A Geometric Interpretation of Some Polynomial Equations

Navid Safaei

Sharif University of Technology, Tehran, Iran

1 Introduction

The goal of this article is to provide a geometric approach for studying polynomial equations of the form $ax^m - bx^n = c$, where c is a real number and m, n are positive integers. Usually such equations are tackled using the polar representation of complex numbers, but this is not always optimal and fairly often leads to incomplete arguments. We propose here a more geometric approach, which still uses in some sense the polar representation, but stays closer to the realm of synthetic geometry.

2 Preliminaries

Let z_1, z_2 be two complex numbers and denote by θ the argument of $z_1 - z_2$ (see figure 1).

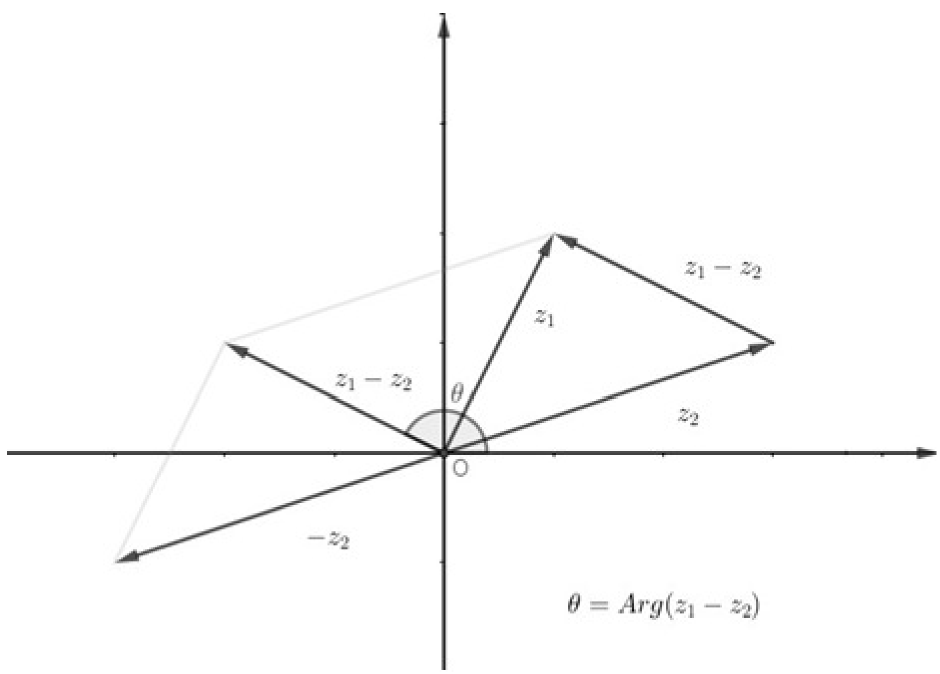

Figure 1

Then $\mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2)$ if and only if $z_1 - z_2$ is a real number, or equivalently θ is an integral multiple of π. Yet another equivalent formulation is that the vector representing $z_1 - z_2$ is horizontal, i.e. parallel to the x-axis. This is clearly depicted in figure 2.

