If $d=3$, then $\Delta=9$ and (1) becomes $5y^2+23y+26=0$, yielding the solution $(x,y)=(1,-2)$.

In conclusion, (x,y) is one of the pairs $(1,1),(1,0),(8,7),(1,-2)$. Coming back to the substitutions we obtain $(m,n)=(1,-7),(-7,1),(0,-12),(-12,0)$.

Remark: after performing the first substitution in the solution above, one can also argue as follows: let $x-y=a$ and $x+y=b$, then the equation $x^3-y^3=(x+y-2)^2$ becomes $a(a^2+3b^2)=4(b-2)^2$. If $a=0$, then $b=2$. If $a=1$, then $b^2-16b+15=0$, with two solutions $b=1$ and $b=15$. The case $a=2$ yields no solution and the case $a=3$ yields $b=-1$. Finally, if $a \geq 4$ then $8(b^2+4) \geq 4(b-2)^2=a(a^2+3b^2) \geq 4(16+3b^2)$, a contradiction.

Also solved by Taes Padhihary, Disha Delphi Public School, India; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran.

Undergraduate problems

U493. Let A, B, C be matrices of order n such that $ABC = BCA = A + B + C$. Prove that $A(B + C) = -BC$ if and only if $(B + C)A = -BC$.

Proposed by Titu Andreescu, University of Texas at Austin, USA

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

The condition $BCA = A + B + C$ implies $B + C = BCA - A$. Then

$$\begin{aligned} A(B + C) &= A(BCA - A) = A(BCA) - A^2 = (ABC)A - A^2 \\ &= (BCA)A - A^2 = (BCA - A)A = (B + C)A. \end{aligned}$$

From this, it follows that $A(B + C) = -BC$ if and only if $(B + C)A = -BC$.

Also solved by Li Zhou, Polk State College, USA; Daniel Lasaosa, Pamplona, Spain; Aratrika Pandey, West Bengal, India; Min Jung Kim, Tabor Academy; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Jenna Park, Blair Academy, Blairstown, NJ, USA; Min Jung Kim, Tabor Academy.

U494. Let m be a real number such that the roots a, b, c of the polynomial $X^3 + mX^2 + X + 1$ satisfy the condition:

$$a^3b + b^3c + c^3a + ab^3 + bc^3 + ca^3 = 0.$$

Prove that a, b, c cannot all be real numbers.

Proposed by Mircea Becheanu, Montreal, Canada

Solution by Li Zhou, Polk State College, USA

By the given information, $a + b + c = -m$, $ab + bc + ca = 1$, and $abc = -1$. Therefore,

$$a^3b + b^3c + c^3a + ab^3 + bc^3 + ca^3 = \left(a^2 + b^2 + c^2\right)(ab + bc + ca) - abc(a + b + c) = m^2 - 2 - m,$$

thus $m = -1$ or $m = 2$. If $m = -1$, then $a^2 + b^2 + c^2 = m^2 - 2 = -1$, so a, b, c cannot be all real. Now consider $m = 2$. Note that $f(x) = x^3+2x^2+x+1 = x(x+1)^2+1 > 0$ if $x \geq -1$. Also, $f'(x) = 3x^2+4x+1 = (3x+1)(x+1) > 0$ for $x < -1$, thus f has only one real zero.

Also solved by Daniel Lasaosa, Pamplona, Spain; Jenna Park, Blair Academy, Blairstown, NJ, USA; Min Jung Kim, Tabor Academy; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Ivko Dimitric, Penn State University Fayette, Lemont Furnace, PA, USA.

U495. Let $g:\mathbb{N}\to\mathbb{N}$ be a one-to-one function such that $\mathbb{N}\backslash g(\mathbb{N})$ is infinite. Let $n\geq 2$ be an arbitrary positive integer. Prove that g admits a functional n^{th} root, that is there is a function $f:\mathbb{N}\to\mathbb{N}$ such that $f\circ\cdots\circ f=g$, where f appears n times.

Proposed by Titu Andreescu, USA, and Marian Tetiva, România

Solution by Daniel Lasaosa, Pamplona, Spain
The result is not necessarily true. Consider function $g:\mathbb{N}\to\mathbb{N}$ defined by $g(0)=1$, $g(1)=0$, and $g(k)=2k$ for all $k\geq 2$, and let $n=2$. Note that g is clearly one-to-one because $0,1$ are the respective images of $1,0$, and each even integer larger than or equal to 4 is the image of its half, the image of g not taking any other value. Note also that $\mathbb{N}\backslash g(\mathbb{N})$ is infinite because it contains exactly 2 and all odd integers larger than or equal to 3. Assume that a functional square root f exists, and let $f(0)=u$. Note that $u\neq 0$ because otherwise we would have $1=g(0)=f(f(0))=f(0)=0$, contradiction. Note that $u\neq 1$ because otherwise we would have $f(1)=f(f(0))=g(0)=1$, for $0=g(1)=f(f(1))=f(1)=1$, contradiction. Therefore $u\geq 2$, hence $1=g(0)=f(f(0))=f(u)$, or $2u=g(u)=f(f(u))=f(1)$, for $0=g(1)=f(f(1))=f(2u)$, and finally $u=f(0)=f(f(2u))=g(2u)=4u$, contradiction since $u\neq 0$. Or at least in this case, a functional square root of g does not exist.

