Ptolemy's sine lemma

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Abstract

We present a lemma that is sometimes useful in Olympiad geometry. It allows us to establish whether or not four points are concyclic. The proof of this lemma is based on the well-known, yet rarely used, Ptolemy's theorem. We therefore called it Ptolemy's sine lemma.

1 Introduction and main lemma

We start by recalling Ptolemy's theorem, a classical result and we present one of its proofs, which we regard as beautiful.

Theorem (Ptolemy's)
In a cyclic quadrilateral $ABCD$, the following relation holds:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$

Proof. Let A', B' and C' be the images of A, B and C under the inversion with center D and radius 1. Then A', B' and C' are collinear, so $A'B' + B'C' = A'C'$. On the other hand, $\triangle DAB \sim \triangle DB'A'$, so $\frac{A'B'}{DB'} = \frac{AB}{DA}$. Since $DB' = \frac{1}{DB}$, we get $A'B' = \frac{AB}{DA \cdot DB}$. One can derive analogous expressions for $B'C'$ and $A'C'$. Multiplying them by $DA \cdot DB \cdot DC$, one easily obtains the desired equality. □

We are now ready to introduce the announced result.

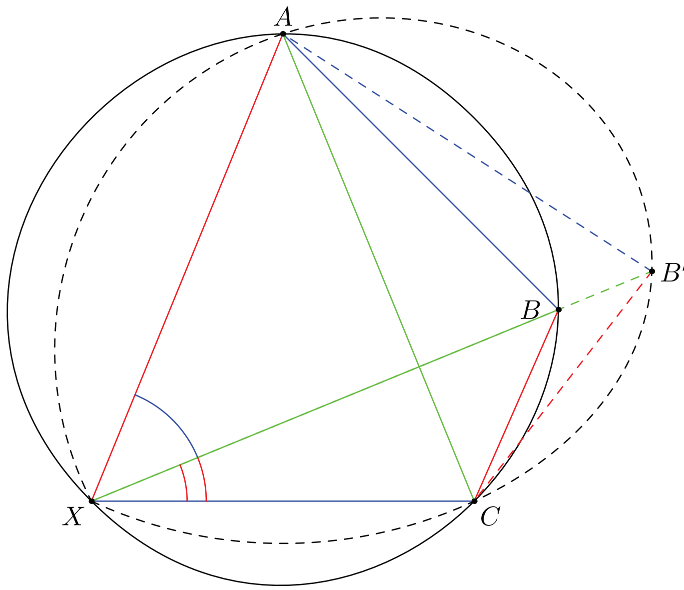
Lemma (Ptolemy's sine lemma)
Points X, A, B and C in the Euclidean plane are concyclic if and only if

$$XA \cdot \sin\measuredangle BXC + XB \cdot \sin\measuredangle CXA + XC \cdot \sin\measuredangle AXB = 0.$$

Proof. WLOG, we can assume that the ray $(XB$ lies between $(XA$ and $(XC$, as in the diagram below. Let B' be the point in which XB intersects the circle (XAC). Then by Ptolemy's theorem, $XA \cdot CB' + XC \cdot AB' = XB' \cdot AC$. By the law of sines,

$$2R = \frac{AB'}{\sin\measuredangle AXB} = \frac{B'C}{\sin\measuredangle BXC} = \frac{AC}{\sin\measuredangle CXA},$$

so that we get $XA \cdot \sin\measuredangle BXC + XB' \cdot \sin\measuredangle CXA + XC \cdot \sin\measuredangle AXB = 0$. Therefore, $XB' = XB$ and $B' = B$, as desired. □



We present the following alternative proof.

Proof. After an inversion with center X and radius 1, we get the following equivalent formulation: A, B and C are collinear if and only if

$$\frac{\sin\measuredangle BXC}{XA}+\frac{\sin\measuredangle CXA}{XB}+\frac{\sin\measuredangle AXB}{XC}=0.$$

Or, after a multiplication by $XA\cdot XB\cdot XC$

$$XB\cdot XC\cdot\sin\measuredangle BXC+XC\cdot XA\cdot\sin\measuredangle CXA+XA\cdot XB\cdot\sin\measuredangle AXB=0$$

This equality above is equivalent to

$$[BXC]+[CXA]+[AXB]=0,$$

which is equivalent to

$$[ABC]=0,$$

meaning that A, B and C are collinear. The expressions in square brackets stand for areas. □

By presenting the following examples, we aim to convince the reader that Ptolemy's sine lemma is a very versatile tool in Olympiad geometry.