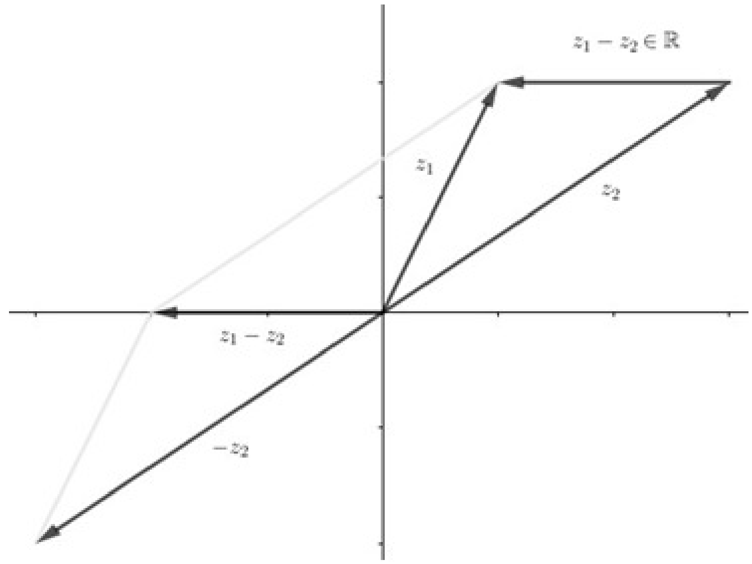

Figure 2

Consider now a relation of the form $A - B = C$, where A, B, C are complex numbers, C being real. The geometric interpretation of this relation is as follows: A is a point on the circle centered at the origin and having radius $|A|$, B is a point on the circle centered at the origin and having radius $|B|$, and the segment connecting A and B is parallel to the x-axis and has length $|C|$. This very simple observation has a certain number of nontrivial consequences, as the next section will show.

3 Applications

We will illustrate now the previous theoretical observations with a certain number of problems from Mathematical Olympiads.

Problem 1. Prove that if the equation $z^{n+1} - z^n - 1 = 0$ has a root lying on the unit circle centered at the origin, then $n + 2$ is a multiple of 6.

Solution. Let z be a solution of the equation with $|z| = 1$ and let θ be the argument of z. Let A be the point represented by z^{n+1}, so that A lies on the unit circle (centered at the origin) and has argument $(n+1)\theta$. Similarly, let B be the point represented by z^n. It also lies on the unit circle and has argument $n\theta$. Since $z^{n+1} - z^n = 1$, our previous discussion shows (see figure 3) that $AB = 1$ and $\angle AOB = \theta$. Thus the triangle OAB is equilateral and $\theta = \pm\frac{\pi}{3}$. Since $z^{n+1} - z^n$ is a real number, we also have $\sin((n+1)\frac{\pi}{3}) - \sin(n\frac{\pi}{3}) = 0$. Considering the various possibilities for n modulo 6, we can easily find that $n \equiv 4 \pmod 6$.

Here is a slightly different argument. Rewrite the equation as $z^n(z-1) = 1$. Since $|z| = 1$, we deduce that $|z-1| = 1$. Thus z lies at the intersection of the unit circle centered at the origin and the circle of radius 1 centered at 1. There are two such points, corresponding to the solutions of the equation $z^2 - z + 1 = 0$, i.e. $z = e^{\pm i\frac{\pi}{3}}$ (algebraically, we have $1 = |z-1|^2 = 2 - (z + \bar{z})$ and $\bar{z} = 1/z$). In particular $z - 1 = z^2$ and the equation becomes $z^{n+2} = 1$. Since $z^3 = -1$ and $z^{n+2} = 1$, one immediately deduces that $6 \mid n + 2$.

Remark. Conversely, if $n + 2$ is a multiple of 6, then the polynomial $X^{n+1} - X^n - 1$ is divisible by $X^2 - X + 1$ and so the equation $z^{n+1} - z^n - 1 = 0$ has a solution on the unit circle.