Editor's Note: The problem statement should have included the additional hypothesis that $g(x)\geq x$ for all $x\in\mathbb{N}$. With this added hypothesis, let $F=\{x:g(x)=x\}$ be all the fixed points of g and let $x_{1,0},x_{2,0},\ldots$ be the elements which are not in the image of g, that is, the elements of $\mathbb{N}\backslash g(\mathbb{N})$. Starting with each of these we can define a chain of values by repeatedly applying g: $x_{i,1}=g(x_{1,0})$, $x_{i,2}=g(x_{i,1}),\ldots$. These chains cannot cycle and cannot merge since g is one-to-one. (If $x_{i,m}=x_{i',m'}$ with $(i,m)\neq(i',m')$ and $m,m'>0$, then $g(x_{i,m-1})=g(x_{i',m'-1})$, contrary to g being one-to-one. Having $x_{i,0}=x_{i',m'}$ would contradict $x_{i,0}$ not being in the image of g.) It is easy to see that these chains and the fixed points F cover all of $\mathbb{N}$. Indeed if we start from any $y\in\mathbb{N}$ and it is neither fixed nor in $\mathbb{N}\backslash g(\mathbb{N})$, then we can find a $y_1<y$ such that $g(y_1)=y$. Clearly y_1 is not a fixed point of g and if it is not in the image, then we can find y_2 with $g(y_2)=y_1$. Continuing in this way we build a decreasing sequence $y=y_0>y_1>\cdots$, which must terminate. Hence there is some y_m which $x_{k,0}$ for some k and hence $y=x_{k,m}$.

To build a functional n-th root f of g we define: $f(x)=x$ if $x\in F$; $f(x_{k,m})=x_{k+1,m}$ if k is not a multiple of n; and $f(x_{k,m})=x_{k+1-n,m+1}$ if k is a multiple of n. Clearly, if $x\in F$, then $f\circ\cdots\circ f(x)=g(x)$. If we apply f to $x_{k,m}$ n times, then on $n-1$ of the steps we will increase k by 1 and leave m unchanged. On the other step we will decrease k by $1-n$ and increase m by 1. Therefore the net effect will be to leave k unchanged, but increase m by one. Hence $f\circ\cdots\circ f(x_{k,m})=x_{k,m+1}=g(x_{k,m})$.

Without the assumption that $g(x)\geq x$ for all x, iterating g on $\mathbb{N}$, breaks the natural numbers not only into fixed points, and one-sided infinite chains beginning at the points not in the image of g (as above), but can also cycles of finite length (the counterexample above has a cycle of length 2) and infinite "cycles", that is, sequences $\cdots,y_{-2},y_{-1},y_0,y_1,y_2,\cdots$ such that $g(y_i)=y_{i+1}$ for all integer i. The existence of a functional n-th root puts restrictions on all the cycles whose length is not relatively prime to n (where we count infinity as being a multiple of n for any integer n) and on the number of one-sided infinite chains. If the cycle length c is not relatively prime to n, then let m be the product of all primes dividing $\gcd(c,n)$ taken with the multiplicity with which they divide n. (For example if $n=360$ and $c=70$, then the gcd is 10. Since 2 divides n three times and 5 once, we take $m=40$.) Then the required condition on cycles of length c is that the number of them must be a multiple of m (where again infinity counts as a multiple of any integer). The restriction on one-sided infinite chains is that the number of them must be a multiple of n. (The hypothesis above that $\mathbb{N}\backslash g(\mathbb{N})$ is infinite guaranteed that this condition held.) If the numbers of cycles of every finite length, the number of infinite cycles, and the number of one-sided infinite chains all satisfy these congruence conditions then a functional n-th root exists.

The counterexample above has 1, hence an odd number, of cycles of length 2 so it cannot have a functional square root.

U496. Prove that the polynomial $X^7 - 4X^6 + 4$ is irreducible in $\mathbb{Z}[X]$.