2 Example problems

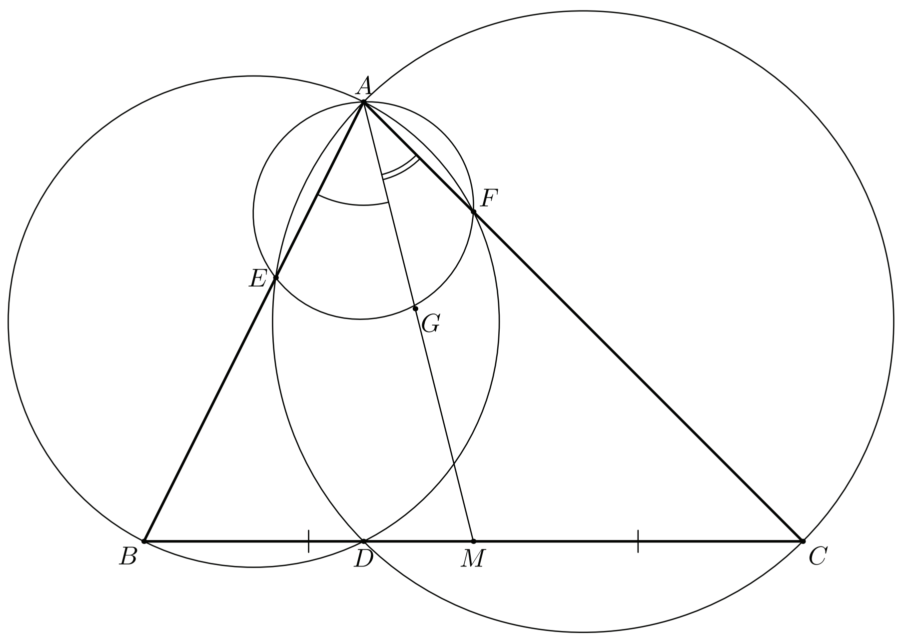
Example 2.1 (ELMO 2013 SL G3)

In $\triangle ABC$, a point D is chosen on the side BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.

Solution. Let M be the midpoint of the side BC. Denote the length of the segment BD by t. One can verify that $BE = \frac{at}{c}$, so $AE = \frac{c^2 - at}{c}$. By a similar computation, one gets $AF = \frac{b^2 - a(a-t)}{b}$. Let G be the second point of intersection between the circle (AEF) and the line AM (other than A). By our lemma, $AF \cdot \sin\angle BAM + AE \cdot \sin\angle MAC = AG \cdot \sin\angle BAC$. It is well-known known that

$$\frac{\sin\angle BAM}{b} = \frac{\sin\angle MAC}{c} = n.$$

Rearranging, we get $AG = \frac{n(b^2 + c^2 - a^2)}{\sin\angle BAC}$, and hence G does not depend on D. □



Example 2.2 (Danylo Khilko)

Let BB_1 and CC_1 be altitudes in $\triangle ABC$. Let M and N be the midpoints of BB_1 and CC_1, respectively. Let P and Q be the intersection points of (BC_1M) and (CB_1N) with BC. Prove that $BP = CQ$.

Solution. WLOG, we can assume that the diameter of the circle (ABC) is 1. Following the notation in the diagram below, one obtains