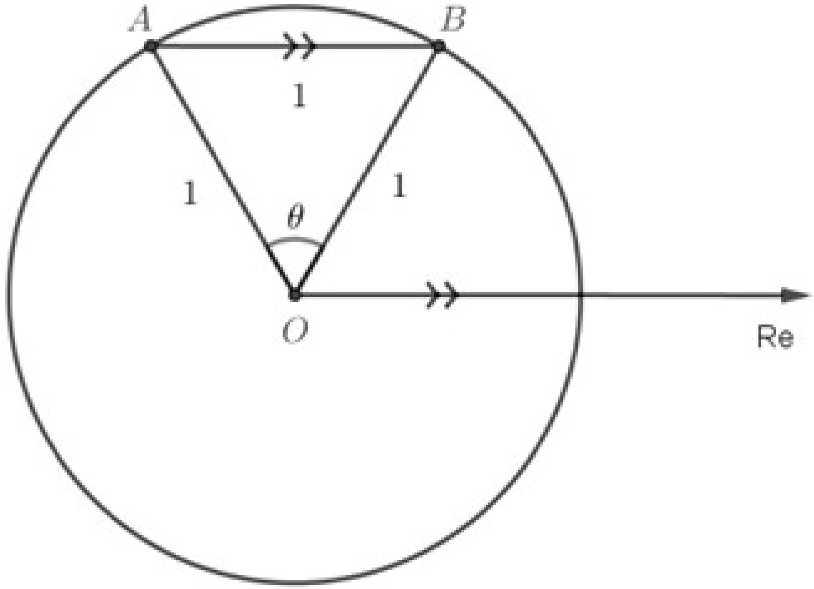

Figure 3

Problem 2. Find all positive integers n for which the polynomial $\sqrt{3}X^{n+1} - X^n - 1$ has a root on the unit circle.

Solution. Let z be a root on the unit circle and let θ be its argument. Let A, B be the points representing $\sqrt{3}z^{n+1}$, z^n, respectively. Then A lies on the circle centered at the origin, of radius $\sqrt{3}$ (see figure 4) and B lies on the unit circle centered at the origin. Since $\sqrt{3}z^{n+1} - z^n = 1$, we deduce from figure 4 that $AB = 1$, $\angle AOB = \theta$. Hence, by the Law of cosines in triangle OAB

$$\cos\theta \ = \frac{OA^2 + OB^2 - AB^2}{2OA.OB} \ = \ \frac{\sqrt{3}}{2},$$

i.e. $\theta = \pm\frac{\pi}{6}$. Combined with the fact that $\sqrt{3}z^{n+1} - z^n$ is a real number, this easily yields $n \equiv 10 \pmod{12}$. Conversely, if this is the case, then the polynomial $\sqrt{3}X^{n+1} - X^n - 1$ is divisible by $X^2 - \sqrt{3}X + 1$; since any root z of the latter satisfies $\sqrt{3}z - 1 = z^2$ and $z^{n+2} = 1$, hence $z^n(\sqrt{3}z - 1) = z^{n+2} = 1$. Thus the solutions of the problem are those positive integers $n \equiv 10 \pmod{12}$.

Here is an alternative way of arguing: rewrite the equation $z^n(\sqrt{3}z - 1) = 1$, showing that $|\sqrt{3}z - 1| = 1$. Combined with $|z| = 1$ this yields $z + \bar{z} = \sqrt{3}$, or equivalently $z^2 - \sqrt{3}z + 1 = 0$. Thus the original equation becomes $z^{n+2} = 1$. Now the solutions of the equation $z^2 - \sqrt{3}z + 1 = 0$ are $e^{\pm i\frac{\pi}{6}}$, and they satisfy $z^{n+2} = 1$ if and only if $12 \mid n+2$.