Proposed by Mircea Becheanu, Montreal, Canada

First solution by Richard Stong, Rice University, USA

Suppose the polynomial factors in $\mathbb{Z}[X]$ as $X^7 - 4X^6 + 4 = q(X)r(X)$ where q has degree $k \geq 1$ and r has degree $l \geq 1$ and $k + l = 7$. Assume without loss of generality that $l \geq k$, hence since $l + k$ is odd, $l > k$. Write

$$q(X) = a_0 + a_1 X + \cdots + a_k X^k \quad \text{and} \quad r(X) = b_0 + b_1 X + \cdots + b_l X^l.$$

Note that since $a_k b_l = 1$, both a_k and b_l are odd. Let i be the least index such that a_i is odd and j the least index such that b_j is odd. Then the coefficient of X^{i+j} in $q(X)r(X)$ is

$$a_i b_j + \sum_{m=0}^{i-1} a_m b_{i+j-m} + \sum_{m=0}^{j-1} a_{i+j-m} b_m,$$

which is odd since the first term is odd and every term in the sums is even. (We take the convention that $b_{i+j-m} = 0$ is $i + j - m > l$ and similarly for a.) But the only odd coefficient in $X^7 - 4X^6 + 4$ is the leading coefficient. Hence $i + j = 7$. Thus $i = k$ and $j = m$. Thus only the leading coefficients of q and r are odd. Note that since $a_0 b_0 = 4$ and both are even, we have $a_0 = b_0 = \pm 2$. Now look at the coefficient of X^k. It is

$$a_k b_0 + \sum_{m=0}^{k-1} a_m b_{k-m}.$$

In the sum every summand is a multiple of 4, but $b_0 = \pm 2$ and a_k is odd. Hence the coefficient is 2 modulo 4. But no coefficient of $X^7 - 4X^6 + 4$ is 2 modulo 4. Thus we have a contradiction and $X^7 - 4X^6 + 4$ is irreducible in $\mathbb{Z}[X]$.

Second solution by Daniel Lasaosa, Pamplona, Spain
Denote $p(x) = x^7 - 4x^6 + 4$. If x is odd then $p(x)$ is odd, hence nonzero. If x is even then $p(x) \equiv 4 \pmod{2^7}$, hence also nonzero. Or $p(x)$ has no integer roots, hence if it is not irreducible, polynomials $q(x)$ and $r(x)$ exist, such that since we may exchange $q(x)$ and $r(x)$ without altering the problem, the degree of $q(x)$ is either 2 or 3, and consequently the degree of $r(x)$ is respectively either 5 or 4. Moreover, the product of independent terms of $q(x)$ and $r(x)$ is 4, or since we may wlog multiply both $q(x)$ and $r(x)$ by -1 without altering the problem, we may assume that the independent term of $q(x)$ is one of $1, 2, 4$.

Case 1: $q(x)$ has degree 2 and independent term 1. Then, integers u, v, a, b, c, d, e exist such that $q(x) = 1 + ux + vx^2$ and $r(x) = 4 + ax + bx^2 + cx^3 + dx^4 + ex^5$, hence $a = -4u$ is a multiple of 4, then $b = -au - 4v$ is also a multiple of 4, $c = -bu - av$ is also a multiple of 4, $d = -cu - bv$ is a multiple of 4, and finally $e = -du - cv$ is a multiple of 4, in contradiction with $ev = 1$.

Case 2: $q(x)$ has degree 3 and independent term 1. Then, integers u, v, w, a, b, c, d exist such that $q(x) = 1 + ux + vx^2 + wx^3$ and $r(x) = 4 + ax + bx^2 + cx^3 + dx^4$, hence $a = -4u$ is a multiple of 4, then $b = -au - 4v$ is a multiple of 4, or $c = -bu - av - 4w$ is a multiple of 4, and finally $d = -cu - bv - aw$ is a multiple of 4, in contradiction with $dw = 1$.

Case 3: $q(x)$ has degree 2 and independent term 2. Then, integers u, v, a, b, c, d, e exist such that $q(x) = 2 + ux + vx^2$ and $r(x) = 2 + ax + bx^2 + cx^3 + dx^4 + ex^5$. Note that $a = -u$, and since $u^2 = 2b + 2v$ is even, then $u = 2n$ is even with n integer, for $a = -2n$ and $b = 2n^2 - v$. Then, $c = -\frac{bu+av}{2} = 2nv - 2n^3$, and finally $d = -\frac{bv+cu}{2} = \frac{v^2}{2} - 3n^2v + 2n^2$ is an integer, or v is even, in contradiction with $ev = 1$.

Case 4: $q(x)$ has degree 3 and independent term 2. Then, integers u, v, a, b, c, d, e exist such that $q(x) = 2 + ux + vx^2 + wx^3$ and $r(x) = 2 + ax + bx^2 + cx^3 + dx^4$. As in the previous case, we have $u = 2n$ with integer n, with $a = -u$ and $b = 2n^2 - v$. It follows that $c = 2nv - 2n^3 - w$, and $v^2 = 2d - 4n^4 + 4n^2v - 4nw$ is even, or $v = 2m$ is even, and $d = 2n^4 - 4n^2m + 4nm + 2m^2$ is even, in contradiction with $dw = 0$.