$$BB_1 = \sin\alpha \cdot \sin\gamma, \quad BC_1 = \sin\alpha\cos\beta.$$

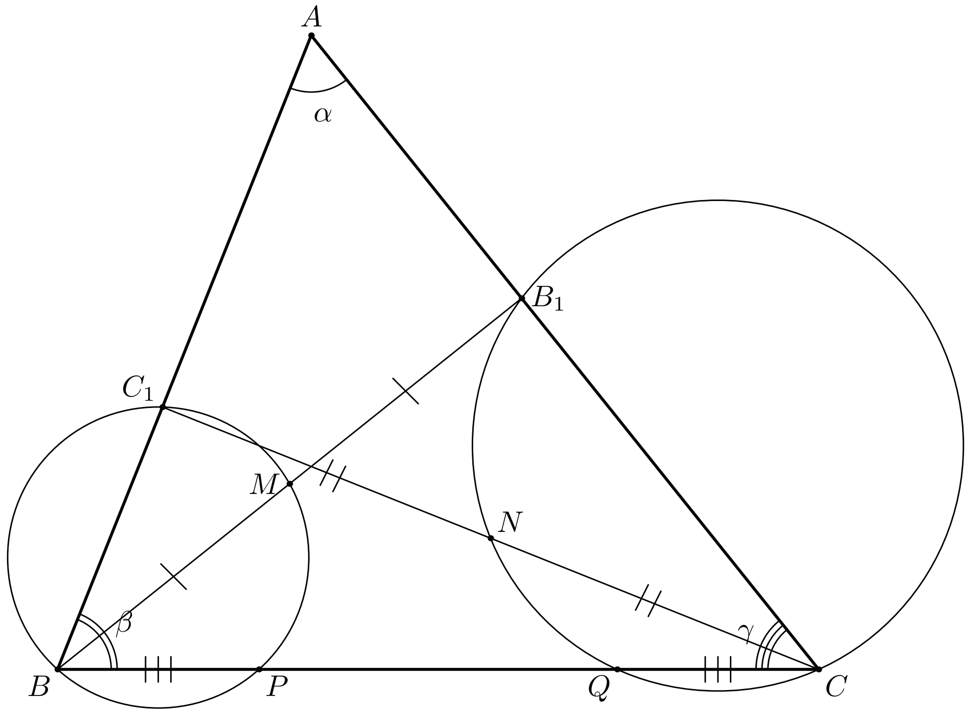
By the main lemma,

$$BP \cdot \cos\alpha + BC_1 \cdot \cos\gamma = \frac{1}{2} BB_1 \cdot \sin\beta,$$

$$BP \cdot \cos\alpha + \sin\alpha\cos\beta\cos\gamma = \frac{1}{2}\sin\alpha \cdot \sin\gamma \cdot \sin\beta,$$

.

By an analogous computation, one can show that an equality identical to the one above holds if BP is replaced by CQ. The conclusion follows. □



Example 2.3 (ISL 2012 G2)
Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.

Solution. Using Ptolemy's sine lemma, it remains to prove that

$$DH \cdot \sin \measuredangle FDG + DG \cdot \sin \measuredangle HDF = DF \cdot \sin \measuredangle HDG.$$

Note that $DH = DE$, $DG = CE$]. Simple angle chasing gives that

$$\measuredangle FDG = \measuredangle DBC, \quad \measuredangle HDF = \measuredangle ADB, \quad \measuredangle HDG = \measuredangle DFC.$$

Our condition can be rewritten as

$$DE \cdot \sin \angle DBC + CE \cdot \sin \angle ADB = DF \cdot \sin \angle DFC$$

By the law of sines,

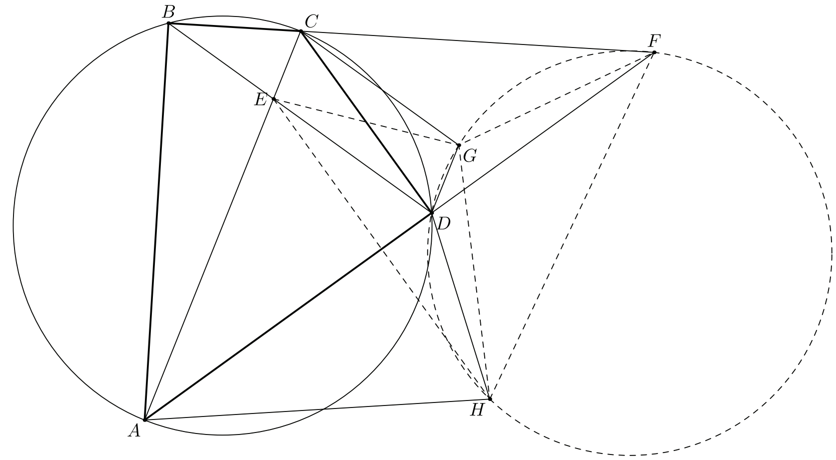
$$DF \cdot \sin \angle DFC = DC \cdot \sin \angle FCD = DC \cdot \sin \angle BCD$$

$$CE \cdot \sin \angle ADB = CE \cdot \sin \angle ECB = EB \cdot \sin \angle EBC$$

and so

$$DE \cdot \sin \angle DBC + EB \cdot \sin \angle EBC = DB \cdot \sin \angle DBC = DC \cdot \sin \angle BCD$$

as desired. □



Example 2.4 (Ukraine RMM TST 2017)