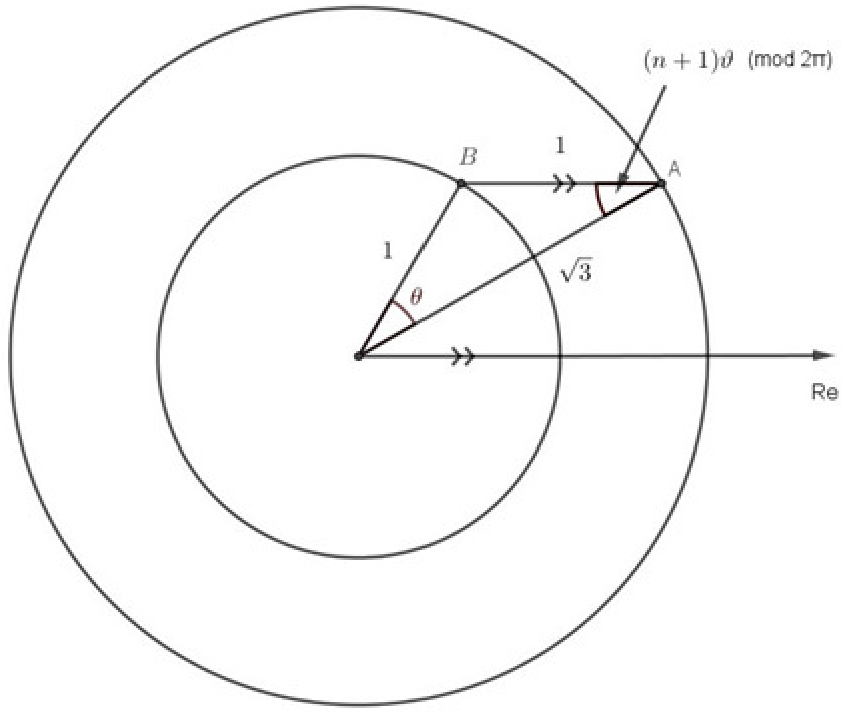

Figure 4

Problem 3. (Mathematical Reflections U296) Let a, b be nonzero real numbers. Prove that any non-real root z of the polynomial $X^{n+1} + aX + nb$ satisfies $|z| \geq \sqrt[n+1]{|b|}$.

Solution. Let $|z| = r,\ \text{Arg}\,(z) = \theta$, then z^{n+1} is a point with the argument of $(n+1)\theta$ on the circle with the radi r^{n+1} (i. e., point A in the figure 5) and $|a|\,z$ is a point with the argument of θ on the circle with the radi $|a|\,r$ (i. e., point B in the figure 5). Since $z^{n+1} + az = -nb$, we find that $AB = nb$. Since $z^{n+1} + az$ is real number, we can easily find that AB should be parallel to the $x-$ axis. Thus, in figure 5, we find that $\angle AOB = -n\theta,\ \angle BAO = (n+1)\,\theta, \angle ABO = \pi - \theta$. Now, by the law of sine in the triangle AOB, we can find that

$$\frac{|\sin\theta\ |}{OA} = \frac{|\sin(n+1)\theta\ |}{OB} = \frac{|\sin n\theta\ |}{AB}.$$

Therefore,

$$\frac{\sin\theta}{r^{n+1}} = \frac{\sin(n+1)\theta}{ar} = \frac{\sin n\theta}{nb},$$

which yields (taking into account the hypothesis that z is not real, thus $\sin\theta \neq 0$)

$$nb = r^{n+1} \cdot \frac{\sin n\theta}{\sin\theta}.$$

To conclude, it suffices to prove that $|\sin n\theta| \leq n|\sin\theta$, which follows by a simple induction on n, using that

$$|\sin(n+1)\theta| = |\sin n\theta \cdot \cos\theta + \cos n\theta \cdot \sin\theta| \leq |\sin n\theta| + |\sin\theta|.$$

We find it instructive to give an alternative argument for the crucial equality

$$nb = r^{n+1} \cdot \frac{\sin n\theta}{\sin\theta},$$

based on the polar representation. Namely, write $z = re^{i\theta}$, then

$$r^{n+1}e^{i(n+1)\theta} + are^{i\theta} + nb = 0$$

is equivalent to

$$r^{n+1}\sin(n+1)\theta + ar\sin\theta = 0 \quad \text{and} \quad r^{n+1}\cos(n+1)\theta + ar\cos\theta + nb = 0.$$

It follows that

$$ar\sin\theta\cos\theta = -r^{n+1}\cos\theta\sin(n+1)\theta\theta(r^{n+1}\cos(n+1)\theta + nb),$$

$$ar\sin\theta\cos\theta = -\sin\theta(r^{n+1}\cos(n+1)\theta + nb).$$

Which yields that

$$nb\sin\theta = r^{n+1}(\cos\theta\sin(n+1)\theta - \sin\theta\cos(n+1)\theta) = r^{n+1}\sin n\theta,$$

as desired.