Case 5: $q(x)$ has degree 2 and independent term 4. Then, integers u, v, a, b, c, d, e exist such that $q(x) = 4 + ux + vx^2$ and $r(x) = 1 + ax + bx^2 + cx^3 + dx^4 + ex^5$, hence $u = -4a$ is a multiple of 4, or $v = -au - 4b$ is a multiple of 4, in contradiction with $ev = 1$.

Case 6: $q(x)$ has degree 3 and independent term 4. Then, integers u, v, a, b, c, d, e exist such that $q(x) = 4 + ux + vx^2 + wx^3$ and $r(x) = 1 + ax + bx^2 + cx^3 + dx^4$, hence $u = -4a$ is a multiple of 4, or $v = -au - 4b$ is also a multiple of 4, and finally $w = -av - bu - 4c$ is also a multiple of 4, in contradiction with $dw = 1$.

Or $p(x)$ is indeed irreducible.

Also solved by Albert Stadler, Herrliberg, Switzerland; Jenna Park, Blair Academy, Blairstown, NJ, USA; Min Jung Kim, Tabor Academy; Aratrika Pandey, West Bengal, India; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran.

U497. Evaluate

$$\int_0^1 \left(2x^3 - 3x^2 + x\right)^{2019} dx$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Henry Ricardo, Westchester Area Math Circle

Make the substitution $x \mapsto -u + 1/2$. Then the integral becomes

$$\int_{-1/2}^{1/2} (-2u^3 + u/2)^{2019}\, du,$$

which has the value 0 since $(-2u^3 + u/2)^{2019}$ is an odd function, and it is well known (and easily proved) that $\int_{-a}^{a} f(x)dx = 0$ if f is an odd function.

Also solved by Daniel López-Aguayo, MSCI, Monterrey, Mexico; Daniel Lasaosa, Pamplona, Spain; Min Jung Kim, Tabor Academy; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Corneliu Mănescu-Avram, Ploiești, Romania; S. Chandrasekhar, Chennai, India; Dumitru Barac, Sibiu, Romania; Ioannis D. Sfikas, Athens, Greece; Aratrika Pandey, West Bengal, India; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania.

U498. Let $f:[0,1]\to\mathbb{R}$ be the function defined by

$$f(x)=x\arctan x-\ln\left(1+x^2\right).$$

Prove that

$$\int_{\frac{1}{2}}^{1} f(x)dx\geq 3\int_0^{\frac{1}{2}} f(x)dx.$$

Proposed by Mihaela Berindeanu, Bucharest, România

Solution by Alexandru Daniel Pirvuceanu, National "Traian" College, Drobeta-Turnu Severin, Romania
Lemma : Let $g:[0,1]\to\mathbb{R}$ be a convex function with $g(0)=0$. Then

$$\int_{\frac{1}{2}}^{1} g(x)dx\geq 3\int_0^{\frac{1}{2}} g(x)dx.$$

Proof: Since g is convex and $g(0)=0$, it is well-known that the function $h:(0,1]\to\mathbb{R}$, $h(x)=\frac{g(x)}{x}$ is increasing.

Hence, $h(x)\leq h\left(\frac{1}{2}\right),\forall x\in\left(0,\frac{1}{2}\right]\iff g(x)\leq 2xg\left(\frac{1}{2}\right),\forall x\in\left(0,\frac{1}{2}\right]$

Let's note that the inequality from above also holds when $x=0$.

So, $g(x)\leq 2xg\left(\frac{1}{2}\right),\forall x\in\left[0,\frac{1}{2}\right]$, and by integrating this inequality on $\left[0,\frac{1}{2}\right]$ we get that

$$\int_0^{\frac{1}{2}} g(x)dx\leq\frac{1}{4}g\left(\frac{1}{2}\right)\tag{1}$$

We also have that $h(x)\geq h\left(\frac{1}{2}\right),\forall x\in\left[\frac{1}{2},1\right]\iff g(x)\geq 2xg\left(\frac{1}{2}\right),\forall x\in\left[\frac{1}{2},1\right]$. By integrating this inequality on $\left[\frac{1}{2},1\right]$ we get that

$$\int_{\frac{1}{2}}^{1} g(x)dx\geq\frac{3}{4}g\left(\frac{1}{2}\right)\overset{(1)}{\geq}3\int_0^{\frac{1}{2}} g(x)dx,$$

which is the desired conclusion.

Back to our problem. Since $f(0)=0$ and f is convex ($f''(x)=\frac{2x^2}{(1+x^2)^2}\geq 0,\forall x\in[0,1]$), the Lemma from above gives us the desired inequality.