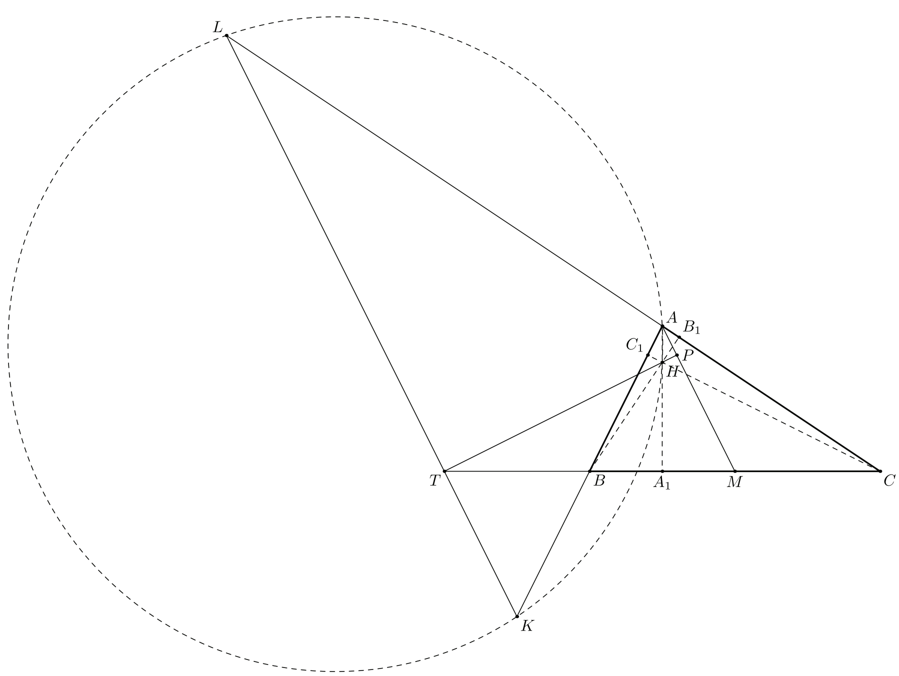
In the acute-angled triangle ABC, let H be the orthocenter and M the midpoint of side BC. Let the line passing through H perpendicular to AM intersect BC at a point T, and the line passing through T parallel to AM intersect lines AB and CA at points K and L respectively. Prove that points A, K, L and H are cyclic.

Solution. Let AA_1, BB_1 and CC_1 be the altitudes. Write P for the projection of H on AM. Then (AB_1HPC_1) and (HA_1MP) are circles with diameters AH and HM respectively, and $(A_1B_1MC_1)$ is the 9-point circle. By the radical axis theorem, the lines B_1C_1, PH and $MA_1 = BC$ concur at T.

From Menelaus' theorem for line B_1C_1T, we find the ratio BT/CT so we also can derive expressions for BT and CT in terms of sides and angles of $\triangle ABC$.

From pairs of similar triangles ($\triangle ABM \sim \triangle KBT$, $\triangle ACM \sim \triangle LCT$) we find AK and AL.

Now, we just have to use Ptolemy's sine lemma to finish our proof. We leave the computation as an useful exercise for the reader. □



3 Practice problems

Problem 3.1. Let ABC be a triangle with orthocenter H and points X, Y and Z on lines AH, BH and CH respectively such that $[XBC] + [AYC] + [ABZ] = [ABC]$, where $[ABC]$ denotes oriented area. Prove that H, X, Y and Z are cyclic.

Problem 3.2 (APMO 2017)**.** Let ABC be a triangle with $AB < AC$. Let D be the intersection point of the internal bisector of angle BAC and the circumcircle of ABC. Let Z be the intersection point of the perpendicular bisector of AC with the external bisector of angle $\angle BAC$. Prove that the midpoint of the segment AB lies on the circumcircle of triangle ADZ.

Problem 3.3 (Ukraine RMM TST 2013)**.** Let ABC be a triangle with orthocenter H and points D, E and F on sides AB, CA and BC, respectively such that $DB = DF$ and $DC = DE$. Prove that A, E, F and H are concyclic.

Problem 3.4 (Russia)**.** Let ABC be a triangle with M being the midpoint of BC and H being the foot of altitude from A. Let the perpendicular from M to AC meet AB at P, and let the perpendicular from M to AB meet AC at Q. Let X be a point symmetric to H with respect to M. Prove that M, P, Q and X are cyclic.

Problem 3.5. Let $ABCD$ be a circumscribed quadrilateral with T being an arbitrary point on side AB. Prove that incentres of ATD, BTC, CTD and T are cyclic.

Problem 3.6 (ELMO 2010 SL G6)**.** Let ABC be a triangle with circumcircle Ω. X and Y are points on Ω such that XY meets AB and AC at D and E, respectively. Show that the midpoints of XY, BE, CD, and DE are concyclic.

Problem 3.7 (China TST 2006)**.** Let ω be the circumcircle of $\triangle ABC$. P is an interior point of $\triangle ABC$. A_1, B_1 and C_1 are the intersections of AP, BP and CP respectively with ω and A_2, B_2 and C_2 are the symmetrical points of A_1, B_1 and C_1 with respect to the midpoints of sides BC, CA and AB. Show that the circumcircle of $\triangle A_2B_2C_2$ passes through the orthocenter of $\triangle ABC$.

Problem 3.8 (ISL 2012 G6)**.** Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

Problem 3.9 (ISL 2009 G8)**.** Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N. Denote by I_1, I_2 and I_3 the incenters of $\triangle ABM$, $\triangle MNC$ and $\triangle NDA$, respectively. Prove that the orthocenter of $\triangle I_1I_2I_3$ lies on g.