Finally, let us present a purely algebraic and fairly simple proof. Using that a, b are real numbers, we obtain $\bar{z}^{n+1} + a\bar{z} + nb = 0$. Eliminating a yields

$$\bar{z}(z^{n+1} + nb) = z(\bar{z}^{n+1} + nb),$$

which can be equivalently written as

$$|z|^2(z^n - \bar{z}^n) = nb(z - \bar{z}).$$

Since z is not real, this is furthermore equivalent to

$$nb = |z|^2(z^{n-1} + z^{n-2}\bar{z} + ... + \bar{z}^{n-2}z + \bar{z}^{n-1}).$$

Using the triangular inequality immediately yields $n|b| \leq n|z|^{n+1}$, as desired.

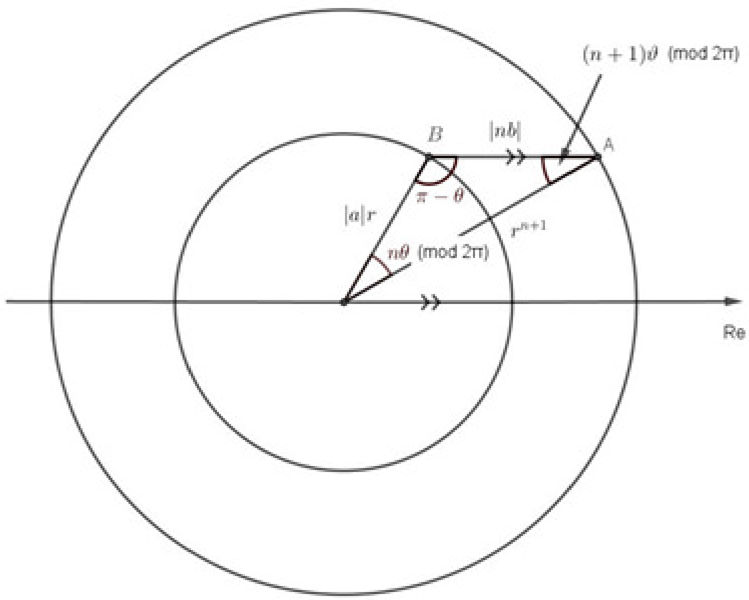

Figure 5

Problem 4. Let a, b be nonzero real numbers such that $b > a$. Prove that the polynomial $bX^n - aX^m + a - b$ has exactly $\gcd(m, n)$ roots lying on the unit circle.

Solution. Let $d = \gcd(m, n)$. Rewriting the polynomial as $b(X^n - 1) - a(X^m - 1)$ makes it clear that all dth roots of unity (i.e. solutions of the equation $z^d = 1$) are roots of the polynomial lying on the unit circle. Thus it remains to prove that if z is a root of the polynomial lying on the unit circle, then $z^d = 1$. But bz^n is a point on the circle centered at the origin, of radius $|b|$ and az^m is a point on the circle centered at the origin, of radius $|a|$. Since $bz^n - az^m = b - a$, the length of the segment between these points is $|b-a| = b-a$, which easily yields $m\text{Arg}\,(z) \equiv n\text{Arg}\,(z) \equiv 0 \pmod{2\pi}$. Therefore $d \cdot \text{Arg}\,(z) \equiv 0 \pmod{2\pi}$ and so $z^d = 1$, as desired.
Here is an alternative, more algebraic proof of the fact that $z^d = 1$ whenever z is a root on the unit circle of the polynomial. Write the equation as $b(z^n - 1) = a(z^m - 1)$. This shows that it suffices to prove that $z^m = 1$ (as then the equation forces $z^n = 1$ and so $z^d = 1$). If this is not the case, taking the complex conjugate of the equation and using that $\bar{z} = 1/z$ yields $b(1 - z^n)/z^n = a(1 - z^m)/z^m$, which combined with the equation gives $z^n = z^m$, but then $(z^m - 1)(b - a) = 0$ and $z^m = 1$, a contradiction.