Also solved by Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Spain; Vicente Vicario García, Sevilla. Spain; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Ivko Dimitric, Penn State University Fayette, Lemont Furnace, PA, USA; Ioannis D. Sfikas, Athens, Greece; Arkady Alt, San Jose, CA, USA.

O493. Let x, y, z be positive real numbers such that $xy + yz + zx = 3$. Prove that

$$\frac{1}{x^2+5} + \frac{1}{y^2+5} + \frac{1}{z^2+5} \le \frac{1}{2}.$$

Proposed by Titu Andreescu, USA, and Marius Stănean, România

Solution by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Applying the Cauchy-Schwarz inequality and the well-known result

$$9(x+y)(y+z)(z+x) \ge 8(x+y+z)(xy+yz+zx)$$

we have

$$\begin{aligned}
\sum_{\text{cyc}} \frac{1}{x^2+5} &= \sum_{\text{cyc}} \frac{3}{3x^2+5(xy+yz+zx)} \\
&= \sum_{\text{cyc}} \frac{3}{3(x+y)(x+z)+2(xy+yz+zx)} \\
&= \frac{1}{3}\sum_{\text{cyc}} \frac{(2+1)^2}{3(x+y)(x+z)+2(xy+yz+zx)} \\
&\le \frac{1}{3}\sum_{\text{cyc}} \left(\frac{4}{3(x+y)(x+z)} + \frac{1}{2(xy+yz+zx)}\right) \\
&= \frac{8(x+y+z)}{9(x+y)(y+z)(z+x)} + \frac{1}{2(xy+yz+zx)} \\
&\le \frac{1}{xy+yz+zx} + \frac{1}{2(xy+yz+zx)} \\
&= \frac{3}{2(xy+yz+zx)} \\
&= \frac{1}{2}.
\end{aligned}$$

This completes the proof. Equality holds for $x = y = z = 1$.

Also solved by Hei Chun Leung, Hong Kong; Daniel Lasaosa, Pamplona, Spain; Taes Padhihary, Disha Delphi Public School, India; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Corneliu Mănescu-Avram, Ploiești, Romania; Dumitru Barac, Sibiu, Romania; Gian Sanjaya; Ivko Dimitric, Penn State University Fayette, Lemont Furnace, PA, USA; Ioannis D. Sfikas, Athens, Greece; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA; Jenna Park, Blair Academy, Blairstown NJ, USA.

O494. Positive real numbers a and b satisfy the following system of equations:

$$a^2 + b = 1$$

$$ab + b^2 = 1.$$

Prove that there is a triangle with side lengths a, a, b, and find the measures of the angles of that triangle.

Proposed by Waldemar Pompe, Warsaw, Poland

Solution by Li Zhou, Polk State College, USA

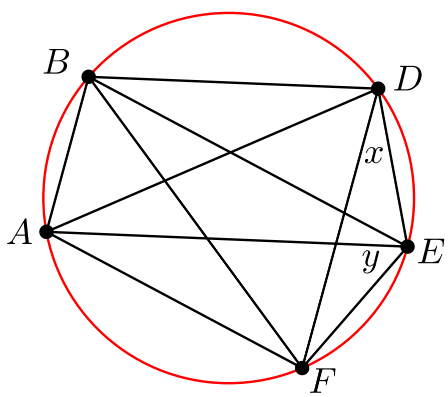
Clearly, $a, b \in (0, 1)$. Also,

$$0 = 2\left(a^2 + b\right) - \left(ab + b^2\right) - 1 = a(2a - b) - (b - 1)^2 < a(2a - b),$$

so $2a > b$, thus there is a triangle with side lengths a, a, and b. Applying Ptolemy's theorem to $a \cdot a + 1 \cdot b = 1 \cdot 1$, we have a unique cyclic quadrilateral $ABDE$ with $AB = DE = a$, $BD = b$, and $EA = AD = BE = 1$. Applying Ptolemy's theorem to $a \cdot b + b \cdot b = 1 \cdot 1$, we have a unique point F on the circumcircle of $ABDE$ such that $DF = FA = b$ and $BF = 1$. Applying Ptolemy's theorem to the quadrilateral $ABEF$ we get

$$1 = AE \cdot BF = AF \cdot BE + AB \cdot EF = b \cdot 1 + a \cdot EF,$$

so $EF = a$. Now let x and y be the acute arc angles subtended by the chords a and b, respectively. Then $y = \angle BFD = \angle DBF = 2x$, so $\angle DEF = 2y + x = 5x$, thus $\triangle DEF$ has interior angles $\pi/7$, $\pi/7$, and $5\pi/7$.



Also solved by Albert Stadler, Herrliberg, Switzerland; Daniel Lasaosa, Pamplona, Spain; Taes Padhihary, Disha Delphi Public School, India; Corneliu Mănescu-Avram, Ploiești, Romania; Gian Sanjaya; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Brian Bradie, Christopher Newport University, Newport News, VA, USA.

O495. Let ABC be an acute triangle. Prove that

$$\frac{h_b h_c}{a^2}+\frac{h_c h_a}{b^2}+\frac{h_a h_b}{c^2}\leq 1+\frac{r}{R}+\frac{1}{3}\left(1+\frac{r}{R}\right)^2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that $h_b=a\sin C$ and $h_c=a\sin B$, or

$$\frac{h_b h_c}{a^2}=\sin B\sin C=\cos B\cos C-\cos(B+C)=\cos A+\cos B\cos C,$$

and similarly after cyclic permutation of A,B,C. Using also Carnot's theorem $\frac{R+r}{R}=\cos A+\cos B+\cos C$ allows us then to rewrite the proposed inequality in its equivalent form

$$3\left(\cos A\cos B+\cos B\cos C+\cos C\cos A\right)\leq\left(\cos A+\cos B+\cos C\right)^2,$$

clearly true by the scalar product inequality, and with equality iff $\cos A=\cos B=\cos C$. The conclusion follows, equality holds iff ABC is equilateral.

Also solved by Vicente Vicario García, Sevilla. Spain; Taes Padhihary, Disha Delphi Public School, India; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Corneliu Mănescu-Avram, Ploiești, Romania; Ivko Dimitric, Penn State University Fayette, Lemont Furnace, PA, USA; Ioannis D. Sfikas, Athens, Greece; Telemachus Baltsavias, Keramies Junior High School Kefalonia, Greece; Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA.

O496. Let M be the set of points with integer coordinates in the plane. Every point (a,b) in M is connected by an edge to all points (ab,c) in M with $c > ab$. Prove that no matter how the points in M are colored with finitely many colors, there is an edge with its end points colored with the same color.

Proposed by Titu Andreescu, USA, and Marian Tetiva, România

Solution by the authors

For any positive integer i, let M_i be the subset of M defined by

$$M_i = \{(i,j) \mid i,j \in \mathbb{Z}_+,\ i < j\}$$

and let C_i be the set of colors used to color the points from M_i.

There are, of course, only a finite number of possibilities for the sets C_i (if the total number of colors is k, C_i can only be one of the $2^k - 1$ nonempty subsets of the set of all colors). So, among the sets $M_{i!}$ (for $i \in \mathbb{Z}_+$), there must be infinitely many to which the same set of colors is assigned. In other words, there exists an infinite sequence of positive integers $i_1 < i_2 < \cdots$ such that

$$C_{i_1!} = C_{i_2!} = \cdots,$$

or, if we denote $i_j!$ by k_j for any positive integer j, we have a strictly increasing sequence $k_1 < k_2 < \cdots$ of positive integers such that each k_s is a divisor of k_t whenever $s \le t$, and

$$C_{k_1} = C_{k_2} = \cdots.$$

A strictly increasing sequence of positive integers is unbounded (actually it goes to ∞), thus there exists n such that $k_n > k_1^2$, hence the point

$$\left(k_1, \frac{k_n}{k_1}\right)$$

belongs to M_{k_1}. (We do not forget that k_1 is a divisor of k_n.) Since $C_{k_1} = C_{k_n}$, the color of this point appears in M_{k_n}, too, meaning that there is a point

$$(k_n, l) \in M_{k_n}$$

(with $k_n < l$) which has the same color as

$$\left(k_1, \frac{k_n}{k_1}\right).$$

And thus we get the points

$$\left(k_1, \frac{k_n}{k_1}\right) \text{ and } (k_n, l)$$

which have the same color and are joined by an edge, as desired.

Editor's Note: The similar problem where each point (a,b) is joined to the points $(a+b,c)$ with $c > a+b$ was proposed by I. Tomescu for a Romanian TST about 40 years ago.

Also note that in the same way we can infer that there are, in fact, infinitely many pairs of points having the same color and being joined by an edge.

Also solved by Li Zhou, Polk State College, USA; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran.

O497. Let $A_1A_2\dots A_{2n+1}$ be a regular $(2n+1)$-gon with center O. Line l passes through O and meets line A_iA_{i+1} at point X_i $(i = 1, 2, \dots, 2n+1,\ A_{2n+2} = A_1)$. Prove that

$$\sum_{i=1}^{2n+1} \overrightarrow{\frac{1}{OX_i}} = 0.$$

Here, $\overrightarrow{\frac{1}{OX_i}}$ is the vector having the orientation of $\overrightarrow{OX_i}$ and the size $\frac{1}{OX_i}$.

Proposed by Waldemar Pompe, Warsaw, Poland

First solution by Ivko Dimitric, Penn State University Fayette, Lemont Furnace, PA, USA

We may suppose that the regular $(2n+1)$-gon lies in the plane of complex numbers and is inscribed in the unit circle centered at the origin. Further, assume that its vertices are $(2n+1)$th roots of unity so that $A_k = \omega^k$, $A_{2n+1} = 1$, where ω is the primitive root with the smallest argument. Represent each point A in the plane by the corresponding complex number denoted by the same letter and any vector $\overrightarrow{XY}$ by the complex number $Y - X$, in particular $\overrightarrow{OX_i} = X_i$. The line through points P and Q has an equation

$$(\overline{Q} - \overline{P})Z - (Q - P)\overline{Z} = P\overline{Q} - \overline{P}Q,$$

so if the line l passes through a point K on the unit circle we have

$$\text{Line}\, l = \overrightarrow{OK}\ :\qquad \overline{K}Z - K\overline{Z} = 0$$

and

$$\text{Line}\, A_iA_{i+1}\ :\qquad (\overline{A_{i+1}} - \overline{A_i})Z - (A_{i+1} - A_i)\overline{Z} = A_i\overline{A_{i+1}} - \overline{A_i}A_{i+1} = \overline{\omega} - \omega.$$

The intersection point X_i of these two lines is found by solving these two equations for Z to get

$$X_i = \frac{K(\overline{\omega} - \omega)}{K(\overline{A_{i+1}} - \overline{A_i}) - \overline{K}(A_{i+1} - A_i)}.$$

Then,

$$\begin{aligned}|\overrightarrow{OX_i}|^2 = X_i \cdot \overline{X_i} &= \frac{K(\overline{\omega} - \omega)}{K(\overline{A_{i+1}} - \overline{A_i}) - \overline{K}(A_{i+1} - A_i)} \cdot \frac{\overline{K}(\omega - \overline{\omega})}{\overline{K}(A_{i+1} - A_i) - K(\overline{A_{i+1}} - \overline{A_i})}\\ &= \frac{(\overline{\omega} - \omega)^2}{[K(\overline{A_{i+1}} - \overline{A_i}) - \overline{K}(A_{i+1} - A_i)]^2}\end{aligned}$$

Therefore,

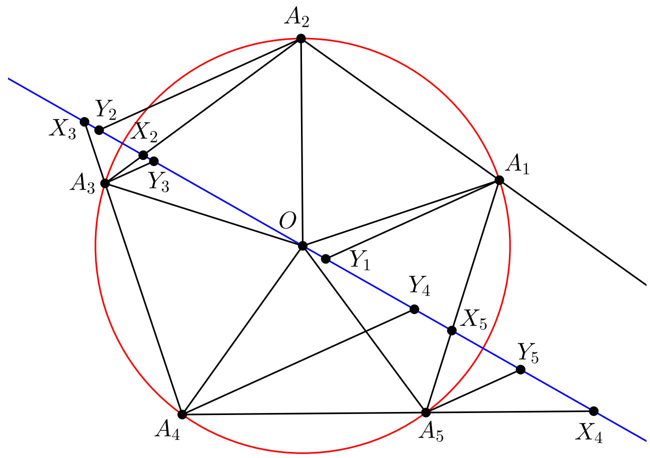
$$\begin{aligned}\sum_{i=1}^{2n+1} \overrightarrow{\frac{1}{OX_i}} &= \sum_{i=1}^{2n+1} \frac{1}{|\overrightarrow{OX_i}|^2}\overrightarrow{OX_i}\\ &= \sum_{i=1}^{2n+1} \frac{[K(\overline{A_{i+1}} - \overline{A_i}) - \overline{K}(A_{i+1} - A_i)]^2}{(\overline{\omega} - \omega)^2} \cdot \frac{K(\overline{\omega} - \omega)}{K(\overline{A_{i+1}} - \overline{A_i}) - \overline{K}(A_{i+1} - A_i)}\\ &= \frac{K^2}{\overline{\omega} - \omega} \sum_{i=1}^{2n+1} (\overline{A_{i+1}} - \overline{A_i}) - \frac{1}{\overline{\omega} - \omega} \sum_{i=1}^{2n+1} (A_{i+1} - A_i) = 0,\end{aligned}$$

since $A_{2n+2} = A_1$ and the sum

$$\sum_{i=1}^{2n+1} (A_{i+1} - A_i) = (A_2 - A_1) + (A_3 - A_2) + \dots + (A_{2n+1} - A_{2n}) + (A_1 - A_{2n+1}) = 0$$

is telescoping.

Second solution by Li Zhou, Polk State College, USA



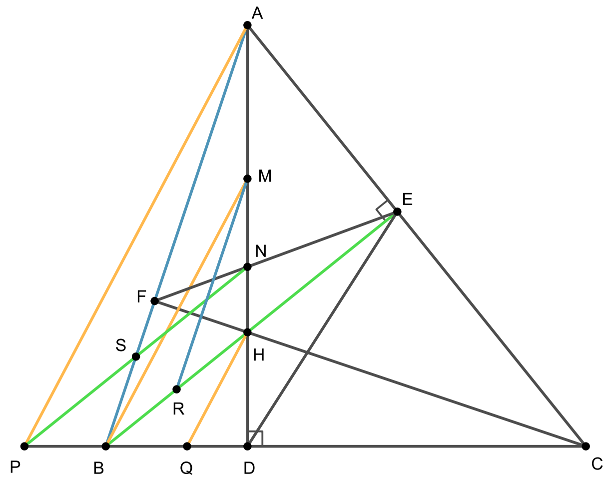
Suppose that $OA_1 = 1$. For each i, let Y_i be the inversive image of X_i with center O and radius 1. Then the claim is equivalent to $\sum_{i=1}^{2n+1} \overrightarrow{OY_i} = 0$. Since $OY_iA_iA_{i+1}$ is cyclic for each i (the inversive image of line A_iA_{i+1}), all A_iY_i are parallel to each other and forming the acute angle $\frac{(2n-1)\pi}{2(2n+1)}$ with l. Since $\overrightarrow{OA_i} = \overrightarrow{OY_i} + \overrightarrow{Y_iA_i}$ for each i and $\sum_{i=1}^{2n+1} \overrightarrow{OA_i} = 0$, we have $\sum_{i=1}^{2n+1} \overrightarrow{OY_i} = 0$ and $\sum_{i=1}^{2n+1} \overrightarrow{Y_iA_i} = 0$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Albert Stadler, Herrliberg, Switzerland.

O498. In triangle ABC, let D, E, F be the feet of the altitudes from A, B, C respectively. Let H be the orthocenter of triangle ABC, M be the midpoint of the segment AH, and N be the intersection point of lines AD and EF. The line through A and parallel to BM intersects BC at P. Prove that the midpoint of the segment NP lies on AB.

Proposed by Titu Andreescu, USA, and Marius Stănean, România

Solution by the authors



First, notice that $AFHE$ and $CEHD$ are cyclic quadrilaterals. Therefore

$$m(\angle HEF) = m(\angle HAF) = 90^\circ - m(\angle B)$$

and

$$m(\angle HED) = m(\angle HCD) = 90^\circ - m(\angle B),$$

hence $(EH$ is the interior angle bisector of $\angle DEF$. Since $m(\angle HEA) = 90^\circ$ it follows that $(EA$ is the exterior angle bisector of $\angle DEF$.

By the angle bisector theorem, respectiv exterior angle bisector theorem, applied in the triangle DEN, we have

$$\frac{HN}{HD} = \frac{AN}{AD} = \frac{EN}{ED},$$

so

$$\frac{HN}{HD} = \frac{AN + HN}{AD + HD} = \frac{AH}{AH + 2HD} = \frac{AH/2}{AH/2 + HD} = \frac{MH}{MD}. \tag{1}$$

Let Q be the intersection point of BC with the line through H and parallel to BM. Note that $AP \parallel MB \parallel HQ$ so since $AM = MH$, we have $PB = BQ$.

Now, from Thales's theorem in triangle MBD ($HQ \parallel MB$), we get

$$\frac{BQ}{BD} = \frac{MH}{MD},$$

or

$$\frac{PB}{BD} = \frac{MH}{MD}.$$

Combining this with (1), we find that

$$\frac{PB}{BD} = \frac{HN}{HD}$$

then by applying the Thales reciprocal theorem in triangle NDP, we deduce that

$$BH \parallel PN,$$

Since $BH \parallel PN$, $MB \parallel AP$ it follows that $\triangle APN \sim \triangle MBH$. Thus, if R, S is the midpoints of the segments (BH), (PN) also, we get $\triangle ASN \sim \triangle MRH$, so $MR \parallel AS$. But, since $MR \parallel AB$ (MR is midline in $\triangle HAB$) we conclude that $S \in AB$ which is exactly the desired result.

Editor's Note: Tthis problem can be generalized. Replace H with any point X in the interior of ABC and let D, E, F be the feet of the Cevians through X, and define the reamining points in the same way. Then the midpoint of NP still lies on AB.

Also solved by Andrea Fanchini, Cantù, Italy; Hei Chun Leung, Hong Kong; Daniel Lasaosa, Pamplona, Spain; Taes Padhihary, Disha Delphi Public School, India; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Corneliu Mănescu-Avram, Ploiești, Romania; Ivko Dimitric, Penn State University Fayette, Lemont Furnace, PA, USA; Konstantinos Konstantinidis, Greece; Pooya Esmaeil Akhondy, Atomic Energy High School, Tehran, Iran; Tovi Wen, USA; Li Zhou, Polk State College, USA.